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Lefschetz exceptional collections in S_k-equivariant categories of $(\mathbb{P}^n)^k$

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The bounded derived category of coherent sheaves is the main homological invariant of an algebraic variety which captures the most essential geometric information. It stands in the focus of many recent research papers. One of the ways to describe it is via an exceptional collection.

Recall that an object E in a $\mathbb{C}$-linear triangulated category $\mathcal{T}$ is *exceptional* if $\mathrm{Ext}^0(E,E)=\mathbb{C}$ and $\mathrm{Ext}^i(E,E)=0$ for $i\neq 0$. Furthermore, a collection $E_1,\dots,E_r$ of objects in $\mathcal{T}$ is an *exceptional collection* if each E_i is an exceptional object and $\mathrm{Ext}^\bullet(E_i,E_j)=0$ for $i>j$. An exceptional collection is *full* if the smallest full triangulated subcategory of $\mathcal{T}$ containing all E_i coincides with $\mathcal{T}$.

Recently a special class of exceptional collections attracted much attention. Recall that an exceptional collection $E_1,\dots,E_r$ in the bounded derived category of coherent sheaves $\mathcal{D}(X)$ of a smooth projective variety X is *Lefschetz* with respect to a line bundle $\mathcal{L}$ if there is a partition $r=r_0+r_1+\dots+r_d$ with $r_0\geq r_1\geq\dots\geq r_d$ such that

$$E_{r_0+r_1+\dots+r_{i-1}+t}\cong E_t\otimes\mathcal{L}^i \qquad \text{for all } 1\leq t\leq r_i \qquad \text{and } 1\leq i\leq d.$$

In other words, if the objects of the collection are obtained by $\mathcal{L}$-twists from the subcollection of the first r_0 objects according to the pattern provided by the partition.

As it is clear from the definition, a Lefschetz collection with respect to a given line bundle $\mathcal{L}$ is determined by its *starting block* $E_1,\dots,E_{r_0}$ and the partition $(r_0,r_1,\dots,r_d)$. It is less evident, but is still true, that if a Lefschetz collection is full, then the partition is itself determined by the starting block of the collection [6, Lemma 4.5]. Thus, extendability to a Lefschetz collection is just a property of an exceptional collection $E_1,\dots,E_{r_0}$.

It follows that there is a natural partial order on the set of all Lefschetz collections in $\mathcal{D}(X)$ with respect to a given line bundle $\mathcal{L}$ — a Lefschetz collection with a starting block $E_1,\dots,E_{r_0}$ is *smaller* than a Lefschetz collection with a starting block $E'_1,\dots,E'_{s_0}$ if $E_1,\dots,E_{r_0}$ is a subcollection in $E'_1,\dots,E'_{s_0}$, see [9, Definition 1.4].

A Lefschetz collection $E_1,\dots,E_r$ with partition $r_0,r_1,\dots,r_d$ is called *rectangular of length* $d+1$, if $r_0=r_1=\dots=r_d$ (equivalently, if the Young diagram representing the partition is a rectangle of length $d+1$). Of course, a necessary condition for the existence of a rectangular Lefschetz collection in $\mathcal{D}(X)$ is a factorization

$$\mathrm{rk}\left(K_0(\mathcal{D}(X))\right)=r_0(d+1) \tag{1.1}$$

for the rank of the Grothendieck group of X. On the other hand, if a rectangular Lefschetz decomposition in $\mathcal{D}(X)$ exists, and if its length $d+1$ has the property that $\mathcal{L}^{d+1}\cong\omega_X^{-1}$ where ω_X is the canonical bundle of X, that is $d+1$ equals the *index* of X with respect to $\mathcal{L}$, then this collection is automatically minimal (this follows easily from Serre duality, see [9, Subsection 2.1]).

Lefschetz collections have many nice properties and are very important for homological projective duality and categorical resolutions of singularities [7]. Especially nice and important are rectangular (resp. minimal) Lefschetz collections. So, the following problem is very interesting.

Problem 1.1. *Given a smooth projective variety X and a line bundle $\mathcal{L}$, construct a full rectangular Lefschetz collection in $\mathcal{D}(X)$ with respect to $\mathcal{L}$ of length equal to the index of X, or, if the above is impossible, a minimal Lefschetz collection.*

There are many varieties X for which the above problem was solved. Among these are projective spaces, most of the Grassmannians, and some other homogeneous spaces [2]. In this dissertation we discuss Problem 1.1 for a very simple variety

$$X = X_k^n := \underbrace{\mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{k \text{ copies}},$$

but replace the category $\mathcal{D}(X_k^n)$ with the equivariant derived category $\mathcal{D}_{S_k}(X_k^n)$ with respect to the natural action of the symmetric group S_k (by permutation of factors). Note that this category can be considered as the derived category of the *quotient stack* $[X_k^n/S_k]$. The line bundle $\mathcal{L}$ here is, of course, the ample generator $\mathcal{O}(1, 1, \ldots, 1)$ of the invariant Picard group $\mathrm{Pic}(X_k^n)^{S_k}$. Note that the index of X_k^n with respect to $\mathcal{L}$ is equal to $n + 1$, so the goal of the dissertation can be formulated as follows.

Problem 1.2. *Find a full rectangular Lefschetz collection of length $n+1$ in $\mathcal{D}_{S_k}(X_k^n)$ with respect to the line bundle $\mathcal{O}(1, 1, \ldots, 1)$ or a minimal Lefschetz collection if the above is impossible.*

Note that without passing to the equivariant category the problem becomes trivial. To construct a rectangular Lefschetz collection in $\mathcal{D}(X_k^n)$ one can just choose any full exceptional collection in $\mathcal{D}(X_{k-1}^n)$ and consider its pullback to X_k^n as the starting block. Using the projective bundle formula it is elementary to check that it extends to a rectangular Lefschetz collection of length $n + 1$. However, the S_k-symmetry in this construction is broken, and it cannot be performed in the equivariant category.

For $k = 1$ the Problem 1.2 is trivial (the desired collection is just the Beilinson exceptional collection $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ of line bundles on $\mathbb{P}^n$). Furthermore, for $k = 2$ the Problem 1.2 was essentially solved in [11].

The main result of the dissertation is a partial solution to the Problem 1.2.

First, we construct a rectangular S_k-invariant Lefschetz exceptional collection of line bundles in $\mathcal{D}(X_k^n)$ whose cardinality in case of coprime k and $n+1$ equals the rank of the Grothendieck group of X_k^n (by Elagin's Theorem this gives an exceptional collection in the equivariant category, whose length equals the rank of its Grothendieck group). The first block of the collection is defined as $\langle \mathcal{O}(e) \rangle_{e \in \mathbb{E}_k^n}$, where

$$\mathbb{E}_k^n = \left\{ S_k \cdot e \;\middle|\; e_1 \geq \cdots \geq e_k = 0 \text{ and } e_i \leq \frac{h(k-i)}{k} \right\} \subset \mathbb{Z}^k. \tag{1.2}$$

So, it is natural to expect that this collection is full and (in the coprime case) gives a solution to Problem 1.2. However, in general we could not prove its fullness.

Our second main result is a proof of fullness of the above collection for $k = 3$ and $n = 3p$ or $n = 3p + 1$ (this ensures that k and $n + 1$ are coprime).

We also perform a first step in the direction of non-coprime k and $n + 1$ by constructing a minimal S_3-invariant Lefschetz exceptional collection in $\mathcal{D}(X_3^2)$ (including a proof of its fullness).

Besides that we also solve the Problem 1.2 for $n = 1$, that is, construct a rectangular S_k-invariant Lefschetz collection of length 2 in $\mathcal{D}(X_k^1)$ when k is odd, and a minimal Lefschetz collection when k is even. However, this case is much more simple than the case $k = 3$ discussed above.

An interesting feature of the Lefschetz collections that we construct is that they resemble very much the minimal Lefschetz collections in the derived categories of the Grassmannians $\mathrm{Gr}(k, n + 1 + k)$ constructed by Anton Fonarev, see [2]. It would be very interesting to understand the relations between these, since on one hand, this suggests a possible solution to the Problem 1.2 for other values of k (by considering analogues of Fonarev's collections), and on the other hand, a solution to the Problem 1.2 can help in dealing with the Grassmannians $\mathrm{Gr}(k, n)$ when k and n are not coprime (in this case there is no rectangular collection on the Grassmannian, and a minimal collection is not quite known).

The results of the thesis are published in two papers

1. M. Mironov, *Lefschetz exceptional collections in S_k-equivariant categories of* $(\mathbb{P}^n)^k$, European Journal of Mathematics 7 (2021), pages 1182–1208

2. M. Mironov, *S_2-invariant exceptional collections on* $\mathbb{P}^n \times \mathbb{P}^n$, Mathemaical Notes 111 (2022), pages 316–319

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