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Invariants and Parameter Space Models for Rational Maps

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The main research area of this thesis is holomorphic dynamics. In holomorphic dynamics we deal with dynamical systems given by analytic functions, and therefore these dynamical systems can be "complexified". Complex numbers are easier to work with than real numbers. This makes methods of holomorphic dynamics so important. In this thesis we are studying holomorphic rational functions of one variable. We focus on the different classes of these functions and on the methods of their classification and parameterization by meaning of geometrical objects using mainly a topological approach.

1 Invariant trees for Thurston map

We consider 1-dimensional holomorphic rational functions $f : \mathbb{S}^2 \to \mathbb{S}^2$. Recall that *critical points* are the points at which the derivative is zero or not defined. Images of critical points are called *critical values*. The first class of 1-dimensional holomorphic rational functions is the class of *post-critically finite* rational functions, which are also called *Thurston functions*. Recall that *the degree of a rational function* is the maximum of the degrees of the numerator and denominator.

Definition 1. Denote the set of critical points of f as $C(f)$. Then the following union $P(f) := \bigcup_{n=1}^{\infty} f^n(C(f))$ is called the *post-critical set* of f. If $P(f)$ is a finite set, then f is said to be *post-critically finite*.

These Thurston maps were introduced by W. Thurston in his work related to the studying of the rational maps. The main celebrated result of this work is *Thurston's characterization theorem* (see [DH93]). This theorem allows to study algebraic objects (rational functions) by topological tools. More precisely, we can look at rational maps as at a class of purely topological objects – branched coverings. It turnes out that on the class of Thurston maps there exists a natural equivalence relation, which is called Thurston equivalence, such that different rational functions are almost never equivalent. Formally,

Definition 2. Two Thurston maps $f : \mathbb{S}^2 \to \mathbb{S}^2$ and $g : \mathbb{S}^2 \to \mathbb{S}^2$ are called (Thurston) *equivalent* if there exist two orientation-preserving homeomorphisms h_0, $h_1 : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h_0 \circ f = g \circ h_1$ and such that h_0 and h_1 are homotopic relative the post-critical set $P(f)$.

Informally, Thurston equivalence can be seen as a combination of the *topological conjugacy* (existence of a homeomorphism $h : S^2 \to \tilde{S}^2$ such that $h \circ f = g \circ h$) on the set of post-critical points and of the "perturbed" topological conjugacy on the other surface.

From the formal definition we can see that *topological conjugacy* is a special case of a Thurston equivalence.

So Thurston theorem is a very powerful tool, which allows to understand if a Thurston map is equivalent to a rational function. The criteria of this equivalence depends on the existence of a purely topological object: *combinatorial obstruction*, which is some special union of simple curves outside the set of *post-critical points*.

Thurston theorem shows that Thurston map is equivalent to a rational function if and only if there is no obstruction. But it is a very complicated and non-trivial problem to show "non-existence" of obstruction, since it basically means that we have to check infinitely many options for sets of curves. Thus, even if there exists a classification theorem, the general problem of classification of Thurston maps up to equivalence remains an important problem. It has been focus of recent developments, for example [BN06, BD17, CG+15, KL18, Hlu17]. We will be interested in degree 2 Thurston maps.

We look on the rational function as on the branched coverings of the sphere, so we write $\mathbb{S}^2$ for the oriented topological 2-sphere. One of the goals of our work is to reduce the dynamics on the sphere to the dynamics on the purely combinatorial object: a *graph in the sphere*, which is exactly a 1-dimensional cell complex embedded into $\mathbb{S}^2$. By its *vertices* and *edges*, we mean 0-cells and 1-cells, respectively. Then for a graph G, we denote the set of its vertices as $V(G)$ and the set of its edges as $E(G)$. A *tree* is a simply connected graph. For a vertex x of a tree $T \subset \mathbb{S}^2$ and an edge e of T if x is in the closure of e, then x is called *incident* to e. In this case e is also *incident* to x. We say that a vertex is *terminal* if it is adjacent to exactly one edge. Vertex a is non-terminal if and only if its complement $T \backslash \{a\}$ consists of at least two connected components. Vertices of the graph T which do not satisfy the last property are called *branch points*. Let $P \subset \mathbb{S}^2$ be some finite subset. Then a tree T in $\mathbb{S}^2$ such that $P \subset V(T)$ is called a *spanning tree* for P if $V(T) - P$ consists of branch points.

Definition 3 (Invariant spanning tree)**.** Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a Thurston map. A spanning tree T for $P(f)$ is called an *invariant spanning tree* for f if:

1. we have $f(T) \subset T$;

2. vertices of T map to vertices of T.

Some examples of invariant spanning trees (or of close similar objects) were also considered in some other works:

1. First example for polynomials could be obtained from the construction of invariant graphs, called *Hubbard tree*, which was introduced by A. Douady and J. Hubbard. These trees are are finite planar trees, containing postcritical set. We can connect *Hubbard trees* to infinity to form invariant spanning trees.

2. For two polynomials p and q we can consider their *formal mating* $f = p \amalg q$. Informally, we identify the dynamical plane of p to the upper hemisphere $\mathbb{H}^+$of $\mathbb{S}$ and the dynamical plane of q to the lower hemisphere $\mathbb{H}^-$of $\mathbb{S}$ and then we consider their "regluing". Then we can obtain invariant spanning trees by joining the two Hubbard trees.

3. We can obtain invariant spanning trees from the geometrical construction of classical *captures* in the sense of [Wit88, Ree92].

We are also interested in some additional structure on a graph.

We want to work not only with embedded graphs, but also with abstract graphs. Then we should define a *ribbon graph (or a fat graph, or a cyclic graph)*:

Definition 4. An abstract graph in which the edges incident to each particular vertex are cyclically ordered is called a *ribbon graph*.

In fact, by [MA41], ribbon trees can be seen as isomorphism classes of embedded trees in $\mathbb{S}^2$.

The following result shows the importance of the invariant spanning trees. It enables us to recover the Thurston equivalence class of f from an invariant spanning tree of f.

For a spanning tree T for $P(f)$, we write $C(T)$ for the set of critical points of f in T.

Theorem 1 (Theorem A in [ST19])**.** *Suppose that f, $g : \mathbb{S}^2 \to \mathbb{S}^2$ are two Thurston maps of degree 2. Let T_f and T_g be invariant spanning trees for f and g, respectively. Suppose that there is a cellular homeomorphism $\tau : T_f \to T_g$ with the following properties:*

1. *The map τ is an isomorphism of ribbon graphs.*

2. *We have $\tau \circ f = g \circ \tau$ on $V(T_f) \cup C(T_f)$.*

3. *The critical values of f are mapped to critical values of g by τ.*

Suppose also that τ can be extended to edges of $f^{-1}(T_f)$ incident to points in $C(T_f)$ to the isomorphism of the new graph (with the edges as mentioned above attached, for which $C(T_f)$ are vertices) and a similar graph constructed for a map g, to preserve the cyclic order of edges incident to a given vertex of $C(T_f)$ and so that to satisfy (2). *Then f and g are Thurston equivalent.*

In other words, to know the Thurston equivalence class of f, it suffices to know the following data:

1. the ribbon graph structure of T_f;

2. the restriction of the map f to the set $V(T_f) \cup C(T_f)$;

3. the cyclic order, in which pullbacks of certain edges of T_f appear around a point of $C(T_f)$.

These data are purely combinatorial. It means that it can be inserted in the computer program to analyse and compare branched coverings.

There exists also a very important algebraic invariant of a Thurston map – a *biset*. A biset is a algebraic object, which fully encodes the Thurston equivalence class. To describe it let us fix some objects for a Thurston map f and notations for them:

- a fixed basepoint $y \in \mathbb{S}^2 - P(f)$;
- the set $\mathcal{X}_f(y)$ denoting the set of all homotopy classes of paths from y to $f^{-1}(y)$ in $\mathbb{S}^2 - P(f)$;
- the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$ denoted as π_f.

It turns out, that there is a structure of a *biset over a fundamental group* π_f on the $\mathcal{X}_f(y)$. The most full definition with algebraic background can be found in [Nek05] (bisets are called *bimodules* there, see Chapter 2). The *iterated monodromy group* of f can be immediately recovered from the biset.

It turns out, the the biset can be recovered from the invariant spanning tree:

Theorem 2. *[Theorem B in [ST19]] Suppose that f is a Thurston map of degree 2, and T is an invariant spanning tree for f. There is an explicit presentation of the biset of f based only on the data* (1) – (2) *listed below:*

1. *the ribbon graph structure on T,*

2. *the restriction of f to $V(T) \cup C(T)$.*

1.1 Dynamical tree pairs

But it is not a trivial problem to find the invariant spanning tree in general case. It turns out that we can generalize Theorem 2. We need to introduce one more object for being able to do it:

Definition 5. For a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$ of degree 2 two spanning trees T^* and T for $P(f)$ such that $f(T^*) \subset T$ are called a *dynamical tree pair* for f if the vertices of T^* are mapped to vertices of T under f. We denote a dynamical tree pair as (T^*, T).

From the definition we can see that dynamical tree pairs generalize invariant spanning trees.

Then to find an actual invariant spanning tree it is natural to look at an iterative process of the transitions from T to T^*. This iterative process will is called an *ivy iteration*.

Moreover, a spanning tree T for $P(f)$ gives rise to a distinguished generating set $\mathcal{E}_T$ of the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$ with $y \in \mathbb{S}^2 - T$. The set $\mathcal{E}_T$ consists of the identity element and the homotopy classes of smooth loops based at y intersecting the edge of T only once and transversely.

We also show that the biset of f is determined by a dynamical tree pair (T^*, T). More precisely, the biset can be explicitly presented knowing the following discrete data:

1. the ribbon graph structures on T^*, T;

2. the map $f : V(T^*) \cup C(T^*) \to V(T)$;

3. how elements of $\mathcal{E}_{T*}$ are expressed through elements of $\mathcal{E}_T$ (or how both $\mathcal{E}_{T*}$, $\mathcal{E}_T$ are expressed through some other generating set of $\pi_1(\mathbb{S}^2 - P(f), y)$).

1.2 The ivy iteration

As the final result we introduce a combinatorial method of finding invariant spanning trees. "Combinatorial" means that using this method we can insert the combinatorial information about the dynamical tree pair of a Thurston map f into the computer program and thus find invariant spanning trees.The *ivy object* is defined as a homotopy rel. $P(f)$ class of spanning trees for $P(f)$. Then we introduce the *pullback relation* $[T] \multimap [T^*]$ on the set $Ivy(f)$ of ivy objects. A similar relation on isotopy classes of simple closed curves in $\mathbb{S}^2 - P(f)$ was discussed in [Pil03, KPS16]. Then let $ivy(f)$ denote the set of all ivy objects for f. Let T be a spanning tree, and $[T]$ be the corresponding ivy object. A symbolic presentation of the biset of f plus a symbolic encoding of the ribbon tree structure on T give rise to several choices of a spanning tree T^* such that (T^*, T) is a dynamical tree pair for f. We can equip the set $Ivy(f)$ with the structure of an abstract directed graph: we connect two vertices corresponding to two ivy objects $[T_1]$ and $[T_2]$ by an oriented arrow from $[T_1]$ to $[T_2]$ if (T_2, T_1) is a dynamical tree pair. We show that all the data corresponding to this graph can be encoded purely combinatorially. Each arrow of the graph is associated is the transition from T to T^*. Moving by these arrows is exactly an *ivy iteration*.

In the end we introduce some examples of the ivy iteration, which were obtained as the results of the computer program.

2 Zakeri slices parametrization

The second class of rational mappings we focus on is the class of complex cubic polynomials. Let us recall that if z_0 is a fixed point of an analytic function f, i.e. $f(z_0) = z_0$, then the number $\lambda = f'(z_0)$ is called the *multiplier* of f at z_0. We assume fixed point to be 0. Then we are interested in the class $\mathbb{C}_\lambda$ of polynomials with a fixed multiplier $|\lambda| \leqslant 1$ of a fixed point. This space $\mathbb{C}_\lambda$ is called λ-*slice*. For a cubic polynomial P we write $[P]$ for its affine conjugacy class. We have some additional requirements to λ in our work.

Definition 6. The number θ is *bounded type* if the continued fraction coefficients of θ are bounded.

If we suppose that the rotational number of fixed $\lambda = e^{i\phi}$ is of bounded type, then for the polynomials in this slice the origin is a fixed Siegel point (in its neighbourhood function is linearizable). Slices with this property were studied by S. Zakeri as parameter spaces, so we call them *Zakeri slices*.

There exists classical and very powerful method of studying polynomials with fixed or periodic points based upon *linearizations*:

Definition 7. A function $f(z)$ is called *linearizable* if there exists a holomorphic change of coordinates h (the linearization of f) such that $h^{-1} \circ f \circ h = \lambda z$, i.e. f is conjugate to λz. The region, where linearization exists is the *Siegel disc* or a *Herman ring*, or a part of *attracting domain*.

Suppose that a cubic polynomial P has a non-repelling fixed point a, since it can always be arranged by a suitable affine conjugacy that $a = 0$. Let us recall again that we consider the set of all affine conjugacy classes $[P]$ of cubic polynomials P with $P(0) = 0$ and $|P'(0)| \leqslant 1$. Then a central part of this parameter space is the *principal hyperbolic component*, which consists of classes $[P]$ for all hyperbolic P with $|P'(0)| < 1$ and Jordan curve Julia set.

Now let us consider a polynomial f from our class with attracting or neutral fixed point a. Then its linearization (if it exists) is a map ψ of an open disk $\mathbb{D}(r)$ of radius $r > 0$ around 0 such that $\psi(0) = a$, and $\psi(\lambda z) = f \circ \psi(z)$ for all $z \in \mathbb{D}(r)$ where $\lambda = f'(a)$. Moreover, we assume that $r > 0$ is the radius of convergence of the power series of ψ at 0.

Then $\psi(\mathbb{D}(r))$ is called the *Siegel disc* $\Delta(f, a)$ of f around a. If $|\lambda| < 1$, then $\Delta(f, a)$ is compactly contained in the attracting basin of a, and $\partial\Delta(f, a)$ contains a critical point. In the case $a = 0$, the domain $\Delta(f, a)$ is denoted by $\Delta(f)$.

As above let $\mathbb{C}_\lambda$ be the space of complex linear conjugacy classes of complex cubic polynomials with fixed point 0 with fixed multiplier λ, such that $|\lambda| \leqslant 1$. For a cubic polynomial $P(z) = \lambda z + \ldots$, let $[P]_0$ be its class *in* $\mathbb{C}_\lambda$. Write $\mathcal{C}_\lambda \subset \mathbb{C}_\lambda$ for the *connectedness locus* in $\mathbb{C}_\lambda$. That is, $[P]_0 \in \mathcal{C}_\lambda$ if the Julia set $J(P)$ of P is connected. Again as it was described above, a central part of $\mathcal{C}_\lambda$ is the set $\mathcal{P}_\lambda$ of all $[P]_0 \in \mathcal{C}_\lambda$ that lie in the closure of the principal hyperbolic component. The main result of the second part of the thesis is related to understanding the topology and combinatorics of $\mathcal{P}_\lambda$ through a comparison with a suitable dynamical object.

For this comparison we consider the space of quadratic polynomials $Q(z) = Q_\lambda(z) = \lambda z(1 - z/2)$. Then λ is the multiplier of the fixed point 0 of Q. Then we suppose that either $|\lambda| < 1$ or $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type. Let $\psi = \psi_Q : \mathbb{D} \to \Delta(Q)$ be the corresponding linearization (here $\mathbb{D} = \mathbb{D}(1)$). The set $\overline{\Delta}(Q)$ is a Jordan disk — this is a classical result of Douady–Ghys–Herman–Shishikura, see [Dou87, Her87, ?]. Therefore, the Riemann map can be extended to a homeomorphism $\overline{\psi} : \overline{\mathbb{D}} \to \overline{\Delta}(Q)$. The finite critical point of Q is 1, thus the linearizatiton domain $\Delta(Q)$ around 0 contains 1 in its boundary. We normalize ψ so that $\overline{\psi}(1) = 1$. If $|\lambda| = 1$, then the map $\overline{\psi}$ conjugates the rigid rotation by angle θ with the restriction of Q to $\overline{\Delta}(Q)$. Consider the quotient $\tilde{K}(Q)$ of the set $K(Q) - \Delta(Q)$ by the equivalence relation $\sim$ defined as follows. Two different points z, w are equivalent if both belong to $\partial\Delta(Q)$, and $Re(\overline{\psi}^{-1}(z)) = Re(\overline{\psi}^{-1}(w))$.

The partially defined correspondence between the dynamical plane of P and that of Q can be described by a following property:

Property. *For any cubic polynomial P with $[P]_0 \in \mathcal{P}_\lambda$, there exist a full P-invariant continuum $X(P)$ (i.e. $P^{-1}(X(P)) = P(X(P)) = X(P)$) containing both critical points of P and a continuous map $\eta_P : X(P) \to K(Q)$ that semi-conjugates $f|_{X(P)}$ with $Q|_{\eta_P(X(P))}$. If both critical points of P are in the Julia set, then the map η_P is monotone.*

Now we can formulate the main result as the following theorem (illustrated in Figure 1).

Theorem 3 (Main Theorem in [BOST22]). *Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type, and $\lambda = e^{2\pi i\theta}$. Let $Q(z) = Q_\lambda(z) = \lambda z(1 - z/2)$ be a quadratic polynomial with a fixed point of multiplier λ. Then there is a continuous map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ taking $[P]_0$ to the η_P-image of some critical point of P.*

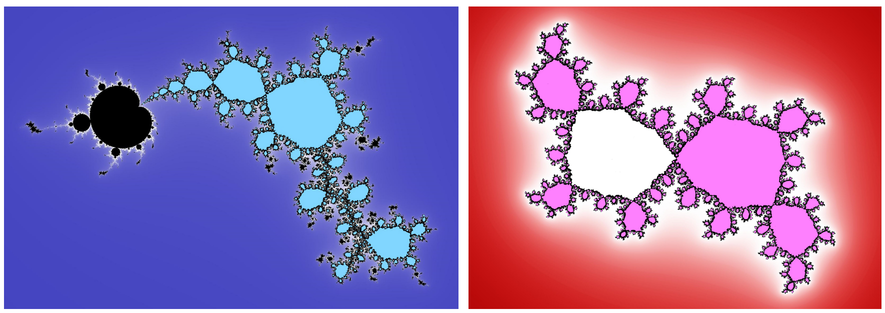

Figure 1: Left: the parameter plane $\mathbb{C}_\lambda$ with $\lambda = \exp(\pi i\sqrt{2})$. We used the parameterization, in which every linear conjugacy class from $\mathbb{C}_\lambda$ is represented by a polynomial of the form $f(z) = \lambda z + \sqrt{a}z^2 + z^3$, where a is the parameter (that is, the figure shows the a-plane). The conjugacy class of f is independent on the choice between the two values of the square root. Regions with light uniform shading are interior components of $\mathcal{P}_\lambda$. There are also various "decorations" of $\mathcal{P}_\lambda$ (that is, components of $\mathcal{C}_\lambda - \mathcal{P}_\lambda$) shown in black; these decorations contain copies of the Mandelbrot set. Right: the dynamical plane of $Q = Q_\lambda$. The bounded white region near the center is the Siegel disk $\Delta(Q)$. A conjectural model of $\mathcal{P}_\lambda$ is obtained from $K(Q)$ by removing this white region and gluing its boundary into a simple curve. Our main theorem provides a continuous map from $\mathcal{P}_\lambda$ to this conjectural model.

Conclusion

First result of this thesis is the theorem about classifying the Thurston maps up to the Thurston equivalence based only on their restriction on the invariant spanning trees:

Theorem 1 (Theorem A in [ST19]). *Suppose that f, $g : \mathbb{S}^2 \to \mathbb{S}^2$ are two Thurston maps of degree 2. Let T_f and T_g be invariant spanning trees for f and g, respectively. Suppose that there is a cellular homeomorphism $\tau : T_f \to T_g$ with the following properties:*

1. *The map τ is an isomorphism of ribbon trees.*
2. *We have $\tau \circ f = g \circ \tau$ on $V(T_f) \cup C(T_f)$.*
3. *The critical values of f are mapped to critical values of g by τ.*

Suppose also that τ can be extended to edges of $f^{-1}(T_f)$ incident to points in $C(T_f)$ to the isomorphism of the new graph (with the edges as mentioned above attached, for which $C(T_f)$ are vertices) and a similar graph constructed for a map g, to preserve the cyclic order of edges incident to a given vertex of $C(T_f)$ and so that to satisfy (2)*. Then f and g are Thurston equivalent.*

After we show, that knowing an invariant spanning tree we can fully describe another important invariant – the biset:

Theorem 2. *[Theorem B in [ST19]] Suppose that f is a Thurston map of degree 2, and T_f is an invariant spanning tree for f. There is an explicit presentation of the biset of f based only on the data* $(1)-(2)$ *listed below:*

1. *the ribbon graph structure on T,*
2. *the restriction of f to $V(T) \cup C(T)$.*

Finally, this thesis provides a new algorithm for searching invariant spanning trees for post-critically finite branched coverings.

In the second part we deal with slices of cubic polynomials obtained by fixing the fixed point multiplier. polynomials obtained by fixing the fixed point multiplier. We parameterize their parts $\mathcal{P}_\lambda$, belonging to the closure of the principal hyperbolic component. This parametrization uses the quadratic reglued model $\bar{K}(Q)$ of the Julia set $K(Q)$. We show that the paremetrizing map Φ_λ satisfy the following property:

Property. *For any $P \in \mathcal{P}_\lambda$, there exist a full P-invariant continuum $X(P)$ (i.e. $P^{-1}(X(P)) = P(X(P)) = X(P)$) containing a critical point c of P and a continuous monotone map $\eta_P : X(P) \to K(Q)$ such that η_P semi-conjugates $f|_{X(P)}$ with $Q|_{\eta_P(X(P))}$, and $\Phi_\lambda(P)$ is the image of $\eta_P(c)$ in $\tilde{K}(Q)$.*

Moreover, we prove the continuity of this parametrization by proving the following Theorem:

Theorem 3 (Main Theorem in [BOST22])**.** *Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is of bounded type, and $\lambda = e^{2\pi i\theta}$. Let $Q = Q_\lambda$ be a quadratic polynomial with a fixed point of multiplier λ. Then there is a continuous map $\Phi_\lambda : \mathcal{P}_\lambda \to \tilde{K}(Q)$ taking $[P]_0$ to the η_P-image of some critical point of P.*

The results of the thesis are published at two papers

• A. Shepelevtseva, V. Timorin, *Invariant spanning trees for quadratic rational maps*, Arnold Mathematical Journal.2019. №5, p.435–481

• A. Blokh, L. Oversteegen, A. Shepelevtseva, V. Timorin, *Modeling core parts of Zakeri slices I*, Moscow Mathematical Journal.2022. №2 p.265–294

References

[BD17] L. Bartholdi, D. Dudko, *Algorithmic aspects of branched coverings*, Ann. Fac. Sci. Toulouse **5** (2017), 1219–1296.

[BN06] L. Bartholdi, V. Nekrashevych, *Thurston equivalence of topological polynomials*, Acta Math. **197** (2006), 1–51

[BOST22] A. Blokh, L. Oversteegen, A. Shepelevtseva, V. Timorin, *Modeling core parts of Zakeri slices I*, Moscow Mathematical Journal.2022. №2 p. 265–294

[Br72] A. Brjuno, *Analytic form of differential equations*, Tr. Mosk. Mat. Obs., **26** (1972), 199–239.

[CG+15] K. Cordwell, S. Gilbertson, N. Nuechterlein, K. Pilgrim, S. Pinella, *The classification of critically fixed rational maps*, Conform. Geom. Dyn. **19** (2015), 51–94.

[Dou87] A. Douady, *Disques de Siegel et anneaux de Herman*, Astérisque, **152-153**, exp. 677 (1987), 151–172.

[DH93] A. Douady, J. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math., 171(2):263 297, 1993.

[GM93] L.R. Goldberg, J. Milnor, "Fixed points of polynomial maps. Part II. Fixed point portraits", Annal. sci. de l'École Normale Supérieure, 4^e série, **26**, n° 1 (1993), 51–98

[Her87] M. Herman, *Conjugaison quasi symétrique des homéomorphismes du cercle à des rotations*, preliminary manuscript, https://www.math.kyoto-u.ac.jp/ mitsu/Herman/qsconj1/

[Hlu17] M. Hlushchanka, *Invariant graphs, tilings, and iterated monodromy groups*, Ph.D. Thesis, Jacobs University Bremen (2017).

[Hub06] J.H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics.* Volume 1. Matrix Editions 2006.

[Kiw00] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, Ergodic Theory and Dynamical Systems, **20** (2000), no. 5, 1391–1403.

[KL18] G. Kelsey, R. Lodge, *Quadratic Thurston maps with few postcritical points*, Geom. Dedicata (2018). `https://doi.org/10.1007/s10711-018-0387-5`

[KPS16] S. Koch, K. Pilgrim, N. Selinger, *Pullback invariants of Thurston maps*. Trans. Amer. Math. Soc. **368** (2016), no. 7, 4621–4655.

[MA41] S. MacLane, V. W. Adkisson, *Extensions of homeomorphisms on the sphere*. Lectures in Topology, University of Michigan Press, Ann Arbor, Mich. 1941, 223–236.

[Ma] S. Marmi, *An Introduction to Small Divisors Problems*, arXiv:math/0009232

[Ma90] S. Marmi, *Critical functions for complex analytic maps*, J. Phys. A: Math. Gen. **23** (1990), 3447-=3474

[MSS83] R. Mañé, P. Sad, D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217.

[Mil09] J. Milnor, *Cubic polynomial maps with periodic critical orbit I*, in: "Complex Dynamics: Families and Friends", ed. D. Schleicher, A.K. Peters (2009), 333–411.

[Nek05] V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, Volume: **117** (2005), AMS.

[Pet96] C. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Acta Math., 177 (1996), 163-224

[Pil03] K. Pilgrim, *An algebraic formulation of Thurston's combinatorial equivalence*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3527–3534.

[Ree92] M. Rees, *A partial description of the Parameter Space of Rational Maps of Degree Two: Part 1* Acta Math. **168** (1992), 11–87

[Si] C. Siegel, *Iteration of analytic functions*, Ann. of Math. **43**(2) (1942), 607–612.

[ST19] A. Shepelevtseva, V. Timorin, Invariant spanning trees for quadratic rational maps, Arnold Math J. **5**, (2019), 435–481

[Sul] D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: Topological conjugacy classes of analytic endomorphisms*, manuscript.

[SW20] W. Shen, Y. Wang, *Primitive tuning via quasiconformal surgery*, preprint `arXiv:2010.05199`, to appear in Israel Journal of Mathematics.

[Wit88] B. Wittner, "On the bifurcation loci of rational maps of degree two", PhD Thesis, Cornell University, 1988

[Yoc95] J.C. Yoccoz, *Petits Diviseurs en Dimension 1*, Astérisque **231** (1995)

[Zak99] S. Zakeri, *Dynamics of cubic siegel polynomials*, Commun. Math. Phys. **206** (1999), 185–233.

[Zak16] S. Zakeri, *Conformal fitness and uniformization of holomorphically moving disks*, Trans. Amer. Math. Soc. **368** (2016), 1023–1049.