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Spectra of Bethe subalgebras in Yangians

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Introduction

Let $\mathfrak{g}$ be a simple Lie algebra. The Yangian $Y(\mathfrak{g})$ is a Hopf algebra, historically one of the first examples of *quantum groups*. It has been defined by V. Drinfeld in [D85].

The simplest case is $\mathfrak{g} = \mathfrak{sl}_n$. The Yangian $Y(\mathfrak{sl}_n)$ can be realized as a factor of the extended Yangian $Y(\mathfrak{gl}_n)$. Thus most of this work concerns the case of $Y(\mathfrak{gl}_n)$. $Y(\mathfrak{gl}_n)$ is in certain sense the unique Hopf algebra deforming the enveloping algebra $U(\mathfrak{gl}_n[t])$, where $\mathfrak{gl}_n[t]$ is the Lie algebra of $\mathfrak{gl}_n$-valued polynomials.

There is a flat family of maximal commutative subalgebras $B(C) \subset Y(\mathfrak{gl}_n)$, called *Bethe subalgebras*, parameterized by invertible diagonal matrices $C \in GL_n$ with pairwise different eigenvalues, which are stable under the $\mathbb{C}$-action by shift automorphisms of $Y(\mathfrak{gl}_n)$. For $\mathfrak{g} = \mathfrak{sl}_n$ this algebra appears in the works of L. Faddeev and St.-Petersburg school in relation to the inverse scattering method, see e.g. [T84, TF]. In full generality this algebra firstly appears in the paper of V. Drinfeld [D85]. The maximality of Bethe subalgebras has been studied in [NO]. This family originates from the integrable models in statistical mechanics and algebraic Bethe ansatz. More precisely, the image of $B(C)$ in a tensor product of evaluation representations of $Y(\mathfrak{gl}_n)$ form a complete set of Hamiltonians of the XXX Heisenberg magnet chain, cf. [B, KBI].

The main problem in the XXX integrable system is the diagonalization of the subalgebras $B(C)$ in the corresponding representation of the Yangian. The standard approach is the *algebraic Bethe ansatz* which gives an explicit formula the eigenvectors depending on auxiliary parameters satisfying some system of algebraic equations called *Bethe ansatz equations*, see for example [KR86].

The questions we address in the present work are closely related to the *completeness* of the algebraic Bethe ansatz, i.e. to the problem whether the eigenvectors obtained by Bethe ansatz form a basis in V. This problem is extensively studied for many years, see e.g. [MV03, MTV07, MTV09, MTV14, T18, CLV, RV]. As the first step towards the solution of this problem, it is necessary that the joint eigenvalues have no multiplicities. The latter is satisfied if and only if the following two conditions hold: first, there is a *cyclic vector* for the Bethe subalgebra in V (i.e. $v \in V$ such that $B(C)v = V$) and, second, the algebra $B(C)$ acts on V semisimply.

In this work we prove the simplicity of spectra in several new cases including tame representations of the Yangian in type A with generic values of the parameters and some Kirillov-Reshetikhin modules in other types.

1. Yangians and their definitions

1.1. **Yangian for simple $\mathfrak{g}$.** Let $\mathfrak{g}$ be a simple complex Lie algebra, G is the corresponding connected simply connected Lie group, T is the maximal torus and T^{reg} is the regular elements of T, $\mathfrak{h}$ is the corresponding Cartan subalgebra, $n = \operatorname{rk}\mathfrak{g} = \dim \mathfrak{h}$.

Let Φ be the root system corresponding to the Lie algebra $\mathfrak{g}$, Φ^+ are the positive roots, $\{\alpha_1, \dots, \alpha_n\}$ are the simple roots, $\{\omega_1, \dots, \omega_n\}$ are the fundamental weights, $(,)$ is the invariant scalar product such that $(\alpha, \alpha) = 2$ for short simple roots, $\mathfrak{g}_\alpha$ are the corresponding root subspaces of $\mathfrak{g}$, $x_\alpha \in \mathfrak{g}_\alpha, x_\alpha^- \in \mathfrak{g}_{-\alpha}$ are such that $(x_\alpha, x_\alpha^-) = 1$, $t_{\omega_i} \in \mathfrak{h}$ is the element corresponding to $\omega_i \in \mathfrak{h}^*$ by the invariant scalar product. In the same way h_i is the element corresponding to $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. Also we define the Casimir elements corresponding to the invarian scalar product

$$\Omega = \sum_{\alpha \in \Phi^+} (x_\alpha^+ \otimes x_\alpha^- + x_\alpha^- \otimes x_\alpha^+) + \sum_i t_{\omega_i} \otimes h_i \in \mathfrak{g} \otimes \mathfrak{g}$$

and

$$\omega = \sum_{\alpha \in \Phi^+} (x_\alpha^+ x_\alpha^- + x_\alpha^- x_\alpha^+) + \sum_i t_{\omega_i} h_i \in U(\mathfrak{g}).$$

Definition 1.1. Yangian $Y(\mathfrak{g})$ is an associative algebra with a unit over $\mathbb{C}$ generated by the elements $\{x, J(x) \mid x \in \mathfrak{g}\}$ with the following relations:

$$xy - yx = [x, y], \qquad J([x, y]) = [J(x), y]$$
$$J(cx + dy) = cJ(x) + dJ(y),$$

$$[J(x),[J(y),z]]-[x,[J(y),J(z)]]=\sum_{\lambda,\mu,\nu\in\Lambda}([x,x_\lambda],[[y,x_\mu],[z,x_\nu]])\{x_\lambda,x_\mu,x_\nu\},$$

$$[[J(x),J(y)],[z,J(w)]]+[[J(z),J(w)],[x,J(y)]]=$$
$$=\sum_{\lambda,\mu,\nu\in\Lambda}\left(([x,x_\lambda],[[y,x_\mu],[[z,w],x_\nu]])+([z,x_\lambda],[[w,x_\mu],[[x,y],x_\nu]])\right)\{x_\lambda,x_\mu,J(x_\nu)\}$$

for all $x,y,z,w\in\mathfrak{g}$ and $c,d\in\mathbb{C}$ where $\{x_\lambda\}_{\lambda\in\Lambda}$ is some orthonormal basis of $\mathfrak{g}$, $\{x_1,x_2,x_3\}=\frac{1}{24}\sum_{\pi\in\mathfrak{S}_3}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$ for all $x_1,x_2,x_3\in Y(\mathfrak{g})$.

The Yangian $Y(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S defined by

$$\Delta(x)=x\otimes 1+1\otimes x,$$
$$\Delta(J(x))=J(x)\otimes 1+1\otimes J(x)+\frac{1}{2}[x\otimes 1,\Omega],$$
$$S(x)=-x,\quad S(J(x))=-J(x)+\frac{1}{4}c_{\mathfrak{g}}x\quad\forall x\in\mathfrak{g},$$
$$\epsilon(x)=0,\quad \epsilon(J(x))=0,$$

where $c_{\mathfrak{g}}$ is the eigenvalue of ω in the adjoint representation.

We will also denote by Δ^{op} the opposite comultiplication of $Y(\mathfrak{g})$; that is, $\Delta^{op}=\sigma\circ\Delta$ where $\sigma=\sigma_{Y(\mathfrak{g}),Y(\mathfrak{g})}$.

There is the shift automorphism τ_c of $Y(\mathfrak{g})$ defined by

$$x\mapsto x,\quad J(x)\mapsto J(x)+cx,\quad\forall x\in\mathfrak{g}.$$

Then we denote $\tau_{c,d}=\tau_c\otimes\tau_d$.

We are now prepared to introduce the universal $\mathcal{R}$-matrix of $Y(\mathfrak{g})$.

Theorem 1.1 ([D85])**.** *There is a unique formal series*

$$\mathcal{R}(u)=1+\sum_{k=1}^{\infty}\mathcal{R}_k u^{-k}\in(Y(\mathfrak{g})\otimes Y(\mathfrak{g}))[[u^{-1}]]$$

satisfying

$$(id\otimes\Delta)\mathcal{R}(u)=\mathcal{R}_{12}(u)\mathcal{R}_{13}(u),$$
$$\tau_{0,u}\Delta^{op}(y)=\mathcal{R}(u)^{-1}(\tau_{0,u}\Delta(y))\mathcal{R}(u)\quad\forall y\in Y(\mathfrak{g}).$$

The series $\mathcal{R}(u)$ is called the universal $\mathcal{R}$-matrix of $Y(\mathfrak{g})$ and it also satisfies the quantum Yang-Baxter equation

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v)=\mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v).$$

Here $\mathcal{R}_{12}(u)=\mathcal{R}(u)\otimes 1\in Y(\mathfrak{g})\otimes Y(\mathfrak{g})\otimes Y(\mathfrak{g})[[u^{-1}]]$, and $\mathcal{R}_{13}(u)$ and $\mathcal{R}_{23}(u)$ are defined similarly.

We can take the image of $\mathcal{R}(-u)$ under $\rho_V\otimes 1$ for some finite-dimensional representation (V,ρ_V) of $Y(\mathfrak{g})$. We will denote $T_V(u)=\rho_V\otimes 1(\mathcal{R}(-u))$ and call it T-operator. We can apply $\rho_V\otimes\rho_V\otimes 1$ to the Yang-Baxter equation and obtain the relations on the T-operator coefficients. The Fourier coefficients of the T-operator can be taken as another set of generators of $Y(\mathfrak{g})$.

Now we will take an algebra $X(\mathfrak{g})$ with such generators and following [W] will obtain a surjective homomorphism $X(\mathfrak{g})\to Y(\mathfrak{g})$.

Let V be a fixed finite-dimensional $Y(\mathfrak{g})$-module with corresponding homomorphism ρ such that V has a non-trivial (not necessarily proper) irreducible submodule. We let $R(u)$ denote the image of the universal R-matrix $\mathcal{R}(-u)$ under $\rho\otimes\rho$:

$$R(u)=(\rho\otimes\rho)\mathcal{R}(-u)\in\mathrm{End}(V\otimes V)[[u^{-1}]].$$

We fix a basis $\{e_1,\dots,e_N\}$ of V and we let $\{E_{ij}\}_{1\leqslant i,j\leqslant N}$ denote the usual elementary matrices with respect to this basis.

The extended Yangian $X(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by elements $\{t_{ij}^{(r)}\mid 1\leqslant i,j\leqslant N, r\geqslant 1\}$ subject to the defining RTT-relation

$$R(u-v)T_1(u)T_2(v)=T_2(v)T_1(u)R(u-v)\quad \textit{in}\quad (\mathrm{End}\,V)^{\otimes 2}\otimes X(\mathfrak{g})[[v^{-1},u^{-1}]],$$

where $T(u) = \sum_{i,j=1}^N E_{ij} \otimes t_{ij}(u)$ with $t_{ij}(u) = \delta_{ij} + \sum_{r\geqslant 1} t_{ij}^{(r)} u^{-r}$ for all $1 \leqslant i,j \leqslant N$, $T_a(u) = \sum_{i,j} 1^{\otimes(a-1)} \otimes E_{ij} \otimes 1^{\otimes(n-a)} \otimes t_{ij}(u)$ and $R(u-v)$ has been identified with $R(u-v)\otimes 1$.

The extended Yangian is a Hopf algebra, with the Hopf algebra structure given by

$$\Delta(T(u)) = T_{[1]}(u)T_{[2]}(u), \quad S(T(u)) = T(u)^{-1}, \quad \epsilon(T(u)) = Id,$$

where $T_{[1]}(u) = \sum_{i,j=1}^N E_{ij}\otimes t_{ij}(u) \otimes 1 \in \operatorname{End} V \otimes (X(\mathfrak{g}))^{\otimes 2}$ and $T_{[2]}(u) = \sum_{i,j=1}^N E_{ij}\otimes 1 \otimes t_{ij}(u) \in \operatorname{End} V \otimes (X(\mathfrak{g}))^{\otimes 2}$.

The RTT-Yangian $Y(\mathfrak{g})$ is the quotient of $X(\mathfrak{g})$ by the two-sided ideal generated by the elements $z_{ij}^{(r)}$, for $1 \leqslant i,j \leqslant N$ and $r \geqslant 1$, defined by

$$Z(u) = \sum_{i,j=1}^N E_{ij} \otimes z_{ij}(u) = S^2(T(u))T(u + \frac{1}{2}c_{\mathfrak{g}})^{-1},$$

where $z_{ij}(u) = \delta_{ij} + \sum_{r\geqslant 1} z_{ij}^{(r)} u^{-r}$ for each pair of indices $1 \leqslant i,j \leqslant N$.

The equivalence of two definitions was stated by V. Drinfeld [D85] and proved by C. Wendlandt in [W].

1.2. **Yangian for $\mathfrak{gl}_n$.** The definition of the extended Yangian $X(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_n$ with the standard representation V of $\mathfrak{gl}_n$ gives us the Yangian of $\mathfrak{gl}_n$.

The algebra $Y(\mathfrak{gl}_n)$ is generated by elements $t_{ij}^{(r)}$, $1 \leqslant i,j \leqslant n$, $r \in \mathbb{Z}_{\geqslant 0}$ and $t_{ij}^{(0)} = \delta_{ij}$. (The elements $t_{ij}^{(r)}$ correspond to $E_{ij}z^r \in \mathfrak{gl}_n[z]$ where $E_{ij} \in \mathfrak{gl}_n$ is the standard matrix unit.) The relations are

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}.$$

Introduce the formal power series in u^{-1}, where u is a formal variable,

$$t_{ij}(u) = \sum_{r\geqslant 0} t_{ij}^{(r)} u^{-r}.$$

These formal power series can be combined into a matrix with values in formal series with coefficients in $Y(\mathfrak{gl}_n)$

$$T(u) = \sum_{i,j} e_{ij} \otimes t_{ij}(u) \in \operatorname{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]],$$

where e_{ij} is the standard matrix unit. Hence the relations can be rewritten as

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

where

$$R(u) = 1 \otimes 1 - u^{-1} \sum_{i,j} e_{ij} \otimes e_{ji}$$

is the R-matrix.

The algebra $Y(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_n$ is the subalgebra of $Y(\mathfrak{gl}_n)$ which consists of all elements stable under the automorphisms $T(u) \to f(u)T(u)$ for all $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$.

For the details and links on the $\mathfrak{gl}_n$ case, we refer the reader to the book [Mo] by A. Molev.

2. Bethe subalgebras

2.1. **Bethe subalgebras, $Y(\mathfrak{g})$ case.** Bethe subalgebras are a family of maximal commutative subalgebras $B(C)$ in $Y(\mathfrak{g})$ defined by the parameter $C \in T^{reg}$.

For any finite dimensional representation (V, ρ_V) of the Yangian $Y(\mathfrak{g})$ we can define the T-operator $T_V(u) = \rho_V \otimes 1(\mathcal{R}(-u))$ (as in the definition of the Yangian in Section 1.1). Then for any element C of G we can define:

Definition 2.1. Bethe subalgebra $B(C)$ is a subalgebra of $Y(\mathfrak{g})$ generated by the coefficients of $\operatorname{tr}_V(\rho_V(C) \otimes 1)T_V(u)$ for all finite-dimensional representations of $Y(\mathfrak{g})$.

For $C = 1$ this definition appeared in the work of V. Drinfeld [D88].

Proposition 2.1 ([IR19]). (1) *Bethe subalgebra $B(C)$ is commutative for any C.*

(2) *For C regular semisimple, $B(C)$ is maximal commutative.*
(3) *For C regular semisimple, $B(C)$ is freely generated by the coefficients of $T_W(u)$ where W ranges over fundamental representations of $\mathfrak{g}$.*

Bethe subalgebras are determined by the following property:

Theorem 2.2 ([I])**.** *Let $C \in T^{reg}$. Subalgebra $B(C)$ contains $\mathfrak{h}$ and coincides with the centralizer of the subspace $Q(C) \subset Y(\mathfrak{g})$ which is the linear envelope of the elements*

$$\sigma_i(C) = 2J(t_{\omega_i}) - \sum_{\alpha\in\Phi^+} \frac{e^\alpha(C)+1}{e^\alpha(C)-1}(\alpha,\alpha_i)x_\alpha x_\alpha^- \in Y(\mathfrak{g}),\ \ i = 1,\dots,n.$$

2.2. **Bethe subalgebras, $Y(\mathfrak{gl}_n)$ case.** We have said that $Y(\mathfrak{gl}_n)$ is the extended Yangian for $Y(\mathfrak{sl}_n)$ and the T-operator for $Y(\mathfrak{gl}_n)$ is defined with the standard representation. We can give a definition of the Bethe subalgebras using only this T-operator.

The symmetric group S_n acts on $Y(\mathfrak{gl}_n)[[u^{-1}]]\otimes(\mathrm{End}\,\mathbb{C}^{\mathrm{n}})^{\otimes\mathrm{n}}$ by permuting the tensor factors. This action factors through the embedding $S_n \hookrightarrow (\mathrm{End}\,\mathbb{C}^{\mathrm{n}})^{\otimes\mathrm{n}}$ hence the group algebra $\mathbb{C}[S_n]$ is a subalgebra of $Y(\mathfrak{gl}_n)[[u^{-1}]]\otimes(\mathrm{End}\ \mathbb{C}^n)^{\otimes n}$. Let S_m be the subgroup of S_n permuting the first m tensor factors. Denote by A_m the antisymmetrizer

$$\sum_{\sigma\in S_m}(-1)^\sigma\sigma \in \mathbb{C}[S_m] \subset Y(\mathfrak{gl}_n)[[u^{-1}]]\otimes(\mathrm{End}\ \mathbb{C}^{\mathrm{n}})^{\otimes\mathrm{n}}.$$

Note that the T-operators corresponding to fundamental representations of $\mathfrak{gl}_n$ are equal to $T_p(u) = A_pT_1(u)\dots T_p(u-p+1)$ since the fundamental representations are the exterior powers of the standard representation.

Suppose that $C \in GL_n$. For any $a \in \{1,\dots,n\}$ denote by C_a the element $i_a(1\otimes C) \in Y(\mathfrak{gl}_n)[[u^{-1}]]\otimes(\mathrm{End}\,\mathbb{C}^n)^{\otimes n}$. For any $1 \leqslant p \leqslant n$ introduce the series with coefficients in $Y(\mathfrak{gl}_n)$ by

$$\tau_p(u, C) = \mathrm{tr}\,A_pC_1\dots C_pT_1(u)\dots T_p(u-p+1),$$

where we take the trace over all p copies of End $\mathbb{C}^{\mathrm{n}}$.

Definition 2.2. *Bethe subalgebra $B(C) \subset Y(\mathfrak{gl}_n)$ is generated by all coefficients of the series $\tau_p(u, C)$ for $p = 1,\dots,n$.*

It follows from the definition that $B(C) = B(a\cdot C)$ for any $a \in \mathbb{C}$. We fix a maximal torus $T \subset GL_n$ (i.e. the subgroup of diagonal matrices) and denote by T^{reg} the set of regular elements of T. Denote by GL_n^{reg} the set of regular elements of the Lie group GL_n. The following Proposition summarizes known algebraic properties of Bethe subalgebras, see e.g. [NO], [IR19], [I].

Proposition 2.3 (Properties of Bethe subalgebras)**.**
(1) *For any $C \in GL_n$, the subalgebra $B(C)$ is commutative.*
(2) *For $C \in T^{reg}$, the subalgebra $B(C)$ is a maximal commutative subalgebra.*
(3) *For any $C \in GL_n^{reg}$ the subalgebra $B(C)$ is freely generated by the coefficients of $\tau_p(u, C)$ with $p = 1,\dots,n$.*
(4) *Let $C \in T^{reg}$ and let $\tilde{C}$ be an arbitrary representative of C in the universal cover of GL_n. Then the subalgebra $B(C)$ generated by all*

$$\mathrm{tr}_V\,\rho(\tilde{C})(\rho\otimes 1)(\mathcal{R}(-u)),$$

where (ρ, V) ranges over all finite-dimensional representations of $Y(\mathfrak{gl}_n)$ and $\mathcal{R}(u)$ is the universal R-matrix for $Y(\mathfrak{gl}_n)$.

In fact there are no doubts that the assertion (2) is true for any $C \in GL_n^{reg}$ and that (4) is true for any $C \in GL_n$ but still there is no rigorous proof in the literature.

An element of the maximal torus can be represented by a point in the Deligne-Mumford space $\overline{M_{0,n+2}}$ of stable rational curves with $n + 2$ marked points: the element of the torus C corresponds to the non-degenerate rational curve with the marked points being $0,\infty$ and the eigenvalues of C. According to [IR18], the family of Bethe subalgebras in $Y(\mathfrak{gl}_n)$ extends to a bigger family $B(X)$ of commutative subalgebras in $Y(\mathfrak{gl}_n)$ with X taking values in $\overline{M_{0,n+2}}$:

the subalgebra $B(C)$ corresponds to the non-degenerate rational curve with the marked points being $0, \infty$ and the eigenvalues of C (same as the point corresponding to C), but there are also some new subalgebras corresponding to degenerate curves $X \in \overline{M_{0,n+2}}$.

In particular, the subalgebra corresponding to the most degenerate *caterpillar* curve is the *Cartan* subalgebra $H \subset Y(\mathfrak{gl}_n)$, also known as the *Gelfand-Tsetlin* subalgebra, generated by all centers of the smaller Yangians $Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \ldots \subset Y(\mathfrak{gl}_n)$ embedded in the standard way.

Proposition 2.4 ([IMR])**.** *Bethe subalgebra $B(C)$ of $Y(\mathfrak{gl}_n)$ is the tensor product $B'(C) \otimes ZY(\mathfrak{gl}_n)$ where $B'(C)$ is a commutative subalgebra in $Y(\mathfrak{sl}_n)$ and $ZY(\mathfrak{gl}_n)$ is the center of $Y(\mathfrak{gl}_n)$.*

$B'(C)$ is the *Bethe subalgebra of* $Y(\mathfrak{sl}_n)$. In the present work we restrict ourselves by $C \in T^{reg}$, i.e. we fix maximal torus and consider the family of Bethe subalgebras parameterized by its regular points.

2.3. **Bethe subalgebras, $Y(\mathfrak{gl}_2)$ case.** In this case the algebra $B(C)$ is generated by all the coefficients of two following formal power series,

$$\operatorname{qdet} T(u) = t_{11}(u)t_{22}(u-1) - t_{21}(u)t_{12}(u-1)$$

and

$$\operatorname{tr} CT(u) = c_{11}t_{11}(u) + c_{12}t_{21}(u) + c_{21}t_{12}(u) + c_{22}t_{22}(u).$$

The algebra does not change under dilations of C, hence the family is parametrized by points in $\mathbb{C}P^3 = \mathbb{P}(Mat_2)$.

If C is a regular matrix, then $B(C)$ is a maximal commutative subalgebra of $Y(\mathfrak{gl}_2)$, as shown in [NO]. Even more, for a regular C, all the coefficients of $\operatorname{qdet} T(u)$ and $\operatorname{tr} CT(u)$ generate $B(C)$ and are algebraically independent. The Poincare series of $B(C)$ with respect to the standard filtration $\deg t_{ij}^{(r)} = r$ is

$$P_{B(C)}(t) = \prod_{k=1}^{\infty} \frac{1}{(1-t^k)^2}.$$

For non-regular C the Poincare series drops. Namely, the coefficients at u^{-1} of $\operatorname{tr} CT(u)$ and $\operatorname{qdet} T(u)$ are both equal to $t_{11}^{(1)} + t_{22}^{(1)}$, while all other coefficients remain algebraically independent, and the Poincare series is equal to

$$P_{B(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))}(t) = \frac{1}{1-t}\prod_{k=2}^{\infty} \frac{1}{(1-t^k)^2}.$$

We study maximal commutative subalgebras, in the sense of [NO], so following [IR18] we complete this smaller subalgebra to have the same Poincare series as for generic C. This completion is defined as the limit

$$\lim_{t\to 0} B(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + tC')$$

and depends on the choice of direction in the tangent space to $\mathbb{C}P^3$ at the point corresponding to the unit matrix, i.e. on C'. We consider the family of Bethe subalgebras $B(C)$ for regular $C \in Mat_2$ and define its closure. We prove the following result:

Theorem 2.5 ([Ma21])**.** *The closure $\mathcal{B}$ of the family of Bethe subalgebras in $Y(\mathfrak{gl}_2)$ is parametrized by the points of the blow up of $\mathbb{C}P^3$ at the point corresponding to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.*

We denote by Z this blow-up of $\mathbb{C}P^3$, i.e. the parameter space of the family $\mathcal{B}$.

Remark. *The family $\mathcal{B}$ is a flat family of commutative subalgebras of $Y(\mathfrak{gl}_2)$ over Z. In* [IR19] *the definition of $B(C)$ is extended to the points of the De Concini-Procesi compactification of the adjoint Lie group of the Lie algebra for which the Yangian is defined. And it is expected that the limit space is some resolution of the De Concini-Procesi compactification. In our case the algebra is $\mathfrak{gl}_2$ and the corresponding group is $PGL(2,\mathbb{C})$. Its De Concini-Procesi compactification is $\mathbb{C}P^3$.*

3. Representations of Yangians

3.1. **Representations of $Y(\mathfrak{g})$.** Since $Y(\mathfrak{g})$ is a Hopf algebra, the action of Yangian on tensor products of representations can be defined.

The shift automorphism τ_u of $Y(\mathfrak{g})$ defines the shift action on representations of $Y(\mathfrak{g})$, i.e.

$$V \mapsto V(u), \qquad \rho \mapsto \rho \circ \tau_u, \qquad u \in \mathbb{C}.$$

We will be considering representations of the Yangian $Y(\mathfrak{g})$ which have the form $\bigotimes\limits_{j=1}^{N} W_j(u_j)$, where W_j are "small" representations.

Definition 3.1. Kirillov-Reshetikhin module $W_{k,r}$ is an irreducible representation of $Y(\mathfrak{g})$ generated by the highest vector v with highest weight $r\omega_k$ for $\mathfrak{g}$ such that $J(\mathfrak{h})v = 0$ and $J(x_\alpha^+)v = 0$ for all $\alpha \in \Phi^+$.

Kirillov-Reshetikhin modules were defined in [KR87]. It is known that

$$W_{k,r}|_{\mathfrak{g}} = V_{k\omega_r} \oplus \bigoplus_s V_{\lambda_s}$$

where λ_s are weights smaller than $k\omega_r$.

3.2. **Representations of $Y(\mathfrak{gl}_n)$.** It is possible to obtain Yangian representations from representations of $\mathfrak{gl}_n$ using the *evaluation homomorphism*

$$\mathrm{ev} : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$$

which gives a structure of $Y(\mathfrak{gl}_n)$ module on every $\mathfrak{gl}_n$-module called *evaluation* $Y(\mathfrak{gl}_n)$-module. The Kirillov-Reshetikhin modules defined in the general case can be obtained in this way.

In the case of $Y(\mathfrak{gl}_n)$ the shift automorphism can be considered as a deformation of the action of $\mathbb{C}$ on $U(\mathfrak{gl}_n[t])$ which shifts the variable t.

It is possible to generalize this construction of $Y(\mathfrak{gl}_n)$-modules using the *centralizer construction* of the Yangian, due to Olshansky [O]. Namely, consider the embedding $\mathfrak{gl}_k \subset \mathfrak{gl}_{n+k}$ as the subalgebra of lower-right block $k \times k$-matrices, then for any $k \geq 0$ there is a homomorphism

$$\eta_k : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{n+k})^{\mathfrak{gl}_k},$$

which is surjective modulo the center of $U(\mathfrak{gl}_{n+k})$ (in particular, we have $\eta_0 = \mathrm{ev}$). Let V_λ be an irreducible representation of $\mathfrak{gl}_{n+k}$ with the highest weight λ. Consider the restriction of V_λ to $\mathfrak{gl}_k$:

$$V_\lambda = \bigoplus_\mu M_{\lambda\mu} \otimes V_\mu,$$

where $M_{\lambda\mu} := \mathrm{Hom}(V_\mu, V_\lambda)$ is the multiplicity space with action of $U(\mathfrak{gl}_{n+k})^{\mathfrak{gl}_k}$ and therefore is an irreducible representation of $Y(\mathfrak{gl}_n)$. Representations of this form are called *skew representation* of $Y(\mathfrak{gl}_n)$ because they depend on the *skew Young diagram* $\lambda \setminus \mu$. If $M_{\lambda\mu}$ is any skew representation of $Y(\mathfrak{gl}_n)$ then we denote by $V_{\lambda\setminus\mu}(z)$ the (irreducible) representation where the action of $Y(\mathfrak{gl}_n)$ is given by $\eta_k \circ \tau_z$. We also call these representations *skew* representations of $Y(\mathfrak{gl}_n)$.

In [NT], Nazarov and Tarasov introduce the class of *tame* representations, i.e. representations of the form $\bigotimes_{i=1}^k V_{\lambda_i\setminus\mu_i}(z_i)$ such that $z_i - z_j \notin \mathbb{Z}$ for all $i \neq j$. This is the class of irreducible representations of $Y(\mathfrak{gl}_n)$ such that the Cartan subalgebra $H \subset Y(\mathfrak{gl}_n)$ acts without multiplicities. So it is natural to expect similar properties for the action of Bethe subalgebras on this class of representations of $Y(\mathfrak{gl}_n)$. The eigenbasis for the Cartan subalgebra $H \subset Y(\mathfrak{gl}_n)$, known as the Gelfand-Tsetlin basis, is naturally indexed by semistandard skew Young tableaux and is described explicitly. The eigenbasis for a general Bethe subalgebra $B(X)$ is then a deformation of the Gelfand-Tsetlin basis (being itself much less explicit).

3.3. **Representations of $Y(\mathfrak{gl}_2)$.** Let $L(a,b)$ denote the evaluation representation of $Y(\mathfrak{gl}_2)$ which comes from the finite-dimensional representation of $\mathfrak{gl}_n$ with highest weight (a,b). Then $\mathcal{B}(x)$ acts on $L(\underline{a},\underline{b}) = L(a_1,b_1)\otimes\ldots\otimes L(a_N,b_N)$ for any $x\in Z$.

Any finite-dimensional irreducible representation of $Y(\mathfrak{gl}_2)$ is isomorphic to $L(\underline{a},\underline{b})$ for some set of weights $\{(a_i,b_i)\}_{i=1}^N$.

4. Hermitian property

We will consider two different Hermitian forms on the representations of the Yangians. One will be considered in the general case and the other only in the case of $\mathfrak{gl}_2$. The first is compatible with the compact form of the group and the second with the split form of the group.

4.1. **Hermitian property compatible with the compact form.** Let θ be the Cartan involution on $\mathfrak{g}$. This is an antilinear involution given by $\theta(x_\alpha) = -x_\alpha^-$, $\theta(x_\alpha^-) = -x_\alpha$, $\theta(h_i) = -h_i$. Fixed points of this antiinvolution are the compact real form $\mathfrak{g}_{comp}$ of the Lie algebra $\mathfrak{g}$. The corresponding subgroup $G_{comp}\subset G$ is compact.

On the Yangian $Y(\mathfrak{g})$ acts the antiautomorphism $\hat{\theta}$ given by $\hat{\theta}(x) = \theta(x)$, $\hat{\theta}(J(x)) = -J(\theta(x))$ $\forall x\in\mathfrak{g}$. This antiautomorphism is generalizing the antiautomorphism defined by Kirillov and Reshetikhin in [Re].

Consider T^{reg}_{comp}, the fixed points of the Cartan involution of T^{reg}. I.e., $T^{reg}_{comp} = T^{reg}\cap G_{comp}$.

Lemma 4.1 ([Ma22])**.** *If $C\in T^{reg}_{comp}$ then $\hat{\theta}(B(C)) = B(C)$.*

Being Hermitian is the sufficient condition for semisiplicity of operators, i.e. on the representation there is a Hermitian scalar product with respect to which $B(C)$ acts with normal operators.

Definition 4.1. The representation $\pi : Y(\mathfrak{g}) \to \mathrm{End}(V)$ in a Hermitian space V we call Hermitian if $\pi(x)^+ = \pi(\hat{\theta}(x))$ for any $x\in Y(\mathfrak{g})$.

4.2. **Hermitian property compatible with the split form.** In the case of $\mathfrak{g} = \mathfrak{gl}_n$ we consider a different Hermitian form on finite dimensional representations of $Y(\mathfrak{gl}_n)$. This form extends the standard form that we have on a representation of $\mathfrak{gl}_n$.

Definition 4.2. A representation V of $Y(\mathfrak{gl}_n)$ is called unitary if there is a positive definite Hermitian form $\langle\cdot,\cdot\rangle$ on V such that for any $v,w\in V$ $\langle t_{ij}(u)v,w\rangle = \langle v, t_{ji}(u)w\rangle$.

We denote by ϕ^+ the Hermitian conjugate operator to an operator ϕ with respect to this Hermitian form.

Lemma 4.2 ([Ma21])**.** *If $C\in T^{reg}_{split}$ then we have $B(C)^+ = B(C)$ in the representation V.*

5. Main results

We have said that as the first step towards the solution of the Bethe Ansatz problem, it is necessary to establish that the joint eigenvalues of a Bethe subalgebra have no multiplicities. As noted, this condition is satisfied if and only if the following two conditions hold: first, there is a *cyclic vector* for the Bethe subalgebra in V (i.e., $v\in V$ such that $B(C)v = V$) and, second, the algebra $B(C)$ acts on V semisimply.

In this section we will discuss under which assumptions these two conditions are satisfied.

Conjecture 5.1. *$B(C)$ acts with a cyclic vector in any irreducible finite dimensional representation of $Y(\mathfrak{g})$ for all $C\in T^{reg}$.*

The condition that can be verified and that guarantees semisimplicity of operators from the Bethe subalgebra $B(C)$ is that they act with normal operators with respect to a Hermitian form, i.e. $B(C)^+ = B(C)$.

Conjecture 5.2. *For all tensor products of Kirillov-Reshetikhin modules $\bigotimes_{j=1}^N W_{k_j,r_j}(u_j)$ where $u_j\in i\mathbb{R}$, $B(C)$ acts with normal operators with respect to the Hermitian form defined in section 4.1 for $C\in T^{reg}_{comp}$.*

In this thesis, we obtain several results supporting these conjectures and these are the main results of our work. In the following subsections we will discuss them.

5.1. $Y(\mathfrak{g})$ **case for simple** $\mathfrak{g}$**.** For the cyclic vector condition we consider representations for which $W_{k,r}|_{\mathfrak{g}} = V_{k\omega_r}$. According to [KR87], for all classical Lie algebras such representations exist. In particular, in type A all Kirillov-Reshetikhin modules are of this form, and in type C all Kirillov-Reshetikhin modules with $r = 1$ are of this form. Also in the orthogonal case the spin representation has such a form.

Theorem 5.3 ([Ma22])**.** *If $W_{k,r}$ is a Kirillov-Reshetikhin module satisfying the condition above, then for all $C \in T^{reg}$ for generic u_j, i.e. a Zariski open subset in $\mathbb{C}^{\otimes N}$, Bethe algebra $B(C)$ acts in representation $V = \bigotimes_j W_{k_j,r_j}(u_j)$ with a cyclic vector.*

Now we state the result on the semismplicity.

Consider T^{reg}_{comp}, the fixed points of the Cartan involution on T^{reg}. I.e. $T^{reg}_{comp} = T^{reg} \cap G_{comp}$.

Theorem 5.4 ([Ma22])**.** *Let V be a representations of the Yangian $Y(\mathfrak{g})$ which is a tensor product of representations $W_{k,r}(u)$ where $u \in i\mathbb{R}$ with the condition that $W_{k,r}$ is irreducible as $\mathfrak{g}$-module. Then V is Hermitian.*

The main result of our work in this general case is the following:

Corollary 5.5. *If $C \in T^{reg}_{comp}$, the spectrum of Bethe subalgebra $B(C)$ is simple in representations that satisfy the conditions of Theorems 5.3 and 5.4.*

5.2. **Cyclic vector and simplicity of spectrum,** $Y(\mathfrak{gl}_n)$ **case.** Let $X \in \overline{M_{0,n+2}}$ and consider the Bethe subagebra $B(X)$. Our conjecture in this setting is as follows:

Conjecture 5.6. *$B(X)$ has a cyclic vector in any tame representation of $Y(\mathfrak{gl}_n)$ for all $X \in \overline{M_{0,n+2}}$.*

In fact, it is easy to see that the Conjecture is true for *generic* $X, z_1, \ldots, z_N$. Indeed, consider the parameter space $\overline{M_{0,n+2}} \times \mathbb{C}^N$. The condition that $B(X)$ acts with a cyclic vector on $\bigotimes_{i=1}^N V_i(z_i)$ determines a Zariski open subset of $\overline{M_{0,n+2}} \times \mathbb{C}^N$, therefore once we have a single point $(X, z_1, \ldots, z_N) \in \overline{M_{0,n+2}} \times \mathbb{C}^N$ such that $B(X)$ acts with a cyclic vector on $\bigotimes_{i=1}^N V_i(z_i)$ we automatically have the same property for generic $(X, z_1, \ldots, z_N)$. On the other hand, according to [NT] the Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ (which is a particular case of $B(X)$) acts without multiplicities on any tame representation, so has a cyclic vector. Hence this Zariski-open subset is non-empty. The problem with this argument is that it does not give any representation such that $B(X)$ acts cyclicly for *all* $X \in \overline{M_{0,n+2}}$.

Theorem 5.7 ([IMR])**.** *There is a Zariski open dense subset of $I \subset \mathbb{C}^N$ such that $B(X)$ has a cyclic vector in V for all $X \in \overline{M_{0,n+2}}$ and $(z_1, \ldots, z_N) \in I$.*

Particularly, in a generic tame representation in the sense of [NT] every Bethe subalgebra $B(X)$ with $X \in \overline{M_{0,n+2}}$ acts with a cyclic vector. This allows to study the joint spectrum of $B(X)$ in a given tame representation as a finite covering of $\overline{M_{0,n+2}}$ and reformulate some properties of this spectrum in terms of geometry of Deligne-Mumford compactifications.

Theorem 5.8 ([IMR])**.** *Let $W_{k,r}$ be a Kirillov-Reshetikhin module such that its weights have no multiplicities. Then for all $C \in T^{reg}$ for generic $u_j \in i\mathbb{R}$ Bethe subalgebra $B(C)$ acts in the representation $V = \bigotimes_j W_{k_j,r_j}(u_j)$ with a cyclic vector.*

Another main result of this work is the following theorem. It was first proved in type A by Reshetikhin [Re] for $\mathfrak{g} = \mathfrak{sl}_3$.

Theorem 5.9 ([Ma22])**.** *Let V be a representation of the Yangian $Y(\mathfrak{g})$ such that V is a tensor product of representations $W_{k,r}(u)$ where $u \in i\mathbb{R}$ and $W_{k,r}$ is irreducible as a $\mathfrak{g}$-module. Then V is Hermitian (in the sense of definition 4.1).*

Theorem 5.7 implies that once $B(C)$ acts semisimply, it has simple spectrum (i.e. the joint eigenvalues have no multiplicities). The usual sufficient condition for this is the existence of a Hermitian scalar product such that $B(C)^+ = B(C)$ i.e. all elements of $B(C)$ act by normal operators. We give sufficient conditions on the representation of the Yangian guaranteeing that such scalar product exists provided that C belongs either to the closure of the set of regular elements of the compact real torus $T_{comp} \subset T$ or to that of the split real torus $T_{split} \subset T$. So we get

Corollary 5.10 ([IMR]). *For $C \in T_{comp}^{reg}$ the spectrum of Bethe subalgebra $B(C)$ in representations satisfying the conditions of Theorems 5.8 and 5.9 is simple.*

The case of the compact torus goes back to Kirillov and Reshetikhin [KR86]. Then $W_{k,r}(u)$ is a different notation for $V_{k\omega_r}(\frac{k_i}{2} - \frac{r_i}{2} + u)$. Note that the closure of the set of regular points of the compact torus T_{comp} in $\overline{M_{0,n+2}}$ is the compact form $\overline{M_{0,n+2}^{comp}}$. Therefore for the $\mathfrak{gl}_n$ case we obtain:

Theorem 5.11 ([IMR]). *Suppose that all V_i's are Kirillov-Reshetikhin modules. Let $k_i \times r_i$ be the size of the corresponding Young diagram. Suppose that $z_i = \frac{k_i}{2} - \frac{r_i}{2} + ix_i$, where $x_i \in \mathbb{R}$. Then, for $(x_1, \ldots, x_N) \in \mathbb{R}^N$ from Zariski open subset, $B(X)$ has simple spectrum on $\bigotimes_{i=1}^N V_i(z_i)$ for all $X \in \overline{M_{0,n+2}^{comp}}$.*

The closure of the set of regular points of the split real torus T_{split} in $\overline{M_{0,n+2}}$ is the real form $\overline{M_{0,n+2}^{split}}$. Our next main result is

Theorem 5.12 ([IMR]). *Let $V_i, i = 1, \ldots, N$ be a set of skew representations of $Y(\mathfrak{gl}_n)$. Then, for $(x_1, \ldots, x_N)$ from a non-empty Zariski open subset in $\mathbb{R}^N$, $B(X)$ has simple spectrum on $\bigotimes_{i=1}^N V_i(x_i)$ for all $X \in \overline{M_{0,n+2}^{split}}$.*

5.3. $Y(\mathfrak{gl}_2)$ **case.** In the case of $Y(\mathfrak{gl}_2)$ we were able to find explicitly the Zariski open subset we discussed before.

A string is a set $S(a, b) = \{a - 1, a - 2, \ldots, b + 1, b\}$ for $a, b \in \mathbb{C}$, $a > b$, $a - b \in \mathbb{Z}$. It is known that the representation $L(\underline{a}, \underline{b}) = L(a_1, b_1) \otimes \ldots \otimes L(a_N, b_N)$ is irreducible if and only if, for any $1 \leqslant i < j \leqslant N$, one of three possibilities hold: $S(a_i, b_i) \cup S(a_j, b_j)$ is not a string, or $S(a_i, b_i) \subset S(a_j, b_j)$, or $S(a_i, b_i) \supset S(a_j, b_j)$.

We have the following two results:

Theorem 5.13 ([Ma21]). *The action of any algebra in the family $\mathcal{B}$ in $L(a_1, b_1) \otimes \ldots \otimes L(a_N, b_N)$ has a cyclic vector, if, for any $1 \leqslant i < j \leqslant n$, $S(a_i, b_i) \cup S(a_j, b_j)$ is not a string.*

Secondly, we restrict to the closure of the subfamily corresponding to real diagonal matrices parametrized by the points of $\mathbb{R}P^1 \simeq Z' \subset Z$ (see Theorem 2.5 for the definition of Z).

Theorem 5.14 ([Ma21]). *For any $x \in Z'$ and any $a_1, b_1 \ldots, a_N, b_N \in \mathbb{R}$ such that (a_i, b_i) and (a_j, b_j) are disjoint and $S(a_i, b_i) \cup S(a_j, b_j)$ is not a string for each pair i, j, the subalgebra $\mathcal{B}(x)$ acts on $L(a_1, b_1) \otimes \ldots \otimes L(a_N, b_N)$ with simple spectrum.*

6. Further development

Theorem 5.11 allows to regard the set of eigenlines for $B(X)$ in $\bigotimes_{i=1}^k V_i(z_i)$ as an *unramified* covering of the space $\overline{M_{0,n+2}^{comp}}$. In particular, we get the monodromy action of the fundamental group $\pi_1(\overline{M_{0,n+2}^{comp}})$ (which is natural to call (pure) *affine cactus group*) on the spectrum of Bethe subalgebras. Moreover, it is possible to define the structure of a Kirillov-Reshetikhin crystal on this spectrum, following the strategy of [HKRW] (see [KMR]). We expect that the action of the affine cactus group on this set is given by partial Schutzenberger involutions on the KR-crystal.

Similarly to the case of the compact real form, Theorem 5.12 gives an action of the usual cactus group $\pi_1(\overline{M_{0,n+2}^{split}})$ on the spectrum of a Bethe algebra. Specializing the parameter of the Bethe algebra to the *caterpillar* point of $\overline{M_{0,n+2}^{split}}$ we get an action of the cactus group on the Gelfand-Tsetlin basis in the tensor product of skew representations. The latter is indexed by collections of semistandard skew Young tableaux, and we conjecture that the action of the cactus group is given by Bender-Knuth involutions, similarly to the construction of Chmutov, Glick and Pylyavskyy [CGP].

The main results of the thesis are presented in two papers: I. I. Mashanova-Golikova *Simplicity of spectra for Bethe subalgebras in* $Y(\mathfrak{gl}_2)$. Arnold Math J. (2021)

II. I. Mashanova-Golikova *Hermitian property and simplicity of spectra of Bethe subalgebras in Yangians.* Funct. Anal. and its Appl. (2022) accepted for publication

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