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The Weight System Related to the $\mathfrak{sl}_2$ Lie Algebra and the Hopf Algebra of Graphs

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Contents

Introduction

The thesis is devoted to the study of the $\mathfrak{sl}_2$ weight system. Weight systems are assigned to invariants of knots of finite order. Weight systems are functions on chord diagrams (combinatorial objects having the form of an oriented circle with a set of chords on it). In particular, the $\mathfrak{sl}_2$ weight system corresponds to the colored Jones polynomial.

Despite the simplicity of the definition, computing the values of the $\mathfrak{sl}_2$ weight system is a cumbersome and meaningful problem. We computed the values of the $\mathfrak{sl}_2$ weight system on certain families of graphs, representing a join of a given graph with discrete graphs, namely, on complete bipartite graphs (joins of two discrete graphs) and on the joins of the cycle on five vertices with discrete graphs. We denote the join of a graph G with a discrete graph on n vertices by (G, n). If G is the C_5 graph, a cycle on 5 vertices, then graphs (G, n) are not intersection graphs of chord diagrams for $n \geq 1$. The family (C_5, n) is the first infinite family of graphs that are not intersection graphs, for which the values of the $\mathfrak{sl}_2$ weight system are known.

In addition, we prove an explicit formula for the projections of graphs of series (G, n) to the subspace of primitive elements along the subspace of decomposable elements in the Hopf algebra of graphs. Our formula expresses the exponential generating function for projections of graphs (G, n), $n = 0, 1, 2, \ldots$ in terms of exponential generating functions for graphs (H, n), $n = 0, 1, 2, \ldots$, where H are all possible subgraphs of the graph G. Applying this formula to the explicit values of the $\mathfrak{sl}_2$ weight system on the specified series of graphs we deduced, we compute its values on projections of these graphs to the subspace of primitive elements. In particular, the results obtained confirm the following conjecture due to S. Lando: the value of the $\mathfrak{sl}_2$ weight system on the projection of the graph onto the subspace of primitive elements is a polynomial of degree at most half of the circumference (that is, the length of the longest simple circle) of the graph.

Part I
General theory

1 Theory of Vassiliev invariants

Vasiliev's wide-reaching theory of complements to discriminants and the topology of these complements [1] has an important part: the theory of Vasiliev knot invariants. A *knot* is an isotopy class of embeddings of the oriented one-dimensional circle to the space $S^1 \to \mathbb{R}^3$. *A knot invariant* is a function on the set of knots. *A singular knot* is a class of isotopy of mappings $u \colon S^1 \to \mathbb{R}^3$, such that

1. $u'(t) \neq 0$ for all $t \in S^1$;
2. all self-intersection points of the image $u(S^1)$ are simple double points with transversal self-intersections (that is, if $u(t_1) = u(t_2)$, $t_1, t_2 \in S^1$, $t_1 \neq t_2$, then there is no point $t_3 \in S^1$ different from t_1 and t_2 such that $u(t_3) = u(t_1) = u(t_2)$, and the two tangent vectors $u'(t_1)$ and $u'(t_2)$ are not collinear).

As it is easy to see, the number of double points of a singular knot is finite. V. A. Vasiliev proposed a way (Vasiliev's skein relation) to extend each invariant of knots with values in an abelian group to an invariant of singular knots. A knot invariant is called *Vasiliev invariant of order at most n* if its extension vanishes on all singular knots having more than n double points.

The notion of a finite type invariant extends naturally to the invariants of links ("multicomponent knots"). Many classical knot invariants, e.g. the Conway polynomial or the HOMFLYPT polynomial, are expressed in some way in terms of finite type invariants, although they are not invariants f finite type themselves.

Vasiliev's knot invariants (also known as finite type invariants) have a number of advantages, which make them a convenient and important tool for studying knots and links. Let us list them following [14]. First, the space of invariants of an order not greater than a given one is finite-dimensional, and there is an a priori upper bound on its dimension. In addition, each space of invariants of a fixed degree is algorithmically computable. Further, for each Vasiliev invariant, there exists a polynomial-time algorithm for computing this invariant. Finally, Vasiliev invariants are stronger than all known classical polynomial knot invariants (Alexander, Jones, Kaufmann, Conway, HOMFLYPT polynomials, etc.). V. A. Vasiliev's conjecture states that for any two different knots there exists an invariant of finite order distinguishing them. Note, however, that until now prospective approaches to the proof of this conjecture are not known, as well as ways to construct counterexamples to it.

2 Hopf algebras of graphs and chord diagrams

Functions on chord diagrams are closely related to graph invariants. The structure of many of the naturally occurring invariants, in turn, is closely related to Hopf algebra structures on spaces of graphs and chord diagrams. In this section, we describe the corresponding structures.

2.1 Hopf algebras of graphs

By a graph we mean an isomorphism class of simple (i.e., having no multiple edges or loops) finite graphs. Formal linear combinations of graphs form a vector space graded by the number of vertices of a graph.

The *product of graphs* G_1 and G_2 is their disjoint union: $G_1G_2 := G_1 \sqcup G_2$. This multiplication is extended to the space of graphs by linearity. It preserves the grading and defines a structure of a graded algebra on this space.

Denote by $V(G)$ the vertex set of a graph G. The *coproduct* μ of a graph G is defined as follows:

$$\mu(G) := \sum_{U \subset V(G)} G|_U \otimes G|_{V(G)\setminus U}.$$

Here, $G|_U$ denotes the subgraph in G induced by a subset $U \subset V(G)$ of its vertex set. As well as the multiplication, the comultiplication is extended to linear combinations of graphs by linearity and preserves the grading, i.e., we have introduced a structure of a graded coalgebra on the space of graphs. Moreover, it is true that

Claim 1. *The multiplication and comultiplication introduced above, together with the naturally defined unit, counit, and antipode, define the structure of the graded commutative and co-commutative Hopf algebra on the space of graphs.*

This structure of Hopf algebra on the space of graphs was introduced in [13], and it was Tutte [24] who suggested to consider the disjoint union of graphs as a multiplication.

Denote by $\mathfrak{G}$ the Hopf algebra of graphs and by $\mathfrak{G}_n$ the homogeneous vector subspace in it, spanned by the graphs with n vertices, $n = 0, 1, 2, \dots$, so that

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \dots$$

In connection with the theory of finite type knot invariants, we will also be interested in the Hopf algebra of graphs modulo the so-called *four-term relations for graphs*:

$$G - G'_{AB} - \widetilde{G}_{AB} + \widetilde{G}'_{AB} = 0,$$

where G is a graph, A, B are two vertices in G, $A, B \in V(G)$, G'_{AB} is the graph G in which the incidence of vertices A and B is reversed; $\widetilde{G}_{AB}$ is the graph G in which for each vertex connected with B its incidence with A is reversed. Note that all the four graphs entering the four-term relation have the same number of vertices. As a consequence, this relation preserves the grading, and the corresponding quotient Hopf algebra is also graded. We denote it by $\mathfrak{F}$. Referring to the elements of the Hopf algebra $\mathfrak{F}$, we will say "a graph" meaning its equivalence class modulo four-term relations.

There are noteworthy Hopf subalgebras in the Hopf algebra of graphs. One such Hopf subalgebra is generated by complete graphs. Another example (a whole family of Hopf subalgebras) is constructed using the construction of graph join. *The join* of two simple graphs G and H is a graph obtained by adding all edges connecting the vertices of G with the vertices of graph H in the disjoint union $G \sqcup H$ of these graphs. For $n = 0, 1, 2, \dots$, we denote by (G, n) the join of G and the discrete graph on n vertices.

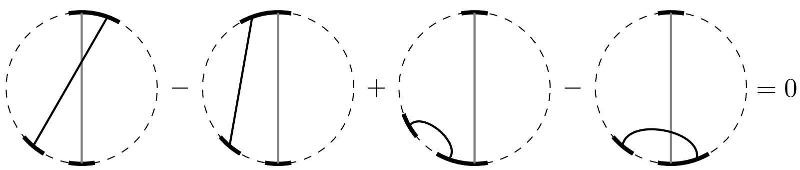
Some families of graphs of this form always consist of intersection graphs of chord diagrams (see below), for example, complete bipartite graphs. However, using this construction it is easy to construct an infinite series of graphs that are not intersection graphs.

Any induced subgraph of a graph (G, n) has the form (H, k), where H is a subgraph of G, and $k \le n$. Graphs of the form (H, n), where H is a subgraph of G, $n \in \mathbb{N} \cup \{0\}$, generate a Hopf subalgebra in the Hopf algebra $\mathfrak{G}$. If $G_0, G_1, G_2, \dots$ is a sequence of graphs in which each graph is an induced subgraph of the next one, then the corresponding Hopf subalgebras form a chain of embedded Hoph subalgebras.

2.2 The Hopf algebra of chord diagrams

A chord diagram of order n is an oriented circle with $2n$ pairwise distinct points in it, split into n pairs, considered up to orientation-preserving diffeomorphisms. Points belonging to the same pair are usually depicted by a chord connecting them. Two chords are said to *intersect* if their ends alternate in the circle.

Formal linear combinations of chord diagrams form a graded vector space. Each homogeneous component in it is spanned by diagrams of the same order. Vasiliev's four-term relation for chord diagrams has the form



(1)

Here and below, unless otherwise indicated, the dashed line indicates the parts of the circle that may contain some set of chords which is the same in all the diagrams of the relation.

The diagrams of the four-term relation are constructed as follows: one of the chords is fixed, one of the ends of the other chord is also fixed, while the other end runs through all the four possible positions close to the two ends of the first chord. (The ends of chords are said to be *close* to each other if there are no ends of other chords between them.)

Let us introduce multiplication and co-multiplication both preserving the grading on the vector space of chord diagrams modulo all possible four-term relations.

Definition 1. *An* arc diagram of order n *is an oriented line with* $2n$ *pairwise distinct points chosen in it, split into* n *pairs, considered up to orientation preserving diffeomorphisms of the line.*

Let us choose a point on a chord diagram different from the ends of the chord, "cut" the circle at this point and unfold it into a straight line. In this way we obtain an *arc representation of the chord diagram* (an example is shown in Fig. 1). A chord diagram of order n can have up to $2n$ different arc representations. In contrast, an arc diagram uniquely defines the corresponding chord diagram.

Product of chord diagrams C_1 and C_2 is the chord diagram corresponding to the arc diagram obtained by the concatenation of two arc representations

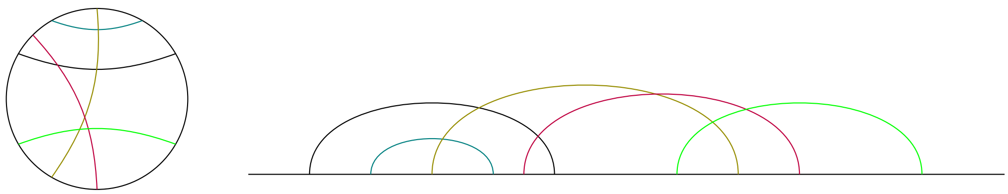


Figure 1: Example of a chord diagram and the corresponding arc diagram

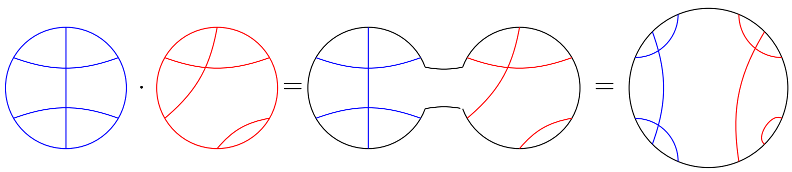


Figure 2: Multiplication of chord diagrams

of of C_1 and C_2 (see Fig. 2). The product of the chord diagrams is well defined (i.e., the result is independent of the choice of the arc representations of the factors) modulo four-term relations.

Denote by $V(C)$ the set of chords of a chord diagram C. The coproduct $\mu(C)$ is defined as follows:

$$\mu(C) := \sum_{U \subset V(C)} C|_U \otimes C|_{V(C)\setminus U}.$$

Here, $C|_U$ denotes the chord diagram formed by a subset $U \subset V(C)$ of the chord diagram set C.

Multiplication and comultiplication are extended to linear combinations of chord diagrams by linearity and preserve the grading.

As proved by Bar-Natan [2], these operations turn the vector space of chord diagrams modulo four-term relations into a Hopf algebra. We denote this Hopf algebra by $\mathfrak{C}$,

$$\mathfrak{C} = \mathfrak{C}_0 \oplus \mathfrak{C}_1 \oplus \mathfrak{C}_2 \oplus \dots,$$

where $\mathfrak{C}_k$ denotes the vector space spanned by chord diagrams with k chords, modulo the four-term relations.

To each chord diagram, a simple graph is assigned using the construction of *intersection graph*. The vertices of this graph correspond to the chords of the chord diagram, and there is an edge between two vertices if and only if the corresponding chords intersect. This mapping is neither injective nor surjective: on one hand, it is easy to give an example of two distinct chord diagrams having the same intersection graph, on the other hand, not every simple graph is an intersection graph. Bouchet [4] presented a complete set of obstacles for a graph to be an intersection graph of any chord diagram. The mapping that maps a chord diagram to its intersection graph is extended to a graded homomorphism of a Hopf algebra $\mathfrak{C}$ to the Hopf algebra $\mathfrak{F}$ (see [19]). Starting with order 7, this homomorphism is not injective, it has a non-trivial kernel; the question whether it is surjective remains open.

2.3 Primitive elements in Hopf algebras

In the study of the structure of Hopf algebras, the so-called primitive elements play an important role. The Milnor–Moore theorem [22] states, that over a field of characteristic zero a connected commutative cocommutative graded bialgebra is isomorphic to the polynomial bialgebra generated by its primitive elements. An element p of a bialgebra with comultiplication μ is called **primitive** if $\mu(p) = 1 \otimes p + p \otimes 1$. A graded bialgebra is called *connected* if its zero homogeneous component is isomorphic to the ground field. As it is easy to see, these conditions are satisfied for the Hopf algebras $\mathfrak{G}$, $\mathfrak{F}$ and $\mathfrak{C}$. In polynomial Hopf algebras, the projection to the subspace of primitive elements along the space of decomposable element is defined. An explicit formula for this projection was proposed by Lando [19]. We consider the projection π in the Hopf algebras of graphs, but there is a similar formula for the Hopf algebra of chord diagrams:

$$\pi(G) := \sum_{V_1 \sqcup \ldots \sqcup V_k} (-1)^{k-1}(k-1)! G|_{V_1} G|_{V_2} \ldots G|_{V_k}. \tag{2}$$

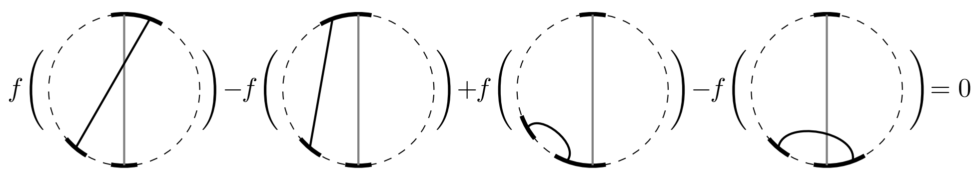
Here and below, we denote by $G|_U$ the subgraph of a graph G induced by a subset $U \subseteq V(G)$ of its vertex set. We sum over all representations of the set $V(G)$ as a disjoint union of nonempty subsets.

The universal formula for this projection in polynomial Hopf algebras represents it as the logarithm of the identity homomorphism (see [18], [23]). However, for graphs of general form computations using this formula are cumbersome. In the special case of graphs of the form (G, n), $n = 0, 1, 2, \ldots$ the calculations related to the projections to the space of primitive elements become significantly simpler.

3 Weight systems

3.1 Constructing weight systems by graph invariants

To each singular knot, its chord diagram is assigned: the ends of its chords are the preimages of double points of the singular knot. The value of a Vasiliev invariant of order n on a singular knot with exactly n double points depends only on the chord diagram of this knot. With such a correspondence, a Vasiliev invariant of order at most n determines a function on chord diagrams with n chords. In this case, it turns out that the resulting functions f must satisfy two relations: the so-called one-term and the essentially more important four-term (3).



(3)

According to the Kontsevich theorem [14], every function on chord diagrams with values in a commutative algebra over a field of characteristic zero that satisfies the one- and four-term relations is obtained from a knot invariant of order at most n. In addition, there exists a renormalization operation which allows one to construct from each function satisfying the four-term relations a function that also satisfies the one-term relations. *A weight system* is a function on chord diagrams that satisfies the four-term relations. For simplicity, we consider weight systems with values in the field $\mathbb{C}$ of complex numbers.

Thus, weight systems are elements of the graded dual Hopf algebra to the Hopf algebra $\mathfrak{C}$. Note that the multiplication and comultiplication of chord diagrams arise naturally: they correspond to the dual operations on the bialgebra of knot invariants.

3.2 Constructing weight systems from Lie algebras

One of the richest sources of weight systems is provided by the construction of a weight system from a finite-dimensional Lie algebra endowed with a nondegenerate invariant bilinear form. This construction was proposed by Bar-Natan [2] and Kontsevich [14].

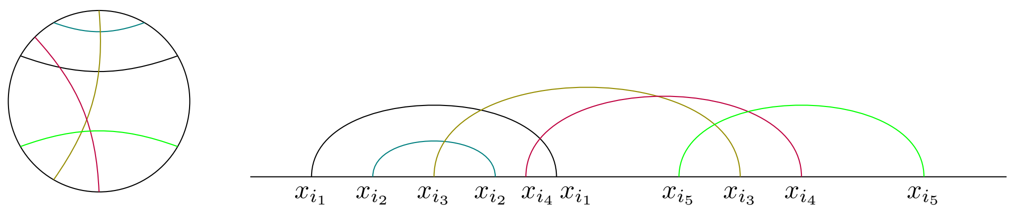


Figure 3: Calculating the value of the weight system corresponding to a Lie algebra with an orthonormal basis $x_1, \dots, x_m$ on an arc diagram corresponding to a chord diagram.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra of dimension m and let $(\cdot,\cdot)$ be a nondegenerate invariant bilinear form on $\mathfrak{g}$. The bilinear form is *invariant* if for any $x, y, z \in \mathfrak{g}$ it is true that $([x, y], z) = (x, [y, z])$. Let $X = \{x_1, x_2, \dots, x_m\}$ be an orthonormal basis of $\mathfrak{g}$, $(x_i, x_j) = \delta_{ij}$. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. We construct the map $w_{\mathfrak{g}}\colon \mathcal{C} \to U(\mathfrak{g})$ as follows.

Let D be a chord diagram, let A be some arc representation of it, let $V(A)$ be the set of arcs of this arc diagram, and let ν be a mapping $\nu\colon V(A) \to \{1, 2, \dots, m\}$. Let us associate with the diagram A and the mapping ν an element $w_X(A, \nu) \in U(\mathfrak{g})$ as follows: for each arc $v \in V(A)$, we write the element $x_{\nu(v)} \in X$ at both its ends and denote by $w_X(A, \nu)$ the result of multiplication of these elements from left to right. Denote by $w_X(A)$ the sum over all such mappings ν:

$$w_X(A) := \sum_{\nu: V(A) \to \{1,\dots,m\}} w_X(A, \nu). \tag{4}$$

For example, the value of the weight system corresponding to a Lie algebra with an orthonormal basis $x_1, \dots, x_m$, on the arc diagram shown in Fig. 3 equals

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{i_3=1}^{m}\sum_{i_4=1}^{m}\sum_{i_5=1}^{m} x_{i_1}x_{i_2}x_{i_3}x_{i_2}x_{i_4}x_{i_1}x_{i_5}x_{i_3}x_{i_4}x_{i_5}.$$

Claim 2 ([14])**.** *1. For any element $C \in \mathfrak{C}$ the result of such an operation is uniquely defined and does not depend on the choice of an arc representation of a chord diagram C.*

2. *For any arc diagram A the element $w_X(A)$ lies in the center of the universal enveloping algebra: $w_X(A) \in Z(U(\mathfrak{g}))$.*

3. *The value of $w_X(A)$ does not depend on the choice of the orthonormal basis.*

4. *The mapping of chord diagrams to $Z(U(\mathfrak{g}))$ thus obtained satisfies the four-term relations and thus is extended to a homomorphism of commutative algebras.*

Since the product of chord diagrams is given by the concatenation of the corresponding arc diagrams, the weight system corresponding to a Lie algebra is multiplicative. Note that the given construction is easily modified for the case of an arbitrary (not necessarily orthonormal) basis in $\mathfrak{g}$: it is only necessary to put x_i at the left end of the arc with index i, and at its right end the element x_i^* of the dual basis. We will apply this construction to the Lie algebra $\mathfrak{sl}_2$ in such a form.

In the thesis we study in detail the properties of the simplest of such weight systems which corresponds to the Lie algebra $\mathfrak{sl}_2$. This weight system will be discussed in detail below.

A weight system corresponding to the Lie algebra $\mathfrak{sl}_3$ is much more complicated and lacks many of the properties of the $\mathfrak{sl}_2$ weight system which, in particular, make computations of the $\mathfrak{sl}_2$ weight system much easier. This weight system was studied, for example, in [17], and in [27] its values at chord diagrams with intersection graph $K_{2,n}$ were calculated. In the recent article [28] a significant progress in understanding of the weight systems constructed from the Lie algebras $\mathfrak{gl}_N$, for arbitrary N, is described.

The construction of weight systems from Lie algebras has been generalized to Lie superalgebras by A. Vaintrob [25]. In [9], this construction was considered in detail for the special case of a Lie superalgebra $\mathfrak{gl}(1 \mid 1)$. In particular, a recurrence relation for the values of $\mathfrak{gl}(1 \mid 1)$ weight system was obtained there.

Part II

The $\mathfrak{sl}_2$ weight system

4 Main properties of the $\mathfrak{sl}_2$ weight system

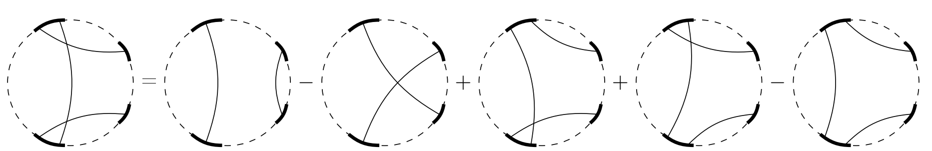
The simplest case of the construction described in the previous section is the weight system corresponding to the Lie algebra $\mathfrak{sl}_2$, or, more briefly, the $\mathfrak{sl}_2$ weight system. The knot invariant to which this weight system corresponds is the colored Jones polynomial. The values of this weight system lie in the center of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, that is, they are polynomials in one variable c (the Casimir element in $\mathfrak{sl}_2$). Its value on a chord diagram with n chords is a monic polynomial of degree n.

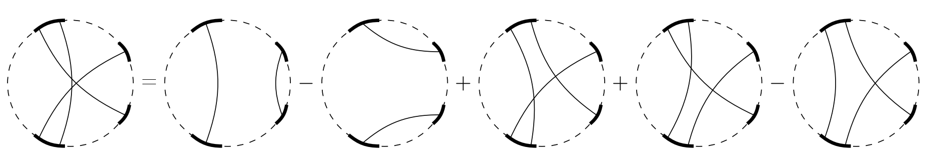
The weight system corresponding to the Lie algebra $\mathfrak{sl}_2$ has been studied in detail in the article by S. V. Chmutov and A. N. Varchenko [8]. In particular, the following relations were derived there:

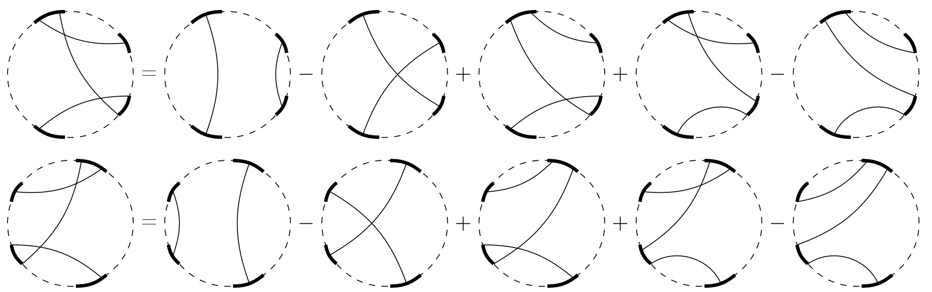
1. if a diagram D contains a leaf, that is, a chord intersecting only one chord, then
$$w_{\mathfrak{sl}_2}(D) = (c-1)w_{\mathfrak{sl}_2}(D'), \tag{5}$$
where D' denotes the chord diagram obtained from D by removing the leaf;

2. if there is no leaf in the chord diagram, then it has a triplet of chords arranged as shown in the left-hand side of one of the following equations;

3. if the chords are arranged this way, then for the values of the $\mathfrak{sl}_2$ weight system the following equations hold:







The six-term relations given here allow one to simplify a chord diagram by reducing the number of chord intersections. In addition, in the same article [8] a recurrence relation is derived from these relations, which allows one to reduce by one the number of chords in a diagram.

5 The $\mathfrak{sl}_2$ weight system on graphs

In the article by S. V. Chmutov and S. K. Lando [7] it is proved, that the value of the $\mathfrak{sl}_2$ weight system on a chord diagram depends only on its intersection graph and thus determines a function on intersection graphs. This result leads to the natural question formulated by S. K. Lando: is there an extension of this weight system to a polynomial invariant of graphs that satisfies the four-term relations for graphs? For all graphs with no more than eight vertices such an extension exists and is unique, as shown by E. S. Krasilnikov [15].

One possible approach to find an extension of the $\mathfrak{sl}_2$ weight system to a polynomial invariant of arbitrary graphs is to define some polynomial invariant of arbitrary graphs satisfying the four-term relations for graphs and coinciding with the $\mathfrak{sl}_2$ weight system on intersection graphs. In order to realize this, it is necessary to have a sufficient number of examples of values of the $\mathfrak{sl}_2$ weight system on different families of graphs.

For instance, the values of the $\mathfrak{sl}_2$ weight system on complete graphs have recently been computed: P. E. Zakorko proved the following conjecture of S. K. Lando (2016, see [3]) about an explicit form of the generating function for these values:

Claim 3. *The generating function for the sequence of values of the weight* *system $w_{\mathfrak{sl}_2}$ on complete graphs $K_0, K_1, K_2, \dots$ is a continuous fraction*

$$\sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}(K_n)t^n = \cfrac{1}{1-\alpha_0(c)t-\cfrac{\beta_1(c)t^2}{1-\alpha_1(c)t-\cfrac{\beta_2(c)t^2}{1-\alpha_2(c)t-\cfrac{\beta_3(c)t^2}{1-\dots}}}},$$

the coefficients of which have the following form

$$\alpha_n(c) = c - n(n+1), \qquad \beta_n(c) = -n^2c + \frac{n^2(n^2-1)}{4}.$$

5.1 Algebra of shares and computation of values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs

The main results of the thesis include explicit formulas for the values of the $\mathfrak{sl}_2$ weight system on some families of graphs and methods for obtaining them. We describe the approach using which we obtained in the thesis a recurrence formula for the generating functions for sequences of the values of the $w_{\mathfrak{sl}_2}$ weight system on complete bipartite graphs.

A share in a chord diagram is a pair of non-intersecting arcs of a chord diagram such that if an end of a chord lies on one of these arcs, then its second end also lies on one of these arcs. Given a share, one may obtain a chord diagram by *closing* a share (adding an empty share as a complement).

All the possible shares span a vector space, and we define the $\mathfrak{sl}_2$ weight system on it. Its values lie in a commutative subalgebra of the tensor square $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. This commutative subalgebra is generated by three elements which we denote by c_1, c_2, ξ. For this weight system, analogues of the four-term and Chmutov–Varchenko relations hold, and the proof of this fact is almost the same as the proof for chord diagrams. We work with the quotient space of the vector space of shares modulo these relations. Using the Chmutov–Varchenko relations, we conclude that each element of this vector space admits a representation in the form of a linear combination of shares of a simpler kind such that the values of the $\mathfrak{sl}_2$ weight system on them have the form $(c_1-1)^{k_1}(c_2-1)^{k_2}c_1^{n_1}c_2^{n_2}\xi^N$, $k_1, k_2, n_1, n_2, N \in \mathbb{N}\cup\{0\}$. On the set of shares, we can define multiplication as the concatenation of the shares and extend it by linearity to all the elements of the quotient space. In this way, we introduce an associative algebra structure in it; we denote this algebra by $\mathcal{S}$. Further, it follows from the existence of such a representation that the $\mathfrak{sl}_2$ weight system gives a homomorphism from the algebra of shares to the algebra of polynomials in ξ whose coefficients are polynomials in c_1, c_2. Moreover, this homomorphism is an isomorphism.

Following the approach proposed by P. E. Zakorko, we introduce operators of adding a chord, which we denote by S_k, $k = 1, 2$, (we use only the operator S_1 further on) and X, on the algebra of shares and corresponding (in the sense of the said homomorphism) operators $\widetilde{S}_1, \widetilde{X}$ on the polynomial algebra. Again using the six-term Chmutov-Varchenko relations and the four-term relations, we derive recurrence formulas for the action of these operators. Using them, we obtain a generating function for the matrix coefficients of the operator $\widetilde{S}_1$ in the basis $1, \xi, \xi^2, \ldots$. (We denote these coefficients by $s_{i,m}, m = 0, 1, 2, \ldots; i = 0, 1, 2, \ldots, m$). In addition, we compute the leading matrix coefficients $s_{m,m}$ explicitly. It turns out that $s_{m,m} = c - \frac{m(m+1)}{2}$.

To deduce formulas for the values of the $\mathfrak{sl}_2$ weight system on chord diagrams with a complete bipartite intersection graph $K_{n,m}$ (we denote such chord diagrams by $B_{n,m}$, $n, m = 0, 1, 2, \ldots$) we introduce a specialization, that is, a mapping from the algebra of shares to the algebra of generating functions that maps the share such that the value of the $\mathfrak{sl}_2$ weight system on it equals ξ^m to the ordinary generating function G_m for the values of the $\mathfrak{sl}_2$ weight system on the chord diagrams $B_{0,m}, B_{1,m}, B_{2,m}, \ldots$. Using this specialization, we obtain a formula that expresses G_m in terms of $G_0, G_1, \ldots, G_{m-1}$ and the matrix coefficients of the operator S_1.

Theorem 1. *A sequence of ordinary generating functions $G_m(t)$ for the values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs satisfies the initial condition $G_0(t) = \frac{1}{1-t\cdot c}$ and the recurrence relation*

$$G_m(t) = \frac{c^m + t\sum_{i=0}^{m-1} s_{i,m} G_i(t)}{1 - t\cdot\left(c - \frac{m(m+1)}{2}\right)},$$

where $s_{i,j}$ stands for the coefficients given by the following generating function:

$$\begin{aligned}\sum_{m=0}^{\infty} S_1(\xi^m)t^m &= \sum_{m=0}^{\infty}\sum_{i=0}^{m} s_{i,m}\xi^i t^m \\ &= \frac{1}{1-\xi t}\left(c_1 + \frac{c_1c_2t^2 - \xi t}{1-(2\xi-1)t-(c_1+c_2-\xi^2-\xi)t^2}\right)\end{aligned}$$

It follows that

Theorem 2. *The ordinary generating function $G_m(t)$ for the values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs $K_{0,m}, K_{1,m}, K_{2,m}, \dots$ is a linear combination of geometric progressions of the form $\sum_{k=0}^{m} \frac{p_{m,k}(c)}{1-t\left(c-\frac{k(k+1)}{2}\right)}$, where $p_{m,k}(c)$ are polynomials of degree at most m.*

Corollary 1. *The exponential generating function*

$$\mathcal{G}_m(t) := \sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}(K_{n,m})\frac{t^n}{n!}$$

for the values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs $K_{0,m}, K_{1,m}, K_{2,m}, \dots$ is a linear combination of the form

$$\sum_{k=0}^{m} P_{m,k}(c)\exp\left(t\left(c-\frac{k(k+1)}{2}\right)\right), \tag{6}$$

where $P_{m,k}(c)$ are polynomials of degree at most m.

Corollary 2. *For any G which is an intersection graph of a chord diagram obtained by closing some share with both ends of each chord lying on different arcs, the exponential generating function*

$$\mathcal{G}_G(t) := \sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}((G,n))\frac{t^n}{n!}$$

for the values of the $\mathfrak{sl}_2$ weight system on graphs of the form $(G,0), (G,1), (G,2), \dots$ is a linear combination of the form

$$\mathcal{G}_G(t) = \sum_{k=0}^{|V(G)|} p_{G,k}(c)\exp\left(t\left(c-\frac{k(k+1)}{2}\right)\right),$$

where $p_{G,k}(c)$ are polynomials of degree at most $|V(G)|$.

5.2 The $\mathfrak{sl}_2$ weight system and the projections to the space of primitive elements

Many graph invariants behave naturally with respect to the structure of the Hopf algebra of graphs. Such invariants are significantly simplified under the projection (2) to the subspace of primitive elements. For example, the value of the chromatic polynomial on $\pi(G)$ is the linear term of the chromatic polynomial of the graph G (see [5]).

S. K. Lando conjectured that the value of the $\mathfrak{sl}_2$ weight system on the projection of a chord diagram is a polynomial of degree at most half the circumference (length of the longest circle of its intersection graph).

In the proof of this statement for some infinite series of graphs the following formula (7) obtained in the thesis proved useful.

For an arbitrary graph G we introduce exponential generating functions

$$\begin{aligned}\mathcal{G}_G(x) &:= x^{|V(G)|}\sum_{n=0}^{\infty}(G,n)\frac{x^n}{n!},\\ \mathcal{P}_G(x) &:= x^{|V(G)|}\sum_{n=0}^{\infty}\pi((G,n))\frac{x^n}{n!}.\end{aligned}$$

Theorem 3. *The generating function $\mathcal{P}_G(x)$ is equal to*

$$\mathcal{P}_G(x) = \sum_{V_1\sqcup\ldots\sqcup V_k=V(G)}(-1)^{k-1}(k-1)!\mathcal{G}_{G|_{V_1}}(x)\mathcal{G}_{G|_{V_2}}(x)\cdots\mathcal{G}_{G|_{V_k}}(x)\left(\exp(-K_1x)\right)^k, \tag{7}$$

where the summation is over all representations $V(G)=V_1\sqcup\cdots\sqcup V_k$ of the set $V(G)$ as a disjoint union of nonempty subsets.

Note that the structure of this formula resembles the formula (2) by Lando for the projection of a graph.

It follows from Theorem 3 and Corollary 1 that the following theorem is true:

Theorem 4. *The value of the $\mathfrak{sl}_2$ weight system on the projection of a complete bipartite graph to the space of primitive elements is a polynomial in c such that its degree is less than or equal to the number of vertices in the smaller of the two parts.*

Thus, for complete bipartite graphs, Lando's conjecture assering that the value of the $\mathfrak{sl}_2$ weight system on the projection of a chord diagram to the space of primitive elements is a polynomial of degree at most half the circumference of its intersection graph is true.

Moreover, it follows from Corollary 2 that the same is true also for a more wide family of graphs:

Corollary 3. *Under the assumptions of Corollary 2, the exponential generating function $\mathcal{P}_G(t)$ for the values of the $\mathfrak{sl}_2$ weight system on the projections of graphs of the form $(G,0),(G,1),(G,2),\dots$ is a linear combination of the form*

$$\sum_{k=0}^{|V(G)|} F_{G,k}(c)\exp(k\cdot t), \tag{8}$$

where $F_{G,k}(c)$ are polynomials of degree at most $|V(G)|$.

Note that in [27] the values of the weight system $\mathfrak{sl}_3$ on the projections of chord diagrams with the intersection graph $K_{2,n}$ are discussed.

5.3 Values of the $\mathfrak{sl}_2$ weight system on a family of graphs that are not intersection graphs

If a graph G is not an intersection graph of any chord diagram, then the value of the $\mathfrak{sl}_2$ weight system on it is undefined. However, such a graph may be equivalent to a linear combination of intersection graphs modulo the four-term relations. In this case, we can determine the value of the $\mathfrak{sl}_2$ weight system on it as the corresponding linear combination of the values of the $\mathfrak{sl}_2$ weight system on intersection graphs. Generally speaking, such a definition can depend on the way in which the graph G is represented as a linear combination of intersection graphs. The question of existence of a well defined extension remains open. At the same time, if we have found such a representation, then it determines the possible extension uniquely.

Let C_5 be a cycle on 5 vertices. If $n>0$, then the graph (C_5,n) is not an intersection graph. However, with the use of a single four-term relation for any $n>0$ such a graph is represented as a linear combination of three intersection graphs and thus the value $w_{\mathfrak{sl}_2}((C_5,n))$ appears as a linear combination of the values of $w_{\mathfrak{sl}_2}$ on the corresponding chord diagrams. This allows us to compute it explicitly.

Theorem 5. *If the $\mathfrak{sl}_2$ weight system admits an extension to graphs (C_5,n), $n=0,1,2,\dots$ that satisfies the four-term relation for graphs, then the gen-*

erating function for its values on these graphs has the form

$$x^5 \cdot \sum_{n=0}^{\infty} \frac{w_{\mathfrak{sl}_2}((C_5), n)x^n}{n!} = \frac{1}{630} c x^5 \big((270c^4 - 540c^3 - 999c^2 + 576c + 324)e^{(c-1)x} + (280c^4 - 1610c^3 + 3234c^2 - 2646c + 756)e^{(c-6)x} + (80c^4 - 1000c^3 + 4065c^2 - 6120c + 2700)e^{(c-15)x}\big) \quad (9)$$

Theorem 6. *Under the assumptions of the previous theorem, the generating function for the values of the extension of the $\mathfrak{sl}_2$ weight system on the projections onto the space of primitive elements of graphs (C_5, n), $n = 0, 1, 2, \dots$ is equal to*

$$x^5 \cdot \sum_{n=0}^{\infty} \frac{w_{\mathfrak{sl}_2}(\pi((C_5, n)))x^n}{n!} = \frac{1}{630} c x^5 \cdot \big((480c^4 + 720c^3 - 159c^2 + 576c + 324)e^{-x} + (-5040c^4 - 5040c^3 + 315c^2)e^{-3x} + (4080c^4 + 1620c^3 - 3990c^2 + 360c)e^{-4x} + 15120c^4 e^{-5x} + (-25760c^4 + 19600c^3 + 1974c^2 - 2646c + 756)e^{-6x} + (8400c^4 - 12600c^3 + 4725c^2)e^{-7x} + (5040c^4 - 13860c^3 + 7560c^2)e^{-8x} + (-1680c^4 + 5880c^3 - 5985c^2 + 1890c)e^{-9x} + (-720c^4 + 4680c^3 - 8505c^2 + 4050c)e^{-11x} + (80c^4 - 1000c^3 + 4065c^2 - 6120c + 2700)e^{-15x}\big) \quad (10)$$

As in the case of complete bipartite graphs, this result confirms the conjecture by Lando asserting that the value of the $\mathfrak{sl}_2$ weight system on the projection of a chord diagram to the space of primitive elements is a polynomial of degree at most half the circumference of its intersection graph.

6 Main results of the thesis

Theorem ([1,3]). *The following recurrence relation for ordinary generating functions for the values of the $\mathfrak{sl}_2$ weight system at complete bipartite graphs holds:*

$$G_0(t) = \frac{1}{1 - t \cdot c},$$
$$G_m(t) = \frac{c^m + t \sum_{i=0}^{m-1} s_{i,m} G_i(t)}{1 - t \cdot \left(c - \frac{m(m+1)}{2}\right)}.$$

where $s_{i,j}$ stands for the coefficients given by the following generating function:

$$\sum_{m=0}^{\infty} S_1(\xi^m)t^m = \sum_{m=0}^{\infty}\sum_{i=0}^{m} s_{i,m}\xi^i t^m = \frac{1}{1-\xi t}\left(c_1 + \frac{c_1c_2t^2 - \xi t}{1-(2\xi-1)t-(c_1+c_2-\xi^2-\xi)t^2}\right).$$

Theorem ([**3**])**.** *The ordinary generating function $G_m(t)$ for the values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs $K_{0,m}, K_{1,m}, K_{2,m}, \dots$ is a linear combination of geometric progressions of the form $\sum_{k=0}^{m} \frac{p_{m,k}(c)}{1-t\left(c-\frac{k(k+1)}{2}\right)}$, where $p_{m,k}(c)$ are polynomials of degree at most m.*

Theorem ([**3**])**.** *The exponential generating function*

$$\mathcal{G}_m(t) := \sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}(K_{n,m})\frac{t^n}{n!}$$

for the values of the $\mathfrak{sl}_2$ weight system on complete bipartite graphs $K_{0,m}, K_{1,m}, K_{2,m}, \dots$ is a linear combination of the form

$$\sum_{k=0}^{m} P_{m,k}(c)\exp\left(t\left(c-\frac{k(k+1)}{2}\right)\right), \tag{11}$$

where $P_{m,k}(c)$ are polynomials of degree at most m.

Theorem. *For any G which is the intersection graph of a chord diagram obtained by closing some share with both ends of each chord lying on different arcs, the exponential generating function*

$$\mathcal{G}_G(t) := \sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}((G,n))\frac{t^n}{n!}$$

for the values of the $\mathfrak{sl}_2$ weight system on graphs of the form $(G,0),(G,1),(G,2),\dots$ is a linear combination of the form

$$\mathcal{G}_G(t) = \sum_{k=0}^{|V(G)|} p_{G,k}(c)\exp\left(t\left(c-\frac{k(k+1)}{2}\right)\right),$$

where $p_{G,k}(c)$ are polynomials of degree at most $|V(G)|$.

Theorem. *Under the assumptions of the previous theorem, the exponential generating function $\mathcal{P}_G(t)$ for the values of the $\mathfrak{sl}_2$ weight system on the projections of graphs of the form $(G,0),(G,1),(G,2),\dots$ is a linear combination of the form*

$$\sum_{k=0}^{|V(G)|} F_{G,k}(c)\exp(k\cdot t), \qquad (12)$$

where $F_{G,k}(c)$ are polynomials of degree at most $|V(G)|$.

Theorem ([**2**])**.** *The generating function for the values of the $\mathfrak{sl}_2$ weight system on the graphs (C_5,n), $n=0,1,2,\dots$ equals*

$$x^5\cdot\sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}((C_5,n))\frac{x^n}{n!} = \frac{1}{630}cx^5\big((270c^4-540c^3-999c^2+576c+324)e^{(c-1)x} \\ +(280c^4-1610c^3+3234c^2-2646c+756)e^{(c-6)x} \\ +(80c^4-1000c^3+4065c^2-6120c+2700)e^{(c-15)x}\big) \qquad (13)$$

Theorem ([**2**])**.** *The exponential generating function*

$$\mathcal{P}_G(x)=\sum_{n=0}^{\infty}\pi((G,n))\frac{x^n}{n!}$$

for the projections of graphs (G,n) to the subspace of primitive elements is given by the formula

$$\mathcal{P}_G(x)= \\ \sum_{V_1\sqcup\ldots\sqcup V_k=V(G)} (-1)^{k-1}(k-1)!\mathcal{G}_{G|_{V_1}}(x)\mathcal{G}_{G|_{V_2}}(x)\cdots\mathcal{G}_{G|_{V_k}}(x)\left(\exp(-K_1x)\right)^k.$$

Theorem ([**2**])**.** *The generating function for the values of the $\mathfrak{sl}_2$ weight system values at the projections to the space of primitive elements of the* *graphs* (C_5,n), $n=0,1,2,\dots$ *equals*

$$x^5\cdot\sum_{n=0}^{\infty} w_{\mathfrak{sl}_2}(\pi((C_5,n)))\frac{x^n}{n!} = \frac{1}{630}cx^5\cdot\big((480c^4+720c^3-159c^2+576c+324)e^{-x}$$
$$+(-5040c^4-5040c^3+315c^2)e^{-3x}+(4080c^4+1620c^3-3990c^2+360c)e^{-4x}$$
$$+15120c^4e^{-5x}+(-25760c^4+19600c^3+1974c^2-2646c+756)e^{-6x}$$
$$+(8400c^4-12600c^3+4725c^2)e^{-7x}+(5040c^4-13860c^3+7560c^2)e^{-8x}$$
$$+(-1680c^4+5880c^3-5985c^2+1890c)e^{-9x}$$
$$+(-720c^4+4680c^3-8505c^2+4050c)e^{-11x}$$
$$+(80c^4-1000c^3+4065c^2-6120c+2700)e^{-15x}\big) \quad (14)$$

7 The main results of the thesis are presented in these papers

1 Filippova (Zinova), P. A., *Values of the $\mathfrak{sl}_2$ Weight System on Complete Bipartite Graphs*, Functional Analysis and Its Applications, 2020, 54:3, 208–223 (arXiv:2102.03487)

2 Filippova(Zinova), P. A., *Values of the $\mathfrak{sl}_2$ weight system on a family of graphs that are not the intersection graphs of chord diagrams*, Sb. Math., 213:2 (2022), 235–267

3 Kazaryan M. E., Zinova P. A. *Algebra of shares, complete bipartite graphs, and the $\mathfrak{sl}_2$ weight system*, submitted

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