Lomonosov Moscow
State University

Faculty of Mechanics and Mathematics

as a manuscript

Andrey Dukov

Generic finite-parameter families of vector fields on two-shpere

Summery of the PhD Thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

Academic supervisor:
Doctor of sciences,
Professor Yu. Ilyashenko

Moscow - 2022

Introduction

This thesis refers to the bifurcation theory of dynamical systems on two-manifolds. It discribes some properties of polycycles of vector fields and their generic unfoldings. The paper is splited into three parts. In these parts we consider generic bifurcations of a 'heart' polycycle, numeric invariants in semilocal bifurcations and the multiplicity of limit cycles appearing after perturbations of hyperbolic polycycles respectively.

Polycycles

Let M be a C^∞-smooth two-dimensional oriented manifold. In most cases the manifold M is a sphere $\mathbb{S}^2$ or its open subset. Consider the space of all C^r-smooth vector fields $Vect^r(M)$, $r \in \mathbb{N} \cup \{\infty\}$, endowed with the standard smooth topology.

Definition 1. A finite directed graph γ imbedded to M is called a *polycycle* of a vector field if it satisfies the following properties:

- its vertices are singular points of the vector field;
- its edges are phase trajectories, different from the singular points; the time determines the direction;
- the graph γ is Eulerian (there exists a path that visits each edge exactly once).

Suppose a generic finite-parameter family V with a base of parameters $B = (\mathbb{R}^k, 0)$ perturbes the field v_0. We suppose that the space of all k-parameter families is endowed with the compact-open topology. A property of points of a topological space (in our case of the space of k-parameter families) is called *generic* if the set of all points (the families) that satisfy this property contains a countable intersection of dense open sets.

Example 1. Consider a polycycle formed by two hyperbolic saddles and two separatrix connections. Suppose their free separatrices are on different sides of the polycycle. This polycycle is called 'heart' polycycle (see Fig. 1). Since it has only two degeneracies (two saddle connections), we see that it can be perturbed in a generic two-parameter family.

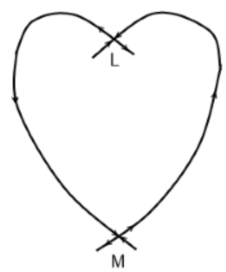

Рис. 1: 'Heart' polycycle.

The theory of bifurcations aims to *describe all generic perturbations of a vector field with any given degeneracy* (for example with a polycycle). A complete study of a polycycle implies a study of the bifurcation diagram of the family. Recall the definition of structural stability of vector fields and the definition of bifurcation diagrams.

Definition 2. Two vector fields are *orbitally topologically equivalent* if there exists a homomorphism H such that it brings the phase portrait of the first vector field to the phase portrait of the second one with respect to trajectory directions. A field v is called *structurally stable* if in the space of all vector fields there exists a neighbourhood U of V such that each field $\tilde{v} \in U$ is orbitally topologically equivalent to the field v.

Definition 3. The *bifurcation diagram* of a family V with a base of parameters B is a subset of the base of parameters B. It consists of values such that the correspondent vector fields of the family are structually unstable.

Today all bifurcations of codimension 1 (i.e. with one degeneracy only) are studied thoroughly by the folowings mathematicians: Andronov, Leontovich (a separatrix loop of a saddle) [1]; Sotomayor (all elementary degeneracies of codimension 1) [2]; Malta, Palis (a parabolic limit cycle with two saddles

[1] A.Andronov, E.Leontovich, I.Gordon, A.Mayer, The theory of bifurcations of dynamical systems on plane, 'Science', 1967

[2] J. Sotomayor, Generic one-parametre families of vector fields on two-dimensional manifolds // Publications Mathématiques de l'IH?S, Volume 43 (1974) , p. 5-46, Zbl 0279.58008 | MR 49 4039.

inside and outside the cycle) [3]; Ilyashenko, Solodovnikov (a separatrix loop with several saddles inside the loop) [4]; Goncharuk, Ilyashenko, Solodovnikov (a parabolic limit cycle with several saddle inside and outside the cycle) [5]; Starichkova (structual stability of elementary bifurcations of codimension 1) [6].

All degeneracies of codimension 1 are splited into six defferent cases but all degeneracies of codimension 2 are splited into about hundred cases. They were studied by Reyn ('lune' polycycle) [7]; Grozovskiy ('apple' and 'half-apple' polycycles) Roitenberg (a separatrix loop with the characteristic number equal to one, 'lune', 'heart', 'apple', 'half-apple' polycycles etc. on orientable and non-orientable manifolds, sparkling saddle connections) [8].

If the codimension is equal to 3 then numeric invariants appear. They are real values such that any equivalence of families preserves them. They were discovered in article of Ilyashenko, Kudryashov and Schurov [9].

The bifurcation diagram of a family perturbing a polycycle of large codimension is very complicated. Hence, for these polycycles we should change the problem: does a given degeneracy appear in a their generic perturbation? The most interesting objects are limit cycles i.e. periodic trajectories isolated from other periodic trajectories. There are some estimations for the count of the limit cycles that appear after any perturbation of a polycycle of codimension

[3]Malta, I. P., Palis, J. [1981] "Families of vector fields with finite modulus of stability," Dynam- ical Systems and Turbulence, Warwick 1980, eds. Rand, D. & Young, L.-S. (Springer Berlin Heidelberg, Berlin, Heidelberg), ISBN 978-3- 540-38945-3, pp. 212–229.

[4]Yu. Ilyashenko, N. Solodovnikov, Global bifurcations in generic one-parameter families with a separatrix loop on S^2 , Moscow Math. J., 2018, pp. 93–115

[5]N. Goncharuk, Yu. Ilyashenko, N. Solodovnikov, "Global bifurcations in generic one-parameter families with a parabolic cycle on S^2", Mosc. Math. J., 19:4 (2019), 709–737

[6]V. Starichkova, Global Bifurcations in Generic One-parameter Families on S^2, Regul. Chaotic Dyn., 23:6 (2018), 767–784

[7]Reyn, J. W. [1980] "Generation of limit cycles from separatrix polygons in the phase plane," Geo- metrical Approaches to Differential Equations, ed. Martini, R. (Springer Berlin Heidelberg, Berlin, Heidelberg), pp. 264–289.

[8]V.Roitenberg, Unlocal 2-parameter bifurcations on surfaces, Doctor of Philosophy thesis, Yaroslavl State Technical University, Yaroslavl, 2000

[9]Yu. Ilyashenko, Yu. Kudryashov and I. Schurov, Global bifurcations in the two-sphere: a new perspective, Invent. math., 2018, Volume 213, Issue 2, pp 461–506

n in any generic k-parameter family [10] [11] [12].

The theory of bifurcations is an actively developing branch of the maths as you can see from the historical review. In the theory there are many unsolved problems. Several recent results change our perception of behavior of the polycyles.

Bifurcation diagram of the 'heart' polycycle

In the first part of the thesis we study the bifurcation of a 'heart' polycycle (see Fig. 1). This polycycle is formed by two separatrix connections. Hence, it can be perturbed in a generic two-parameter family.

Definition 4. The *characteristic number* of a hyperbolic saddle is the modulo of the ratio of the its eigenvalues, the negative one is in the numerator. A saddle is called *(non-) dissipative* if its characteristic number is greater (less) than one.

There exist two qualitatively different scenarios of the 'heart' polycycle bifurcation, depending on the dissipativity of the saddle points. The key result of this part is presented in the following two theorems.

Theorem 1. *For dissipative and non-dissipative saddles, the bifurcation diagram of a generic two-parameter family which perturbes a vector field with a 'heart' polycycle is the union of C^1-curves approaching the origin comprising (for special parameters) two axes, two curves corresponding to the vector fields with a separatrix loop, one curve corresponding to the vector fields with a parabolic limit cycle and two countable series of curves corresponding to the sparkling separatrix connections.*

The bifurcation diagram for this case is shown in the Figure 2a.

Theorem 2. *The bifurcation diagram for a generic 2-parameter family perturbing a vector field with a 'heart' polycycle is, in the case of two non-dissipative*

[10] Ilyashenko Yu., Yakovenko S., Finite cyclicity of elementary polycycles in generic families, Concerning the Hilbert 16th Problem, Amer. Math. Soc. Transl. Ser. 2, vol. 165, Amer. Math. Soc., Providence, RI, 1995, 21-65.

[11] V.Kaloshin, The Existential Hilbert 16-th problem and an estimate for cyclicity of elementary polycycles, Invent. math. 151, 451–512 (2003) DOI: 10.1007/s00222-002-0244-9

[12] P. Kaleda, I. Schurov, Cyclitcity of elementary polycycles with a fixed count of singular points in generic k-parameter families, Algebra and analysis, 2010, vol.22, serie 4, 57–75

saddles, is a union of C^1-curves tending to the origin, comprising (for special parameters) two axes, two curves corresponding to the vector fields with separatrix loops and two countable series of curves corresponding to the sparkling separatrix connections.

The bifurcation diagram in this case is shown in Figure 2b.

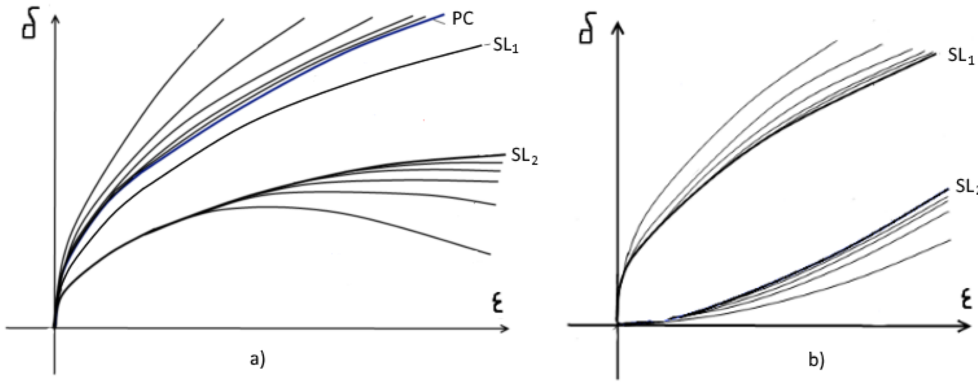

Рис. 2: Two possible bfurcation diagrams. The curve PC corresponds to the vector fields with a parabolic limit cycle, the curves SL_1 and SL_2 correspond to the vector fields with separatrix loops, another countable set of curves corresponds to the vector fields with sparkling separatrix connections.

Definition 5. Consider an outcoming separatrix of a saddle and an incoming separatrix of another saddle. In the base of parameters without the origin select the set of the values such that these separatrices form the separatrix connection. If the germ of this set splits into infinite number of components then in the family there is a sequence of *sparkling saddle connections*.

In the paper we proved that two sequences of sparkling separatrix connections appear after a perturbation of the 'heart' polycycle in any case. Thus the bifurcation of a 'heart' polycycle is a simplest semilocal bifurcation (i.e. it occurs in a arbitrary small neighbourhood of the polycycle) containing sparkling separatrix connections.

Numerical invariants of semilocal bifurcations

Definition 6. A *glocal* family of vector fields on a manifold M is a germ of a map of the family base $(\mathbb{R}^k, 0)$ to the space $Vect^r(M)$.

The term *glocal* is produced by gluing together two terms: *global* and *local*. The glocal family is local with respect to the parameter but global on the phase space.

Definition 7. Two representatives of glocal families $V = \{v_\alpha | \alpha \in (\mathbb{R}^k, 0)\}$ and $W = \{w_\beta | \beta \in (\mathbb{R}^k, 0)'\}$ are topologically equivalent, provided that there exists a map

$$H : B \times M \to B' \times M',\ (\alpha, x) \mapsto (h(\alpha), H_\alpha(x)),$$

where $h : B \to B'$ is a homeomorphism of representatives of the bases, H_α is a homeomorphism of the phase spaces that brings the phase portrait of v_α to that of $w_{h(\alpha)}$.

If nothing more is required then the equivalence is called *weak*.

If H is a homeomorphism then the equivalence is called *strong*.

If H (and H^{-1}) is continuous in α, x on the union of all singular points, periodic orbits and separatrices of the vector field v_0 (respectively, of w_0) on the fiber $\{0\} \times M$ ($\{0\} \times M'$), then H is called *sing*-equivalence of glocal families V and W.

The main result of the second part of the thesis is the following.

Theorem 3. *There exists an open set in the space of all C^∞-smooth 5-parameter families such that any family V from this set contains a vector field v_0 with the following properties. The field v_0 has a hyperbolic polycycle γ_5 with five vertices (two of them coincide) and five edges whose semilocal bifurcations have numeric invariants of sing-equivalence.*

A numeric invariant of sing-equivalence is a continuous function on the space of all families such that its set of values is an open subset of the real line. Any sing-equivalence preserves its value for each family.

Multiple limit cycles

Definition 8. Let Γ be a cross-section to a limit cycle. The Poincaré map of the cycle is a monodromy map along the orbits from the cross-section Γ to itself. *A limit cycle is of multiplicity* m if the Poincaré map has a fixed point of multiplicity m.

Let a vector field $v_0 \in Vect^\infty(M)$ has a polycycle γ. Consider a k-parameter family $V = \{v_\delta\}$, $\delta \in B = (\mathbb{R}^k, 0)$ perturbing the field v_0.

Definition 9. A limit cycle (of multiplicity m) appears from the polycycle γ of the field v_0 in the family V if there exists a tending to zero (that corresponds to the field v_0) sequence of parameter values $\{\delta_\alpha\}_{\alpha\in\mathbb{N}}$ such that for any α the field v_{δ_α} has a limit cycle $LC(\delta_\alpha)$ (of multiplicity m) and the sequence of the limit cycles $LC(\delta_\alpha)$ tends to the polycycle γ as $\delta_\alpha \to 0$ with respect to Hausdorff metric.

Let a field v_0 has a polycycle γ. The polycycle is formed by n separatrix connections of hyperbolic saddles $S_1, \ldots, S_n$ (some saddles may coincide). By $\lambda_1, \ldots, \lambda_n$ denote the characteristic numbers of the saddles $S_1, \ldots, S_n$ respectively.

Theorem 4. *For any positive integer n there exists a non-trivial polynomial $\mathcal{L}_n(\lambda_1, \ldots, \lambda_n)$ such that the following statement holds. Let v_0 be a vector field with a hyperbolic polycycle γ and the characteristic numbers $\lambda_1, \ldots, \lambda_n$ of the saddles satisfy the inequation*

$$\mathcal{L}_n(\lambda_1, \ldots, \lambda_n) \neq 0. \tag{1}$$

Then for any C^∞-smooth finite-parameter family the multiplicity of any appearing limit cycle is not greater than n.

In the paper we proved that any multiple limit cycle implies that a polynomial system of homogeneous equations with coefficients depending on characteristic numbers $\lambda_1, \ldots, \lambda_n$ has a non-trivial solution. The polynomial $\mathcal{L}_n$ connects with the resultant of the polynomial system.

In the case of small codimension, we can write the polynomial $\mathcal{L}_n$. For this we need to define some polynomials.

For any positive integer n denote by Λ_n the following polynomial in characteristic numbers $\lambda_1, \ldots, \lambda_n$:

$$\Lambda_n(\lambda_1, \ldots, \lambda_n) = \prod_{I \neq (0,\ldots,0)} (\lambda^I - 1),$$

where $I = (i_1, \ldots, i_n)$ is a multiindex, $i_j \in \{0, 1\}$. By λ^I we denoted the product $\lambda_1^{i_1} \ldots \lambda_n^{i_n}$. For example, $\Lambda_2(\lambda_1, \lambda_2) = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1\lambda_2 - 1)$.

In addition, by $M(\lambda_1, \lambda_2, \lambda_3)$ denote the following polynomial:

$$M(\lambda_1, \lambda_2, \lambda_3) = 4(\lambda_1\lambda_2\lambda_3 - 1) - (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1).$$

Theorem 5. *For any $n = 1, 2, 3, 4$ the following polynomials $\mathcal{L}_n$ satisfy Theorem 4:*

1. $\mathcal{L}_1(\lambda_1) = \Lambda_1(\lambda_1)$;
2. $\mathcal{L}_2(\lambda_1, \lambda_2) = \Lambda_2(\lambda_1, \lambda_2)$;
3. $\mathcal{L}_3(\lambda_1, \lambda_2, \lambda_3) = \Lambda_3(\lambda_1, \lambda_2, \lambda_3)$;
4. $\mathcal{L}_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \Lambda_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\cdot$
   $\cdot M(\lambda_1, \lambda_2, \lambda_3) M(\lambda_1, \lambda_2, \lambda_4) M(\lambda_1, \lambda_3, \lambda_4) M(\lambda_2, \lambda_3, \lambda_4)$.

Approbation of the results

The results of the thesis were presented at the seminar 'Dynamical Systems' (the faculty of maths of Higher School of Economics) under the supervision of Ilyashenko Yu. in 2017, 2018, 2020 and 2021; at the Summer School at Dubna in 2018, 2019; at the seminar 'Dynamical Systems and Applications' of Higher School of Economics at Nizhniy Novgorod in 2021; at the Shilnikov Conference in 2020.

Publications of the author of the thesis

[1] A.Dukov, Bifurcation of the 'heart' polycycle in generic 2-parameter families, Trans. Moscow Math. Soc., 2018, 209–229.

[2] A.Dukov, Yu.Ilyashenko, Numeric invariants in semilocal bifurcations, J. Fixed Point Theory Appl. 23, 3 (2021). https://doi.org/10.1007/s11784-020-00837-x.

[3] A.Dukov, Multiplicity of limit cycles appearing after a perturbation of the hyperbolic polycycles, Sbornik: Mathematics, 2023, accepted to publication.