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Lebedev Mikhail

Stationary states for nonlinear Schrödinger equation with periodically modulated nonlinearity: mathematical and numerical study

Ph. D. Thesis summary

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Doctor of Sciences, professor
Georgy L. Alfimov

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General Description of the Work

Introduction

Since the 90s of the last century, the Nonlinear Schrödinger Equation (NLS) with additional spatial non-autonomous terms has been an object of thorough studies. For the one-dimensional case, this equation can be written as follows:

$$i\Psi_t + \Psi_{xx} - U(x)\Psi + P(x)|\Psi^2|\Psi = 0. \tag{1}$$

Specific interest to this class of equations has been caused by progress in experimental study of Bose–Einstein Condensate[1] (BEC) as well as advances in photonics and its applications.

In the context of BEC, equation (1) is called the *Gross–Pitaevskii equation* (GPE). It describes the dynamics of the condensate in so-called mean-field approximation. Here $\Psi(t, x)$ is the dimensionless wave function of the condensate cloud, that is assumed to be elongated along the axis x. The function $U(x)$ describes the trap potential that is used to confine BEC, and $P(x)$ corresponds to nonlinear potential (also called *pseudopotential*). The pseudopotential describes the spatial dependence of scattering length that may be non-constant due to various reasons. Intervals with positive values of pseudopotential, $P(x) > 0$, correspond to the case of attraction of the condensate particles, while intervals with negative values, $P(x) < 0$, correspond to the interatomic repulsion. The prototypical examples of $U(x)$ are the harmonic potential $U(x) = Ax^2$ (the magnetic trap), periodic potential $U(x) = A\cos 2x$ (the optical trap), and various types of potential wells. As examples of the pseudopotential $P(x)$, different functions has been used, including periodic ones. The prototypical example is the cosine pseudopotential, $P(x) = A + B\cos\Omega x$. In the latter case one says that there exists a *nonlinear lattice*[2] interacting with the condensate cloud.

[1] A. Einstein, "Quantentheorie des einatomigen idealen Gases", Preussische Akademie der Wissenschaften, Berlin, 1924.

[2] H. Sakaguchi, B. A. Malomed, "Matter-wave solitons in nonlinear optical lattices", Phys. Rev. E, Vol. 72, P. 046610, 2005.

In optical applications, equation (1) describes the propagation of a light beam in an optical fiber. In this case the function $\Psi(t,x)$ corresponds to the amplitude of the electromagnetic wave, where t is the propagation distance and x is a transverse spatial coordinate. Function $U(x)$ corresponds to a local perturbation of the refractive index, which accounts for the optical inhomogenity of the medium[3]. Function $P(x)$ describes the spatial modulation of the Kerr coefficient, that can be achieved by adding resonant dopants into the fiber[4]. Also, periodical dependence of $P(x)$ naturally arises while considering a multilayer periodic system of thin film nonlinear waveguides[5]. Such dependence of the Kerr coefficient implies the presence of a *nonlinear lattice* in the inhomogeneous optical medium.

For different physical applications the solutions of equation (1) of the special form, so-called *stationary localized solutions* (*stationary localized modes*, SLMs), play an important role. Such solutions can be obtained by putting into (1) an ansatz

$$\Psi(t,x) = u(x)e^{-i\omega t}, \tag{2}$$

where the function $u(x)$ satisfies the localization conditions of the form:

$$\lim_{x\to\infty} u(x) = 0. \tag{3}$$

Here ω is a real parameter that stands for a chemical potential of the condensate. The profile $u(x)$ of the stationary localized solution is a real-valued function[6], satisfying the equation

$$u_{xx} + Q(x)u + P(x)u^3 = 0; \quad Q(x) = \omega - U(x). \tag{4}$$

It should be noted that not all localized solutions of Eq. (4) are equally interesting from a physical point of view. The stability is a critically important property

[3] Y. V. Kartashov, B. A. Malomed, and L. Torner, "Solitons in nonlinear lattices", Rev. Mod. Phys. Vol. 83, P. 247, 2011.

[4] J. Hukriede, D. Runde, and D. Kip, "Fabrication and application of holographic Bragg gratings in lithium niobate channel waveguides", J. Phys. D, Vol. 36, R1, 2003.

[5] Y. S. Kivshar, G. P. Agrawal, "Optical Solitons", Academic Press, P. 386–424, 2003.

[6] G. L. Alfimov, V. V. Konotop, and M. Salerno, "Matter solitons in Bose–Einstein condensates with optical lattices", Europhys. Lett., Vol. 58, P. 7–13, 2002

of the localized solutions. If SLM is unstable, a small perturbation leads to its destruction during the temporal evolution. Therefore, stable localized solutions are especially valuable from the perspective of physical applications. So, the analysis of stability is an essential part of the theoretical study of SLMs.

Formulation of the problem

While studying the dynamics described by equation (1) the following questions naturally arise:

1. Is it possible to describe *completely all* stationary localized solutions of equation (1) that coexist for under given parameters?

2. How to identify stable solutions among them?

A survey of the current state of the field

It's worth noting that in the majority of works devoted to this topic the problem of finding / describing of *all* SLMs has not been raised. Instead, only specific classes of solutions corresponding to particular physical structures has been described, see the comprehensive review??. At the same time, despite the questions above seem a little bit "challenging", the combination of rigorous analytical methods with numerical computations makes it possible to achieve significant progress in this direction. Let us note some related results.

For equation (4) with potential $U(x)$ of the form of infinite potential well, in the case of repulsive interparticle interactions, $P(x) \equiv -1$, the computational descriptive procedure has been proposed[7]. This procedure provides a computational evidence and can guarantee the complete description of *all* bounded solutions of equation for the given set of parameters. The proposed method was afterwards generalized to systems of several coupled Gross–Pitaevskii equation, in which the corresponding pseudopotential do not depend on the spatial coordinate[8].

[7] G. L. Alfimov, D. A. Zezyulin, "Nonlinear modes for the Gross–Pitaevskii equation — a demonstrative computational approach", Nonlinearity, Vol. 20, P. 2075–2092, 2007.

[8] G. L. Alfimov, I. V. Barashenkov, A. P. Fedotov, V. V. Smirnov, D. A. Zezyulin, "Global search for localised modes in scalar and vector nonlinear Schrödinger-type equations", Physica D, Vol. 397, P. 39–53, 2019.

It was shown, that for the periodic potential $U(x)$ in the case of repulsive interactions of the condensate particles, $P(x) \equiv -1$, there exist sufficient conditions that allow to describe exhaustively *all* bounded solutions of equation (4). Moreover, it was shown that under these conditions there exist one-to-one correspondence between the bounded solutions and all possible bi-infinite sequences of symbols of some finite alphabet[9]. Such sequences are called *codes of solutions*, and the algorithm of assigning of such codes can be called *coding of solutions*. In the above mentioned paper, the verification of the sufficient conditions was performed by means of numerical computations. Results of this paper were further extended[10], specifically: there has been proposed an algorithm that allows to reconstruct numerically the profile of the solution by its symbolic code.

It's also worth to mention the mathematical works of F. Zanolin and co-authors[11],[12], in which the existence of some types of solutions in related problems is proved. Such solutions also can be classified by means of methods of nonlinear dynamics. Authors of these works use an approach that relies on topological argumentation and differs from the one presented in the dissertation.

Relevance of the research topic

Generalization of the above mentioned results to the case of non-constant pseudopotential, $P(x) \neq \text{const}$, is an important actual problem. Application of the "coding approach" to the Gross-Pitaevskii equation with periodic pseudopotential yields a classification of nonlinear stationary states in BEC in nonlinear lattice. This classification opens up the possibility of experimental finding of new, previously unknown

[9] G. L. Alfimov, A. I. Avramenko, "Coding of nonlinear states for the Gross–Pitaevskii equation with periodic potential", Physica D, Vol. 254, P. 29–45, 2013.

[10] G. L. Alfimov, P. P. Kizin, D. A. Zezyulin, "Gap solitons for the repulsive Gross-Pitaevskii equation with periodic potential: Coding and method for computation", Discrete and Continuous Dynamical Systems — Series B, Vol. 22, P. 1207–1229, 2017.

[11] Ch. Zanini, F. Zanolin, "Complex Dynamics in One-Dimensional Nonlinear Schrödinger Equations with Stepwise Potential", Complexity, Vol. 2018, Article ID 2101482, 2018.

[12] Ch. Zanini, F. Zanolin, "An Example of Chaos for a Cubic Nonlinear Schrödinger Equation with Periodic Inhomogeneous Nonlinearity", Advanced Nonlinear Studies, Vol. 12, No. 3, P. 481–499, 2012.

stable stationary states.

Tasks and objectives of the study

The main object of the study in the dissertation is the set of stationary solutions of one-dimensional Gross – Pitaevskii equation (1) with *periodic pseudopotential*. Tasks and objectives of the study can be formulated as follows:

1. To formulate sufficient conditions that allow to generalize the method of coding of SLMs[9] to the case of periodic pseudopotential; to specify the ways of verification of these conditions (analytically or with numerical computations).

2. To study the set of stationary solutions of equation (1) with periodic pseudopotential in the case when the trapping potential can be neglected, $U(x) \equiv 0$.

3. For the case of harmonic potential well, $U(x) = Ax^2$, to investigate the effect of periodic pseudopotential on the structure of the set of stationary localized solutions and their stability.

Methods

In order to study possible types of SLMs the *method of excluding of singular solutions*[9] is used. We call a solution of equation (4) *singular* if it goes to infinity at a finite point $x = x_0$ of the real axis, i.e.

$$\lim_{x \to x_0} u(x) = \infty. \tag{5}$$

Obviously, such solutions cannot describe a profile of stationary state, so they should be excluded from the consideration. The main idea is that under certain conditions, “the most part” of solutions of Eq. (4) are singular. The set of remaining solutions, called *regular*, turns out to be quite “poor” and it can be fully described in terms of symbolic dynamics.

In order to compute profiles of localized solutions of equation (4) the shooting method is used involving standard procedures for solving Cauchy problems for ODE. The obtained solutions are checked for linear stability by solving corresponding eigenvalue problem in the Fourier space (Fourier Collocation Method[13]), and also via simulation of dynamics of (1) with a conservative finite-difference scheme[14]. All the algorithms and numerical methods are implemented in `MATLAB` using `MEX` extension for high performance computing support.

Scientific novelty

In the thesis, a number of exact statements about regular and singular solutions of equation (4) are proved. The conditions that ensure the existence of singular solutions, or their absence are formulated. In particular, it was shown that if the pseudopotential is negative at least at one point x_0, $P(x_0) < 0$, then there exist two one-parametric families of solutions which tend to infinity at this point x_0. Asymptotic expansions for these families are given.

The method of excluding of singular solutions was further developed. The dissertation proposes sufficient conditions for existence of one-to-one correspondence between regular solutions of equation (4) and bi-infinite symbolic sequences over some alphabet. In contrast to the previously obtained results[9], the proposed conditions admit effective numerical verification. An algorithm of the numerical verification is provided in the dissertation along with its theoretical justification.

For the case $U(x) \equiv 0$ and cosine periodic pseudopotential of the form $P(x) = A + B\cos 2x$ the set of stationary localized solutions has been studied. When applying the above-mentioned techniques, the set of SLMs was effectively described, and, eventually, new stable localized solution, named *dipole soliton*, was found. This solution has been previously unknown.

Finally, in the case of harmonic trapping potential, $U(x) = Ax^2$, the effect of periodic pseudopotential of the form $P(x) = A + B\cos\Omega x$ on the set of SLMs was studied. It was shown that in comparison with well-studied case $P(x) = \text{const}$, the set of stationary localized solutions is much richer. Namely, there exist essentially

[13] J. Yang, "Nonlinear Waves in Integrable and Nonintegrable Systems", Philadelphia: SIAM, 2010.

[14] V. Trofimov, N. Peskov Comparison of finite-difference schemes for the Gross-Pi taevskii equation // Mathematical Modelling and Analysis. — 2009. — Mar. — Vol. 14. — P. 109–126.

nonlinear solutions which cannot be predicted by low-amplitude approximation. The dependence of the SLMs stability on the frequency Ω of the pseudopotential was studied. For the pseudopotential with zero mean, $P(x) = B\cos\Omega x$, it was found that the increase of frequency Ω allows to stabilize low-amplitude solutions, whose counterparts in the model with $P(x) = \text{const}$ are unstable.

The highlights of the thesis are:

1. The statements on the presence and absence of singular solutions of equation (4) are proved. It is shown that in the case $P(x) > 0$ all solutions of (4) are regular. If $P(x)$ is negative at least at one point x_0, $P(x_0) < 0$, then there exist two one-parametric families of solutions, which tend to infinity at the point x_0. The asymptotic description of these families are given. In the case $Q(x) < 0$ and $P(x) < 0$, it is shown that all solutions of equation (4) are singular.

2. For equation (4) sufficient conditions for coding of regular solutions are formulated. An effective algorithm for their numerical verification is presented.

3. For the case $U(x) \equiv 0$, $P(x) = A + \cos 2x$ the set of SLMs of Eq. (1) are described. This study reveals new stable localized solution, named *dipole soliton*.

4. It is shown that the model that includes both trapping harmonic potential $U(x) = Ax^2$ and the nonlinear lattice admits new classes of SLMs, in comparison with the case when the nonlinear lattice is not taken into account. For the periodic pseudopotential with zero mean, it is shown that increasing of the frequency of pseudopotential can stabilize low-amplitude localized solutions.

Confidence level and approbation of the results

The Gross – Pitaevskii model is a classical model of physics of ultra-cold temperatures and its confidence is beyond any doubt. SLMs in this model correspond to localized stationary solutions of the Gross–Pitaevskii equation. In the thesis, SLMs are constructed, and their stability is investigated numerically. Numerical computation of SLMs is performed by means of standard numerical methods for ODEs with controlled accuracy. The analysis of stability of SLMs is fulfilled by means of the spectral method which is generally recognized for the similar problems. Results of the stability analysis are verified by solution of the time-dependent Gross–Pitaevskii equation employing a conservative finite-difference scheme. The key findings of the thesis were reported at various scientific seminars and conferences, including:

1. "Фундаментальная математика и ее приложения в естествознании", BSU, Ufa, September 2015, talk "Стационарные моды нелинейного уравнения Шрёдингера в присутствии линейного и нелинейного потенциалов".

2. "Dynamics, Bifurcations and Chaos III", Lobachevsky State University of Nizhni Novgorod, Nizhni Novgorod, July 2016, talk "Stable dipole solitons and soliton complexes in the nonlinear Schrödinger equation with periodically modulated nonlinearity".

3. "Complex Analysis, Mathematical Physics and Nonlinear Equations", Bashkortostan, Bannoe Lake, March 2018, talk "Steady-states for the Gross-Pitaevskii equation with nonlinear lattice pseudo- potential".

4. "Nonlinear Phenomena in Bose Condensates and Optical Systems", Tashkent, Uzbekistan, August 2018, talk "Steady-states for the Gross-Pitaevskii equation with nonlinear lattice pseudopotential".

5. "Complex Analysis, Mathematical Physics and Nonlinear Equations", Bashkortostan, Bannoe Lake, March 2019, talk "Coding of solutions for the Duffing equation with non-homogeneous nonlinearity".

6. "Complex Analysis, Mathematical Physics and Nonlinear Equations", Bashkortostan, Bannoe Lake, March 2021, talk "Coding of bounded solutions of equation $u_{xx} - u + \eta(x)u^3 = 0$ with periodic piecewise constant function $\eta(x)$".

Publications

Materials of the thesis were presented in 3 articles of peer-reviewed journals included in the international citation system Scopus:

1. Alfimov G. L., Lebedev M. E. On regular and singular solutions for equation $u_{xx} + Q(x)u + P(x)u^3 = 0$ // Ufa Mathematical Journal, 2015, Vol. 7, no. 2, P. 3–16, DOI: 10.13108/2015-7-2-3 (Scopus Q2).

2. Lebedev M. E., Alfimov G. L., Malomed B. Stable dipole solitons and soliton complexes in the nonlinear Schrödinger equation with periodically modulated nonlinearity // Chaos, 2016, Vol. 26, P. 073110, DOI: 10.1063/1.4958710 (Scopus Q1).

3. Alfimov G. L., Gegel L. A., Lebedev M. E., Malomed B. A., Zezyulin D. A. Localized modes in the Gross-Pitaevskii equation with a parabolic trapping potential and a nonlinear lattice pseudopotential // Communications in Non-li-near Science and Numerical Simulation, 2019, Vol. 66, P. 194–207, DOI: 10.1016/j.cnsns.2018.06.019 (Scopus Q1).

Personal contribution of the author

The main finding of the thesis was obtained either by the applicant in person, or in collaboration with co-authors where the role of the applicant was dominant. The numerical implementations of all the algorithms and other computer programs was fulfilled by the applicant personally.

Structure and volume of the dissertation

The thesis consists of introduction, four chapters, conclusion, three appendices, and a bibliography. Total volume of the dissertation is 131 page. Among them there are 115 pages of text, including 35 figures, 4 tables and 1 algorithm scheme. Bibliography consists of 61 titles.

The Content of the Work

In **Introduction** the relevance of the dissertation work is justified. The author formulates the goals of the research and the problems to be solved. The practical significance of the obtained results is discussed, and the main findings of the thesis are announced.

Also in Introduction the main concepts that are related to the object of study are presented. *Stationary localized solutions* (or *stationary localized modes*, SLMs) are defined as solutions of equation (1) of the form (2) that satisfy the localization conditions (3). Solutions $u(x)$ of equation (4) is called *singular* if for some finite point x_0 the relation $u(x) \to \infty$ as $x \to x_0$ takes place. Point x_0 is called a collapse point for the solution $u(x)$. Alternatively, one can say that such solution *collapses* at the point x_0. On the contrary, solution is called *regular* if there is no such point x_0.

In **Chapter 1** the propositions about regular and singular solutions of equation (4) are formulated and proved.

Proposition 1. *Let for equation* (4) *functions* $Q(x)$, $P(x) \in C^1(\mathbb{R})$, *and for all* $x \in \mathbb{R}$

(a) $P(x) \geq P_0 > 0$, $|P'(x)| \leq \widetilde{P}$;

(b) $Q(x) \geq Q_0$, $|Q'(x)| \leq \widetilde{Q}$.

Then a solution of the Cauchy problem for equation (4) *with arbitrary initial conditions can be continued to the whole real axis* R.

Proposition 1 establishes the conditions for absence of singular solutions. One of these conditions requires function $P(x)$ to be positive. On the other hand, if function $P(x)$ takes a negative value at least at one point x_0, $P(x_0) < 0$, then there exist two families of solutions of equation (4) that collapse at this point. In order to formulate the corresponding statement it's convenient to assume that $P(x_0) = -1$ (it's always can be achieved by a suitable renormalization).

Proposition 2. *Let $P(x_0) = -1$ and Ω be a neighbourhood of the point x_0, $Q(x) \in C^2(\Omega)$, and $P(x) \in C^4(\Omega)$. Then there exist two C^1-smooth one-parametric families of solutions for equation (4), collapsing at the point $x = x_0$ (while approaching from the left, $x \to x_0 - 0$), and connected by a symmetry $u \to -u$. Each of these families can be parametrized by a free variable $C \in \mathbb{R}$, and, moreover, the following asymptotic expansions are valid:*

$$\pm u(x) = \frac{\sqrt{2}}{\eta} + A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 \ln|\eta| + C\eta^3 + A_4\eta^4 \ln|\eta| + \ldots. \quad (6)$$

Here $\eta = x - x_0$, and coefficients A_n can be expressed in terms of coefficients Q_n, P_n of the corresponding expansions for the functions $Q(x)$, $P(x)$,

$$Q(x) = Q_0 + Q_1\eta + Q_2\eta^2 \ldots, \quad P(x) = -1 + P_1\eta + P_2\eta^2 + \ldots. \quad (7)$$

Similar one-parametric families of collapsing solutions also exist to the right side of the point $x = x_0$. Finally, the conditions are given that ensure that equation (4) has no regular non-zero solutions at all.

Proposition 3. *Let for all $x \in \mathbb{R}$ the conditions $P(x) \leq P_0 < 0$, $Q(x) \leq Q_0 < 0$ take place. Then all solutions of equation (4) are singular except for the zero one.*

It follows from Proposition 2 that if pseudopotential $P(x)$ is a sign-altering function then singular behaviour is common for solutions of equation (4). This fact makes it possible to apply the method of excluding of singular solutions[9].

In **Chapter 2** the basic concept of the theory for coding of stationary states is introduced. It is assumed that the potential and the pseudopotential in Eq. (1) are L-periodic functions, $U(x + L) = U(x)$, $P(x + L) = P(x)$. It turns out that under certain conditions the set of regular solutions of equation (4) can be fully described by the structure of the so-called *coding sets*: $\mathscr{U}_L^+$, $\mathscr{U}_L^-$, $\mathscr{U}_L$. These sets are defined on the plane (u, u') of initial conditions for equation (4) as follows. The set $\mathscr{U}_L^+$ consists of all initial conditions to the Cauchy problem such that the corresponding solutions are bounded on the interval $[0; L]$. Similarly, the set $\mathscr{U}_L^-$ consists of such initial conditions that the corresponding solutions are bounded in the interval $[-L; 0]$. The set $\mathscr{U}_L$ represents an intersection of the two sets above, $\mathscr{U}_L = \mathscr{U}_L^+ \cap \mathscr{U}_L^-$. The relation between the points of these sets can be described in terms of Poincaré map. It's defined as follows. Let $\mathbf{p} \in \mathscr{U}_L^+$, then $\mathcal{P}(\mathbf{p}) = \mathbf{q} = (u(L), u'(L))$, $\mathbf{q} \in \mathscr{U}_L^-$ where $u(x)$ is a solution of Cauchy problem for equation (4) with initial conditions $u(0) = u_0$, $u'(0) = u'_0$. The sequence of points $\{\mathbf{p}_n\}$ constructed by iterating the Poincaré map is called an *orbit*.

The approach is based on *two hypotheses* about the action of the map $\mathcal{P}$ on the set $\mathscr{U}_L$.

(I) The set $\mathscr{U}_L$ is a union of finite or infinite number of connected components, $\mathscr{U}_L = \bigcup_{k \in S} D_k$, where S is a set of indices, and each component D_k is a curvilinear quadrangle with monotonic boundaries. Moreover, for any i, j, sets $H_{ij} = \mathcal{P}(D_i) \cap D_j$ and $V_{ij} = \mathcal{P}^{-1}(D_j) \cap D_i$ are non-empty.

(II) Maps $\mathcal{P}$, $\mathcal{P}^{-1}$ in some way preserve the monotonicity properties of strips, called *h- and v-strips*, connecting the opposite sides of D_k, and the width of these strips decreases under the action of corresponding maps.

Let's clarify Hypothesis II. If it's valid, then the above defined sets H_{ij} and V_{ij} represent h- and v-strips correspondingly. It follows from Hypothesis II that the set $\mathcal{P}(H_{ij}) \cap D_k$ is also an h-strip, and $d_\mathrm{h}(\mathcal{P}(H_{ij}) \cap D_k) < d_\mathrm{h}(H_{ij})$, where $d_\mathrm{h}(\cdot)$ is the h-strip width measured along the vertical line. Similarly, the set $\mathcal{P}^{-1}(V_{ij}) \cap D_k$ represents a v-strip, and $d_\mathrm{v}(\mathcal{P}^{-1}(V_{ij}) \cap D_k) < d_\mathrm{v}(V_{ij})$. Here $d_\mathrm{v}(\cdot)$ is the v-strip width measured along the horizontal line.

It was shown in this chapter that under Hypotheses I and II, the action of $\mathcal{P}$ on $\mathscr{U}_L$ is of the Smale horseshoe type[15], and there exists a one-to-one correspondence between *orbits* of regular solutions of equation (4) and *bi-infinite sequences* of symbols of some alphabet (the number of symbols is equal to the number of com-

[15] S. Wiggins, "Introduction to Applied Nonlinear Dynamical Systems and Chaos", New York: Springer-Verlag, 2003.

ponents D_k of the set $\mathscr{U}_L$). In this part of the study, author follows the approach that goes back to 60-70-es, the works of L. P. Shilnikov and V. M. Alekseev. In the seminal papers[16],[17] the symbolic dynamics approach was applied for description of behaviour of trajectories near a homoclinic loop and for three-body problem. In this thesis, the horseshoe dynamics approach is used for the mapping of sets $\mathscr{U}_L^{\pm}$ that is a new application of this theory.

Verification of the hypotheses is performed numerically. Hypothesis I is verified by direct *scanning* of properly chosen area of the plane of initial conditions. Verification of Hypothesis II requires more sophisticated approach. In this chapter two theorems (called *Theorems on h- and v-strips mapping*) are formulated. These theorems allow to reduce verification of Hypothesis II to an effective numerical procedure. The procedure consists in estimating of some constants at the points $\mathbf{p}$ belonging to special subsets of the set $\mathscr{U}_L$. The algorithm is as follows.

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[16] Л. П. Шильников, «Об одной задаче Пуанкаре-Биркгофа», Математический сборник, Т. 74, №4, С. 378–397, 1967.

[17] В. М. Алексеев, «Финальные движения в задаче трех тел и символическая динамика», Успехи математических наук, Т. 36, Вып. 4, С. 161, 1981.

Algorithm 1. Numerical Check of Hypothesis II

Input: Hypothesis I takes place for equation (4); $\mathscr{U}_L = \bigcup_{k\in S} D_k$.

Step (1). Set up a numerical grid for computations. For all $i, j \in S$ construct sets $H_{ij} = \mathcal{P}(D_i) \cap D_j$ and $V_{ij} = \mathcal{P}^{-1}(D_j) \cap D_i$ numerically on the defined grid.

Step (2). Checking the signs of elements in Jacobi matrices $D\mathcal{P}_{\mathbf{p}}$, $D\mathcal{P}^{-1}_{\mathbf{q}}$ for maps $\mathcal{P}$, $\mathcal{P}^{-1}$.

- **(a)** For each point $\mathbf{p} \in V_{ij}$ compute 2×2 matrix of the operator $D\mathcal{P}_{\mathbf{p}} = (a_{mn})$ and check that $\forall \mathbf{p} \in V_{ij}$ the sings of matrix elements match exactly one of the configuration specified in Theorem on h-strips mapping.
- **(b)** For each point $\mathbf{q} \in H_{ij}$ compute 2×2 matrix of the operator $D\mathcal{P}^{-1}_{\mathbf{q}} = (b_{mn})$ and check that $\forall \mathbf{q} \in H_{ij}$ the signs of matrix elements match exactly one of the configurations specified in Theorem on v-strips mapping.

Step (3). Estimation of the width decrease for h- and v-strips.

- **(a)** Estimate value $\mu_* = \min\limits_{\mathbf{p}\in V_{ij}} a_{11}(\mathbf{p})$ on the numerical grid; check that $\mu_* > 1$.
- **(б)** Estimate value $\nu_* = \min\limits_{\mathbf{q}\in H_{ij}} b_{22}(\mathbf{q})$ on the numerical grid; check that $\nu_* > 1$.

The result of the work of the algorithm above is illustrated (see Figure 1) by the example of equation (4) with $Q(x) \equiv -1$ and $P(x) = \eta(x)$, where $\eta(x)$ is a piecewise constant periodic function of the period $L = L_* + L_0$, defined in a following way:

$$\eta(x) = \begin{cases} -1, & x \in [0; L_*); \\ +1, & x \in [L_*; L_* + L_0). \end{cases} \tag{8}$$

It turns out (see Proposition 6 in the thesis) that the set $\mathscr{U}_L$ on the plane of initial conditions for such equation is unbounded. Nevertheless, one can restrict the consideration to some bounded subset $\mathcal{D} \subset \mathscr{U}_L$. If Hypotheses I and II are valid for $\mathcal{D}$, then it's possible to describe completely a subset of regular solutions, whose orbits do not leave the considered subset $\mathcal{D}$ on the plane of initial conditions. In Figure 1 example of the set $\mathcal{D}$ is shown. This set consists of three connected components. Hypotheses I and II are valid for $\mathcal{D}$, hence, there exists a subset of regular solutions which can be exhaustively coded with three symbols $\{-1, 0, +1\}$.

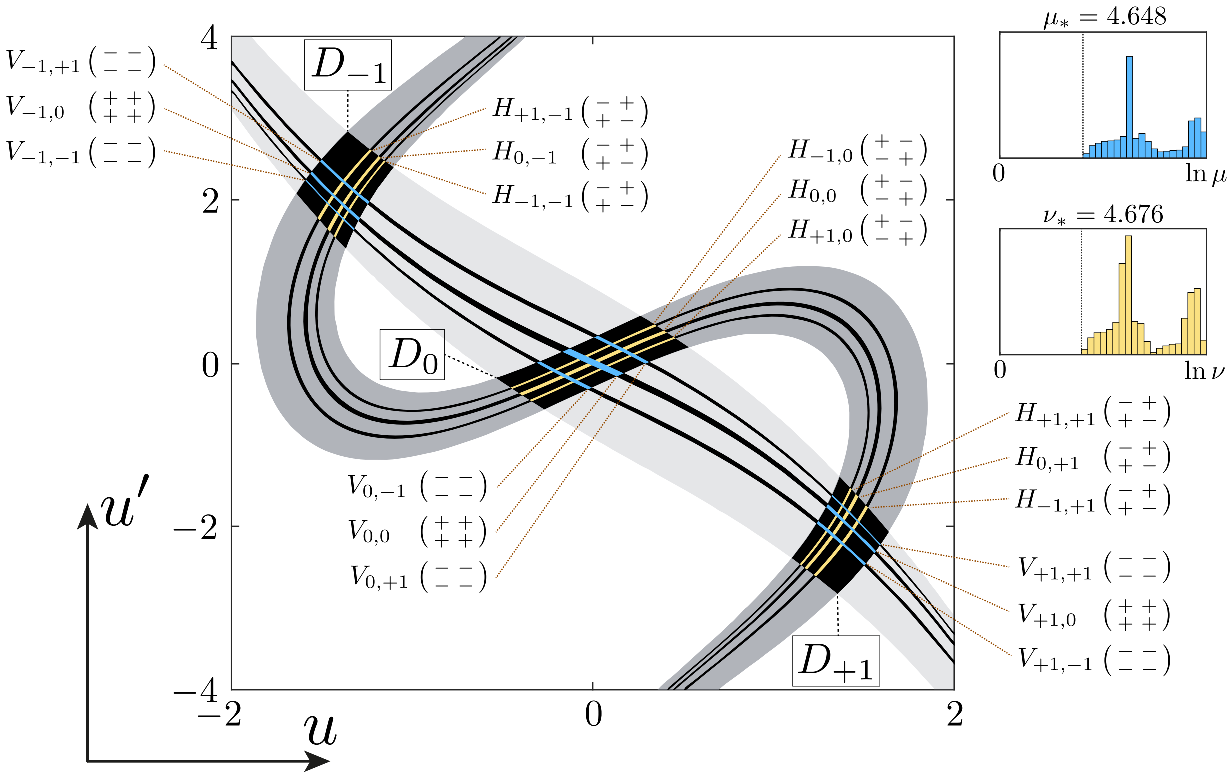

Figure 1: Verification of hypotheses for equation (4) for the case $Q(x) \equiv -1$, $P(x) = \eta(x)$ with parameters $(L_*, L_0) = (2, 1)$. Set $\mathscr{U}_L^+$ (light gray), $\mathscr{U}_L^-$ (darks gray), and their intersection $\mathcal{D} = \{D_{-1}, D_0, D_{+1}\} \subset \mathscr{U}_L$ (black) are depicted. For the sets V_{ij} (blue) and H_{ij} (yellow) the sings of elements in corresponding operators $D\mathcal{P}_{\mathbf{p}}$, $D\mathcal{P}_{\mathbf{q}}^{-1}$ are shown.

In **Chapter 3** the set of stationary localized solutions has been studied for the case of equation (1) in which the trapping potential is absent, $U(x) \equiv 0$, and the pseudopotential has a cosine form, $P(x) = A + \cos 2x$. Such model has been previously considered in literature. It was reported[2] that this model admits a stationary localized bell-shaped solution, called *fundamental soliton*, which is stable for certain parameters of the equation.

In Chapter III the coding approach is applied to this model. Structure of the coding sets is shown in Figure 2. In turns out that the sets $\mathscr{U}_\pi^\pm$ $(L = \pi)$ are infinite spirals. Their intersection $\mathscr{U}_\pi$ is unbounded and consists of infinite number of connected components.

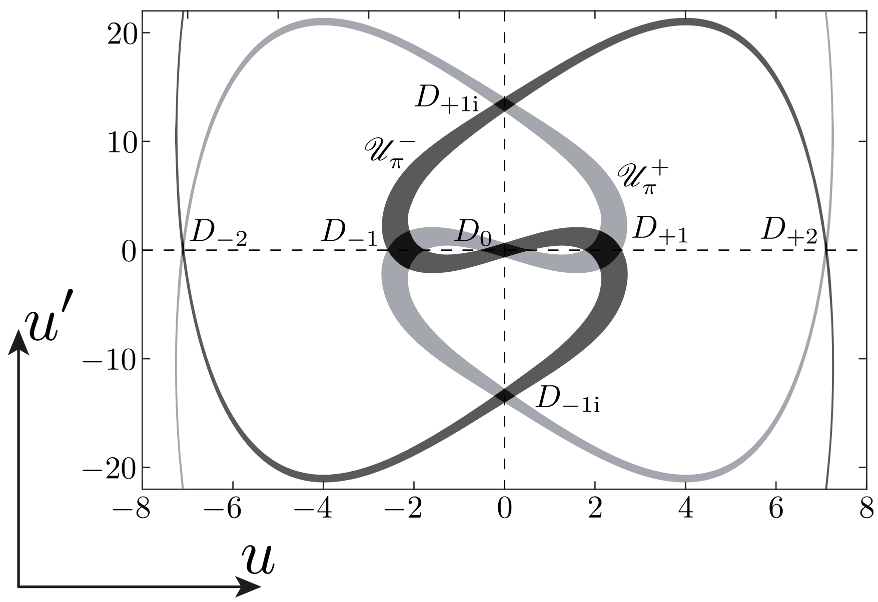

Figure 2: Set $\mathcal{D} \subset \mathscr{U}_\pi$ consisting of seven connected components $\{D_{-2}, D_{-1i}, D_{-1}, D_0, D_{+1}, D_{+1i}, D_{+2}\}$ (black), formed by intersection of the sets $\mathscr{U}_\pi^\pm$ for equation (4) with $Q(x) = -1.5$, $P(x) = \cos 2x$.

Verification of Hypotheses I and II was performed numerically for a subset $\mathcal{D} \subset \mathscr{U}_\pi$ consisting of seven central connected components. The numerical procedure allowed to conclude that both of the hypotheses take place. Hence, there is a one-to-one correspondence between solutions of equation (4) and symbolic codes derived from the structure of coding set. Existence of such correspondence allows to conclude that the set of stationary localized solutions of the considered equation is extremely rich. In Figure 3 different solutions along with their symbolic codes are shown.

Analysis of the linear stability by means of spectral method showed that the majority of solutions are unstable. However, there is a number of stable solutions, and some of them were previously unknown. One of these new stable solutions is the so-called *dipole soliton*. Its profile is depicted in Figure 3 (e).

In **Chapter 4** the set of SLMs is studied for equation (1) where, along with the periodic pseudopotential, the trapping potential in the form of harmonic potential well is included. After the suitable renormalization, the functions of potential and pseudopotential take the forms $U(x) = x^2$, $P(x) = A + B\cos\Omega x$. The equation

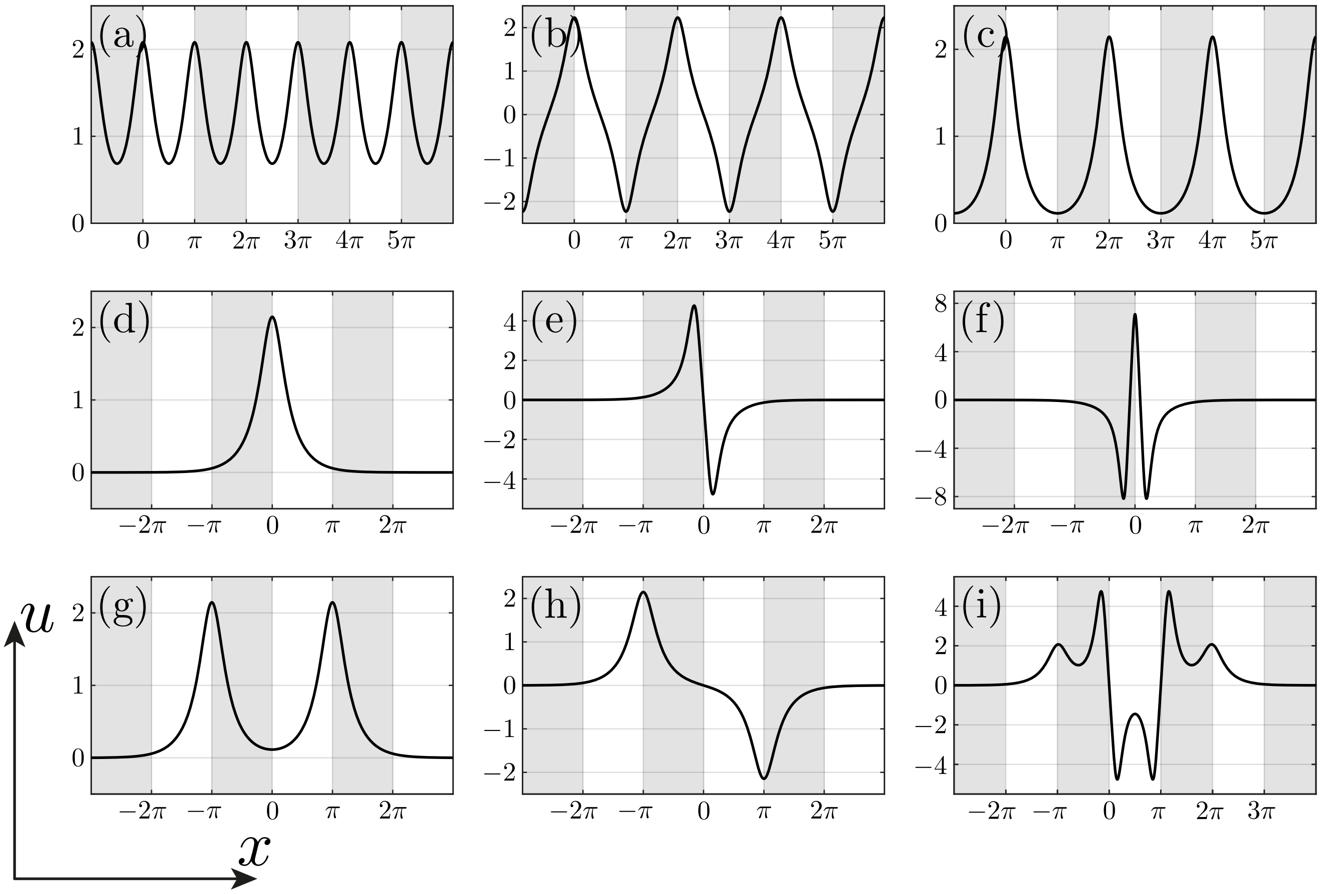

Figure 3: Different solutions of equation (4) for $Q(x) = -1.5$, $P(x) = \cos 2x$. Each solution has its own symbolic code that identify the solution uniquely. Periodic solutions: (a) π-periodic solution $\{\dots,+1,+1,+1,\dots\}$; (b) 2π-periodic solution $\{\dots,+1,-1,+1,-1,\dots\}$; (c) 2π-periodic solution $\{\dots,+1,0,+1,0,\dots\}$. Localized solutions (solitons): (d) fundamental soliton $\{\dots,0,+1,0,\dots\}$; (e) dipole soliton $\{\dots,0,-1\mathrm{i},0,\dots\}$ (f) elementary soliton of code $\{\dots,0,+2,0,\dots\}$ (g) $\{\dots,0,+1,0,+1,0,\dots\}$ (h) $\{\dots,0,+1,0,-1,0,\dots\}$ (i) $\{\dots,0,+1,-1\mathrm{i},+1\mathrm{i},+1,\dots\}$.

admits so-called *stationary solutions with linear counterpart*. These solutions arise from small-amplitude limit, $|u(x)| \ll 1$. In this case one can omit nonlinear part in equation (4) and obtain the harmonic oscillator equation,

$$u_{xx} + (\omega - x^2)u = 0, \tag{9}$$

that is an eigenvalue problem for the values ω. Its solutions are well-known:

$$\tilde{\omega}_n = 2n + 1; \quad \tilde{u}_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2}x^2}; \quad n = 0, 1, \ldots, \tag{10}$$

where functions $H_n(x)$ are Hermite polynomials. When the nonlinearity is turned on, each eigenvalue $\tilde{\omega}_n$ bifurcates and originates one-parametric family of solutions $\Gamma_n = (\omega_n, u_n(x))$ for the nonlinear problem. Such families are called families of solutions with linear counterpart.

It's known[7] that solutions with linear counterpart exhaust the set of SLMs for the case $P(x) \equiv -1$. However, in the case of periodic pseudopotential, there exist solutions without linear counterpart. In Chapter IV the branches of such solutions are constructed numerically, see Figure 4. Linear stability analysis showed that almost all of them are unstable.

The second part of the fourth chapter is devoted to the analysis of the stability for low-amplitude solutions with linear counterpart. It is shown that in the case of pseudopotential with zero-mean, $P(x) = B \cos \Omega x$, increase of the frequency Ω leads to stabilization of low-amplitude solutions with linear counterpart. That is, for each branch Γ_n there exists a threshold Ω_n, such that for $\Omega > \Omega_n$ the corresponding branch of solutions is stable in the vicinity of its bifurcation point $N \ll 1$, $\omega_n \approx \tilde{\omega}_n$.

In **Conclusion** the results of the thesis are summarized. The main outputs can be presented in brief as follows:

1. The statements about the presence and absence of singular solutions of equation (4) are formulated and proved.

2. The sufficient conditions for coding of regular solutions for equation (4) are formulated. An efficient algorithm for numerical verification of these conditions is proposed.

3. For the case $U(x) \equiv 0$, $P(x) = A + \cos 2x$ the set of SLMs for equation (1) has been studied. A new stable solution, called *dipole soliton*, is found.

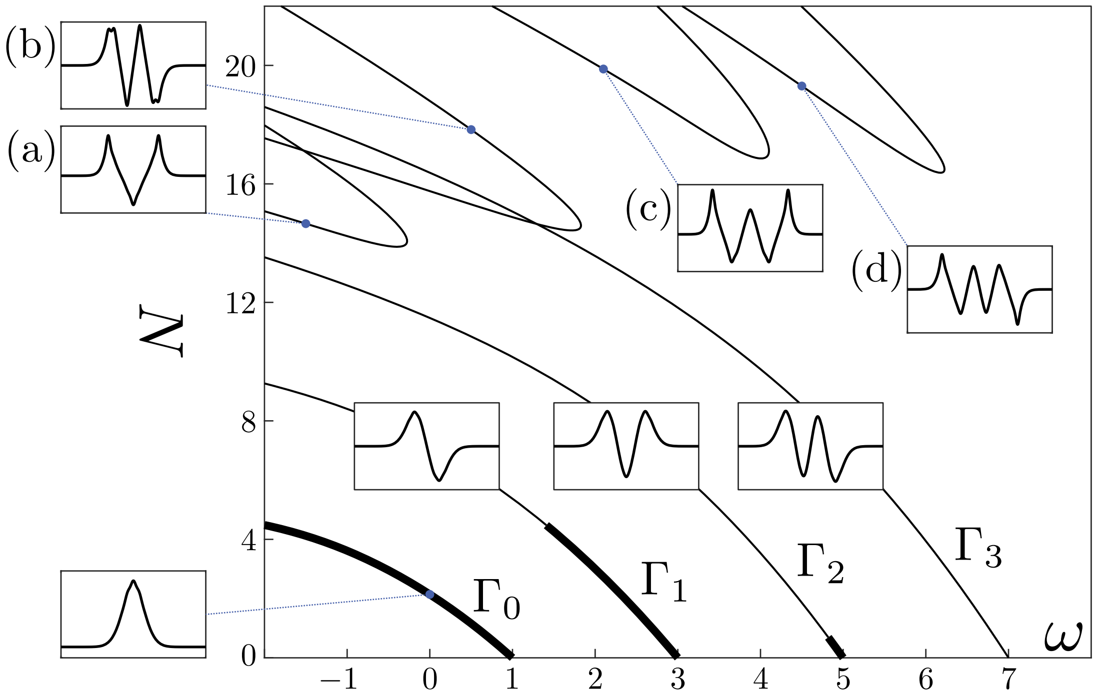

Figure 4: Diagrams of the dependence of the norm of solutions N on the chemical potential ω for equation (4); $Q(x) = \omega - x^2$, $P(x) = 1 + 2\cos(12x)$. Segments of curves $N(\omega)$ that correspond to stable solutions are marked with bold black lines. Branches Γ_n, $n = 0, 1, 2, 3$ correspond to families of solutions with linear counterart.

4. In the case of harmonic potential well it is shown that including of periodic pseudopotential results in new classes of SLMs without linear counterpart. For the pseudopotential with zero mean, it is concluded that the increase of pseudopotential frequency stabilize low-amplitude solutions.

In **Appendix A** Lemma on bounded solutions is proved. This lemma is used in Chapter I.

Appendix B contains explicit solutions for two equations of the Duffing oscillator type:

$$u_{xx} - u + u^3 = 0; \quad u_{xx} - u - u^3 = 0, \tag{11}$$

that are used in Section 2.3.

In **Appendix C** Theorems on h- and v-strips mapping are proved. These theorems are used in Chapter II.