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Irina Bobrova

Non-Abelian generalizations of the Painlevé IV equation

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Vladimir Poberezhny

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Abstract

Six classes of famous Painlevé equations are one of the important objects of modern mathematical physics due to the ubiquitous appearance in physical problems. Their general solutions define new special functions that are known as *the Painlevé transcendents*. Since there exist several examples of different non-abelian analogs for the Painlevé equations that appear in various branches of non-commutative integrable models, a natural question about the classification of integrable non-abelian generalizations arises.

This thesis is devoted to a classification of non-commutative analogs for the fourth Painlevé equation:

$$y'' = \frac{1}{2\,y} y'^2 + \frac{3}{2} y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y}, \qquad y(z),\ z,\ \alpha,\ \beta \in \mathbb{C}. \qquad \mathrm{P}_4$$

Our aim is to discover integrable matrix and non-abelian analogs of the P_4 equation. Their solutions can be understood as new matrix and non-commutative transcendental functions. In addition, such equations can be regarded as quantum or non-abelian versions of the Painlevé IV equation.

The classification is based on two methods that allow us to detect possible candidates for integrable non-abelian analogs of the P_4 equation: (1) a verification of the matrix Painlevé-Kovalevskaya test and (2) a construction of solutions in terms of an infinite non-commutative Toda system. Using the first method, we have found three non-equivalent classes of matrix systems with parameters being constant arbitrary matrices. The second method turns out to be effective for constructing a fully non-commutative analog of the Painlevé IV equation. To prove an integrability of these generalizations, we provide their isomonodromic Lax representations.

1 Historical review

The celebrated Painlevé equations appeared as a result of a classification of complex second-order differential equations of the form

$$y''(z) = P\left(z, y(z), y'(z)\right), \qquad y(z),\ z \in \mathbb{C}, \tag{1}$$

where $P\left(z, y(z), y'(z)\right)$ is a meromorphic function in z and is a rational function in $y\,(z)$, $y'(z)$. In order to study these equations, *the Painlevé property* was introduced. Namely, their general solutions have no movable singular points except of poles[1]. For the first time this property was

[1]In the case of an m-th order ODE with $m > 2$, this criteria is formulated in a different way. Namely, general solutions do not have critical movable points.

used by S. Kovalevskaya in [Kow89]. 50 classes of such equations were found by P. Painlevé and his school [Pai00], [Pai02]. Later B. Gambier [Gam10] proved that only six of them define new special functions. The general solutions of the Painlevé equations are currently known as the most general class of special functions called *the Painlevé transcendents*. These six classes of equations of the form (1) are known as *the Painlevé equations*. They arise in a wide range of applications in mathematics and physics and have surprisingly rich mathematical structures.

In particular, they are related to a system of scalar differential equations [Fuc07], [Gar12] integrable in the sense of the Frobenius theorem. In the paper [Fuc07], R. Fuchs studied the case of the sixth Painlevé equation. The result of R. Fuchs was generalized by R. Garnier who considered irregular singularities [Gar12] and, as a result, found such a representation for other Painlevé equations. Let us note that L. Schlesinger and B. Malgrange also worked in this area (see, e.g. [Sch12], [Mal74]). In the paper [JM81], it was established that the Painlevé equations can be linearized. This fact is connected with monodromy preserving deformations related to vector bundles of rank 2. Thanks to the isomonodromic property, the space of solutions of the Painlevé equations can be parameterized by the monodromy data[2]. Namely, each of the equations can be associated with the zero-locus of an affine cubic which is usually called *the monodromy surface* (e.g., [VDPS09]).

One of the reasons behind the ubiquitous appearance of the Painlevé equations is that they are innately linked to the Toda hierarchy. In [DZ04], B. Dubrovin and Y. Zhang proved that the τ-function of a generic solution to the extended Toda hierarchy is annihilated by some combinations of the Virasoro operators. It is such Virasoro constraints that regulate the correlation functions of many systems in random matrix theory, in string theory and topological field theory. For instance, in [DZ05], expressions for the genus $g \geq 1$ total Gromov–Witten potential were obtained via the genus zero quantities derived from the Virasoro constraints.

Note that the isomonodromic τ-function is closely connected with the so-called sigma form for the Painlevé equations. This issue was investigated by K. Okamoto [Oka81], who developed the Hamiltonian theory of the Painlevé differential equations [Oka80] and showed that all Bäcklund transformations can be obtained as natural affine Weyl groups actions on the sigma form [Oka87a], [Oka87b], [Oka86], [Oka87c].

Regarding applications of the Painlevé equations in integrable systems, it turns out that these equations can be obtained as reduced ODEs of some integrable PDEs. The Ablowitz-Ramani-Segur conjecture [ARS80] states that a nonlinear PDE is solvable by the inverse scattering method [ZS74] only if every nonlinear ODE obtained by an exact reduction has the Painlevé property. For example, the first and second Painlevé equations are the reductions of

[2] Informally speaking.

the KdV equation, the ODE reduction of the sine-Gordon equation is the third Painlevé equation, and the sixth Painlevé equation is the reduction of the nonlinear Schrödinger equation. In the paper [JKT07], it was shown that the Painlevé III–VI equations are the reductions of the three-wave resonant system.

In recent years, quantum, or more generally, non-abelian extensions of various integrable systems have acquired considerable attention. It was motivated by problematics and needs of modern quantum physics as well as by a natural attempts of mathematicians to extend and to generalize the "classical" integrable structures and systems. In particular, the Painlevé transcendents provide a good example of this phenomena.

Quantum versions of the Painlevé equations were obtained in [NGR+08], where the authors quantized the Poisson brackets related to the so-called symmetric form of the Painlevé equations. There is also an interesting non-commutative family of non-autonomous many-particle integrable systems that were introduced in [BCR18]. The authors defined an isomonodromic representation of Takasaki Hamiltonian systems of Painlevé-Calogero families [Tak01]. A fully[3] non-abelian version for the second Painlevé equation was presented in [RR10]. The authors investigated a Hankel quasideterminant structure of the P_2 solutions caused by their relation to the non-commutative Toda equations [GR92]. While it is already known [Kaw15] that each of the Painlevé equations has only one matrix Hamiltonian analog, in the recent paper [AS21], it was shown that the Painlevé II equation has at least three non-equivalent matrix generalizations. In order to derive them, the authors have used the matrix Painlevé-Kovalevskaya test introduced in [BS98]. Since the Painlevé equations are close to the orthogonal polynomials, several authors have derived their matrix analogs, by using matrix generalizations of the orthogonal polynomials (e.g., [CM14], [CM+18]).

This historical review do not pretend to provide full overview of the Painlevé theory, so we refer the reader to excellent books [CM08], [FIN+06], as well as reviews cited in them.

2 Statement of the problem

The main classification problem for integrable systems is related to the question: "What is the notion of integrability?". In order to construct useful definition, we have divided our classifications into two steps.

(1) We first detect non-abelian analogs by using one of the approaches from Table 1.

[3] In the authors terminology.

(2) For a given analog we will present its zero-curvature representation:

$$A_z - B_\zeta = [B, A],$$

where $A(\zeta, z)$, $B(\zeta, z)$ are some matrices, ζ is a spectral parameter, and z is a parameter of deformation (an "independent" variable).

	Approach 1 [BS98]	**Approach 2** [RR10]
Setting	• $\mathcal{A} = \mathrm{Mat}_n(\mathbb{C})$ is an associative unital algebra over $\mathbb{C}$. • "dependent" variables $\in \mathcal{A}$. • the independent variable $\in \mathbb{C}$. • constants $\in \mathcal{A}$ or $\mathbb{C}$. • spectral parameter $\in \mathbb{C}$.	• R is an associative unital division ring over a field $\mathbb{F}$ with a derivation. • all variables $\in R$. • there is an element $t \in R$ s.t. $t' = 1$. • constants $\in \mathbb{F}$. • spectral parameter $\in \mathbb{F}$.
Criterion	Matrix generalizations should pass a matrix Painlevé-Kovalevskaya test.	Analogs should possess solutions in terms of an infinite ncToda system.

Table 1: Approaches

Remark 2.1. We assume that $\operatorname{char} \mathbb{F} = 0$.

Therefore, we have the following

Definition 2.1. A matrix or a non-abelian generalization of the Painlevé IV equation is *integrable*, if

(a) it satisfies a criterion from Approach **1** or Approach **2**, respectively,

(b) and admits the zero-curvature representation.

So, the classification problems can be formulated as follows.

- Apply the matrix Painlevé-Kovalevskaya test for searching new examples of matrix generalizations of the Painlevé IV equation. These examples may contain arbitrary matrix constants. Find zero-curvature representations for all new equations.

- Find fully non-commutative analogs for the Painlevé IV equation, by expressing their solutions in terms of the infinite non-abelian Toda system[4]. For a given analog present its isomonodromic pair.

[4] This point is a generalization of the results from the paper [JKM06] to the non-commutative case.

Remark 2.2. We note that there exist other methods for constructing non-abelian analogs:

(a) a quantization of the Poison brackets [NGR+08];

(b) reductions of integrable non-abelian systems [OS98], [Adl20], [AK22];

(c) a construction of the matrix Sclesinger systems [Kaw15];

(d) a resolution of the matrix Riemann-Hilbert problem [CM14];

(e) a study of autonomous systems [GR19], [BS22b], [BS22a].

3 Main results

3.1 Matrix analogs

In Approach **1**, the dependent variable $y(z)$ belongs to the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$ with the unit element $\mathbb{I}$[5] and the independent variable z is an element of $\mathbb{C}$. By the matrix generalization of the Painlevé-Kovalevskaya test introduced by S. Balandin and V. Sokolov in [BS98], we are able to verify that a matrix ODE may satisfy the Painlevé property (that is a necessary condition). Recall that a scalar ODE of the m-th order is passed the Painlevé-Kovalevskaya test, if all its general formal solutions have m arbitrary constants. We will call such solutions *maximal*.

Example 3.1. Consider the matrix equation

$$y'(z) = -y^2(z), \qquad y(z) \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \tag{2}$$

Suppose that $y(z) \in GL_n(\mathbb{C})$. Then equation (2) can be integrated:

$$-y^{-1}\, y'\, y^{-1} = \mathbb{I}; \qquad (y^{-1})' = \mathbb{I}; \qquad y^{-1} = z\,\mathbb{I} + \Lambda; \qquad y = (z\,\mathbb{I} + \Lambda)^{-1},$$

where $\mathbb{I}$ is a unit matrix and $\Lambda \in \mathrm{Mat}_n(\mathbb{C})$ is a constant of integration. Suppose that Λ can be diagonalized, i.e. $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$. Then, it is obvious that the residue at $z = -\lambda_k$, $k = 1, \dots, n$, of the general solution is a matrix of rank 1.

There is a general observation that maximal solutions of matrix ODEs have a leading coefficient of rank 1. Due to this observation, we prefer to work with polynomial ODEs only. Due to these technical constraints arising from the matrix generalization of the test, we consider homogeneous matrix polynomial ODEs with scalar coefficients resolved w.r.t. the highest derivative,

$$y^{(m)}(z) = P\left(z, y(z), \dots, y^{(m-1)}(z)\right), \qquad y(z) \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \tag{3}$$

[5] Sometimes we will omit it.

Note that this method allows one to implement constant matrix coefficients into (3), if we consider some deformation of a given ODE that possesses the same maximal solutions. Thus, to construct matrix analogs for the Painlevé equations we use the following two ingredients:

(1) matrix generalizations of the principal homogeneous part of the scalar systems that are equivalent to the Painlevé equation and pass the matrix Painlevé-Kovalevskaya test;

(2) linear deformations with arbitrary matrix coefficients of systems from the point (1) that have the same maximal solutions as in the homogeneous case.

As a result, we may find new matrix analogs with constant matrix coefficients, that are connected by some algebraic relations if necessary. Such examples for the first and second Painlevé equations were established in the papers [BS98], [AS21], respectively.

In the case of the Painlevé IV equation, the classification results are given in Theorem 3.1. As we mentioned above (for more details, see Subsection 2.1 in the main text), we are able to work with polynomial ODEs only. Because of this fact, we classify matrix analogs of the Painlevé IV system

$$\left\{\begin{array}{lcl} u' &=& -u^2 + 2uv - 2zu + c_1, \\ v' &=& -v^2 + 2uv + 2zv + c_2, \end{array}\right. \tag{4}$$

where $u(z)$, $v(z)$, $z \in \mathbb{C}$ and c_1, c_2 are arbitrary constants. Using the matrix Painlevé-Kovalevskaya test, we find all integrable[6] matrix generalizations for system (4) of the form

$$\left\{\begin{array}{lcl} u' &=& -u^2 + 2\,uv + \alpha(uv - vu) - 2zu + b_1 u + u b_2 + b_3 v + v b_4 + b_5 \\ v' &=& -v^2 + 2\,vu + \beta(vu - uv) + 2zv + c_1 v + v c_2 + c_3 u + u c_4 + c_5, \end{array}\right. \tag{5}$$

where $u(z)$, $v(z) \in \mathrm{Mat}_n(\mathbb{C})$, α, β, $z \in \mathbb{C}$, and b_i, c_i are constant matrices.

Note that there exist some transformations that preserve the class of the systems of the form (5), changing the parameters. Such transformations are given by the formulas[7]

$$(u,\, v) \mapsto (v,\, u), \tag{6}$$

$$(u,\, v) \mapsto (u^T,\, v^T), \tag{7}$$

$$(u,\, v) \mapsto (-u,\, v - u - 2z) \tag{8}$$

[6] In the sense of Definition 2.1.

[7] For transformations (7) and (8) one should use the scaling $z \mapsto -iz$, $u \mapsto iu$, $v \mapsto iv$ to correct the signs before $2z$ in the resulting system (5) and after transformation (8) also a shift of u and v is needed to bring the result to the form (5).

and

$$u \mapsto e^{zK}\,(u+Q_1)\,e^{-zK}, \qquad\qquad v \mapsto e^{zK}\,(v+Q_2)\,e^{-zK}, \tag{9}$$

where K and Q_i are constant matrices. Let us stress that a transformation (9) takes a system of the form (5) outside of the class, but in very special cases transformations (9) can be applied.

Remark 3.1. According to (6) – (8), the parameters (α, β) change as

$$(\alpha,\beta)\mapsto(\beta,\alpha), \qquad (\alpha,\beta)\mapsto(-\alpha-2,-\beta-2), \qquad (\alpha,\beta)\mapsto(\alpha,-\alpha-\beta-3), \tag{10}$$

respectively. These transformations form a group structure isomorphic to the dihedral group D_{12} of symmetries of a regular 6-gon.

So, the main result of our classification is the following

Theorem 3.1. *Any system (5) that satisfies the matrix Painlevé–Kovalevskaya test can be reduced by transformations (6), (7), (8), and (9) to one of the following:*

$$\begin{cases} u' &=& -u^2+uv+vu-2zu+hu+\gamma_1\,\mathbb{I}, \\ v' &=& -v^2+vu+uv+2zv-vh+\gamma_2\,\mathbb{I}, \end{cases} \qquad \mathrm{P}_4^0$$

$$\begin{cases} u' &=& -u^2+2uv-2zu+h, \\ v' &=& -v^2+2uv+2zv+h+\gamma\,\mathbb{I}, \end{cases} \qquad \mathrm{P}_4^1$$

$$\begin{cases} u' &=& -u^2+2uv-2zu+h_2, \\ v' &=& -v^2+3uv-vu+2zv+h_1u+2h_2+\gamma\,\mathbb{I}. \end{cases} \qquad \mathrm{P}_4^2$$

Here γ, $\gamma_i\in\mathbb{C}$, h is an arbitrary matrix and two constant matrices h_1, h_2 are connected by $[h_2,h_1] \;=\; -\,2\;h_1$.

Remark 3.2. In the case of the scalar matrix h, the P_4^0 system was found by H. Kawakami [Kaw15]. General system P_4^0 can be derived from a natural matrix generalization of the dressing chain with $N=3$ investigated in [VS93].

Remark 3.3. The system P_4^0 is Hamiltonian with respect to the symplectic Poisson structure with the Hamiltonian

$$H(u,v)=\mathrm{tr}\,\big(vu\,(v-u-2z)+vhu+\gamma_1 v-\gamma_2 u\big)\,, \qquad \{u_{ij},v_{kl}\}=\delta_{il}\,\delta_{jk}.$$

When h is a scalar matrix, this Hamiltonian coincides with that presented in [Kaw15]. Remaining systems, P_4^1 and P_4^2, have no Hamiltonians of the form $\mathrm{tr}(Q)$, where $Q(u,v)$ is a non-commutative polynomial, with the constant Poisson bracket.

Remark 3.4. The P_4^1 system is equivalent to the matrix P_4 equation presented in Conclusion of the paper [AS21]. In the case of scalar coefficients, systems P_4^1 and P_4^2 are equivalent to those found in [Adl20].

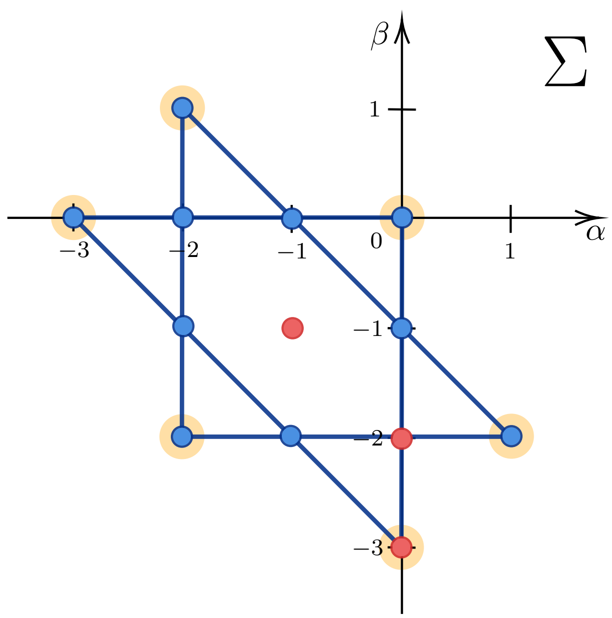
Remark 3.5. Note that the P_4^0 system can be written in a fully non-abelian form. A detailed discussion of this fact can be found in Section 4 in the main text.

To prove this theorem, we divide our research into two steps.

(1) We first do the Painlevé analysis for the homogeneous part of system (5).

(2) Then we consider inhomogeneous system (5) that possess the same maximal solutions as at the first stage. It allows us to find admissible matrix coefficients b_i and c_i.

Note that at the first step we have distinguished 13 points on the (α, β)-plane:



Red points correspond to the systems from Theorem 3.1. They are representatives of the group action generated by (10) (see also Subsection 2.2 in the main text).

In order to construct isomonodromic Lax pairs, we use a procedure of non-abelianization (see Subsection 2.4 in the main text). It turns out that each of systems P_4^0 – P_4^2 admits an isomonodromic Lax representation. Namely, we have

Theorem 3.2. *Let $A(\zeta, z)$ and $B(\zeta, z)$ be 2×2-matrices depending on the spectral parameter ζ as follows*

$$A(\zeta, z) = A_1 \zeta + A_0(z) + A_{-1}(z)\zeta^{-1}, \qquad B(\zeta, z) = B_1 \zeta + B_0(z). \tag{11}$$

Then each of the P_4 *systems listed in Theorem 3.1 has the zero-curvature representation*

$$A_z - B_\zeta = [B, A],$$

where matrices A_1, $A_0(z)$, $A_{-1}(z)$, B_1, *and* $B_0(z)$ *are given by*

- P_4^0 *system*:

$$A_1 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \qquad A_0 = \begin{pmatrix} z\,\mathbb{I} & -uv - \gamma_1\,\mathbb{I} \\ -\mathbb{I} & -z\,\mathbb{I} + h \end{pmatrix}, \qquad A_{-1} = \frac{1}{2}\begin{pmatrix} uv + \frac{1}{2}\gamma_2\,\mathbb{I} & -uvu - \gamma_2 u \\ v & -vu - \frac{1}{2}\gamma_2\,\mathbb{I} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} v - z\,\mathbb{I} + h & uv + \gamma_1\,\mathbb{I} \\ \mathbb{I} & z\,\mathbb{I} \end{pmatrix};$$

- P_4^1 *system*:

$$A_1 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad A_0 = \begin{pmatrix} z\,\mathbb{I} & -uv - h \\ -\mathbb{I} & -z\,\mathbb{I} \end{pmatrix}, \quad A_{-1} = \frac{1}{2}\begin{pmatrix} uv + \frac{1}{2}\gamma\,\mathbb{I} & -u^2v - u(h + \gamma\,\mathbb{I}) \\ v & -uv - h - \frac{1}{2}\gamma\,\mathbb{I} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -z\,\mathbb{I} & uv + h \\ \mathbb{I} & -v + z\,\mathbb{I} \end{pmatrix};$$

- P_4^2 *system*:

$$A_1 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \qquad A_0 = \begin{pmatrix} z\,\mathbb{I} & -uv - \frac{1}{2}h_1 u - h_2 \\ -\mathbb{I} & -z\,\mathbb{I} \end{pmatrix},$$

$$A_{-1} = \frac{1}{2}\begin{pmatrix} uv + \frac{1}{2}h_1 u + h_2 + \frac{1}{2}\gamma\,\mathbb{I} & \begin{matrix} -u^2v - \frac{1}{2}uh_1u - \frac{1}{2}h_1uv + zh_1u \\ -h_2u - uh_2 - \gamma u - \frac{1}{2}h_1h_2 \end{matrix} \\ v & -uv - \frac{1}{2}h_1u - h_2 - \frac{1}{2}\gamma\,\mathbb{I} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} u - z\,\mathbb{I} & uv + \frac{1}{2}h_1u + h_2 \\ \mathbb{I} & u - v + z\,\mathbb{I} \end{pmatrix}.$$

According to the classical degeneration scheme of the Painlevé equations [Gam10], there exists a limiting transition from the P_4 equation to the P_2 equation. Similarly, the P_4^0 – P_4^2 systems degenerate to the matrix P_2^0 – P_2^2 equations [AS21] (see Subsection 2.5 in the main text).

Theorem 3.3. *One can degenerate matrix Painlevé-4 systems and their Lax pairs to matrix Painlevé-2 systems and their Lax pairs as*

$$\mathrm{P}_4^0 \to \mathrm{P}_2^0, \qquad \mathrm{P}_4^1 \to \mathrm{P}_2^1, \qquad \mathrm{P}_4^2 \to \mathrm{P}_2^2.$$

Note that one can degenerate the matrix P_2 systems to a matrix analog for the P_1 equation (see Remark 2.10 in the main text). These limiting procedures extend for the Lax pairs too.

3.2 Fully non-Abelian analogs

In Approach **2**, developed by V. Retakh and V. Rubtsov, we consider an associative unital division ring R over a field $\mathbb{F}$, $\operatorname{char}\mathbb{F} = 0$. Let $D : R \to R$ be a derivation of R, i.e. an $\mathbb{F}$-linear map that satisfies the Leibniz rule. For any $f \in R$ we set $D(f) = f'$. Below we will often refer to the elements of the ring R as *functions*. Let us fix an element $t \in R$ such that $t' = 1$ (so we assume that the differential equation $f' = 1$ in R has solutions) and for any scalar parameter $\alpha \in \mathbb{F}$ we have $\alpha' = 0$. In their paper [RR10], the authors have constructed the so-called *fully non-commutative analog* for the P_2 equation. It can be written as

$$y'' = 2y^3 + \tfrac{1}{2}t\,y + \tfrac{1}{2}y\,t + \alpha, \qquad y,\ t \in R, \qquad \alpha \in \mathbb{F}. \qquad \mathrm{P}_2^{\mathrm{NC}}$$

It is well-known [JKM04] that in the commutative case solutions of the commutative P_2 equation have solutions in a Hankel determinant form as one can derive the Toda chain for the τ-function of the P_2 equation [KMN$^+$01]. Using an appropriate non-commutative generalization for the Toda equation suggested in [GR92], the authors of the paper [RR10] have constructed the solutions of the $\mathrm{P}_2^{\mathrm{NC}}$ equation in terms of the Hankel quasideterminants. Note also that the $\mathrm{P}_2^{\mathrm{NC}}$ equation has an isomonodromic representation found in [Irf12].

In [RR10], the authors constructed solutions of the *infinite Toda system* in terms of quasideterminants of Hankel matrices over R (see Theorem 2.1 in [RR10]). This system contains two parts, "positive" and "negative", and can be written as

$$(\theta_n' \theta_n^{-1})' = \theta_{n+1}\theta_n^{-1} - \theta_n\theta_{n-1}^{-1}, \qquad n \geq 0, \tag{12}$$

$$(\eta_m^{-1}\eta_m')' = \eta_m^{-1}\eta_{m-1} - \eta_{m+1}^{-1}\eta_m, \qquad m \leq 0, \tag{13}$$

where $\theta_1 = \eta_0^{-1} = \kappa_1$ and $\theta_0 = \eta_{-1}^{-1} = \kappa_{-1}^{-1}$ for some generic initial functions κ_{-1} and κ_1. It turns out that if we impose some conditions on the functions κ_{-1} and κ_1, then one can derive the solutions of the $\mathrm{P}_2^{\mathrm{NC}}$ in terms of the Hankel quasideterminants.

Going further in this direction, we will suggest a fully non-commutative version of the commutative Painlevé IV equation in the symmetric form

$$\begin{cases} f_0' &= f_0 f_1 - f_0 f_2 + \alpha_0, \\ f_1' &= f_1 f_2 - f_0 f_1 + \alpha_1, \\ f_2' &= f_0 f_2 - f_1 f_2 + \alpha_2, \end{cases} \tag{14}$$

where $f_i = f_i(t)$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Thanks to Theorem 3.2 in [BRRS22], the solutions of this equation are expressed in terms of the solutions of the Toda equations. Generalizing this result to the fully non-commutative case, we arrive at the fact that, in contrast to the matrix case, in the fully non-abelian setting, there exists only one analog of the fourth Painlevé equation. It generalizes the quantum fourth Painlevé equation [NGR+08] and the matrix P_4^0 system [BS22c]. This analog can be written as a system of the third order and was derived by using the solutions of the infinite non-commutative one-dimensional Toda system. This analog reads as

$$\begin{cases} f_0' &= f_0 f_1 - f_2 f_0 + \alpha_0, \\ f_1' &= f_1 f_2 - f_0 f_1 + \alpha_1, \\ f_2' &= f_2 f_0 - f_1 f_2 + \alpha_2, \end{cases} \tag{15}$$

where f_i, $t \in R$, $\alpha_i \in \mathbb{F}$, and $\alpha_0 + \alpha_1 + \alpha_2 = 1$, and can be regarded as *a fully non-commutative generalization of the* P_4 *symmetric system*[8]. This system admits the same Bäcklund transformations as in the commutative case (see Table 3 in the main text).

	α_0	α_1	α_2	f_0	f_1	f_2
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	f_0	$f_1 + \alpha_0 f_0^{-1}$	$f_2 - \alpha_0 f_0^{-1}$
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	$f_0 - \alpha_1 f_1^{-1}$	f_1	$f_2 + \alpha_1 f_1^{-1}$
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$f_0 + \alpha_2 f_2^{-1}$	$f_1 - \alpha_2 f_2^{-1}$	f_2
π	α_1	α_2	α_0	f_1	f_2	f_0

Table 2: Bäcklund transformations of (15)

In order to construct solutions of (15) in terms of the non-commutative Toda equations, we use non-commutative analogs of the translation operators. Therefore, the existence of Bäcklund

[8] In the commutative case, the system (15) defines 3-periodic solutions of the dressing chain [VS93].

transformations, compositions of which form an extended affine Weyl group $\tilde{W}$ of type $A_2^{(1)}$, is significant for determining solutions of (15). Unlike the $\mathrm{P}_2^{\mathrm{NC}}$ equation, the rhs of system (15) is not written in the anticommutator form. It means that a non-commutative analog of the P_4 equation cannot be obtained by using the Weyl ordering.

One can verify that the system (15) possesses quasideterminant solutions for generic initial functions θ_0, θ_1 and η_0, η_{-1} (see Proposition 3.1 in [BRRS22]). Then we can generalize this result to the arbitrary n and m, using translation operators.

Let the T_1-operator $T_1 = \pi s_2 s_1$ be a non-commutative translation operator associated with the affine Weyl group $\tilde{W}$ whose generators are defined in Table 2. Applying it to (15) and to conditions from Proposition 3.1 in [BRRS22], we arrive at the non-commutative $\mathrm{P}_4[f_{i,n};n]$ symmetric form

$$\begin{cases} f'_{0,n} &= f_{0,n}f_{1,n} - f_{2,n}f_{0,n} + (\alpha_0 + n), \\ f'_{1,n} &= f_{1,n}f_{2,n} - f_{0,n}f_{1,n} + (\alpha_1 - n), \\ f'_{2,n} &= f_{2,n}f_{0,n} - f_{1,n}f_{2,n} + \alpha_2, \end{cases} \tag{16}$$

where $T_1^n(f_i) = f_{i,n} \in R$. Namely, the main result of our classification is based on the following

Theorem 3.4. *Let the functions θ_n, $n \geq 0$ and η_m, $m \leq 0$ satisfy the non-commutative Toda equations (12) – (13) and the following equations*

$$\theta''_{n+1} + t\,\theta'_{n+1} + 2\theta_{n+1}\,\theta_n^{-1}\,\theta_{n+1} + (\alpha_0 - \alpha_1 + 2n)\,\theta_{n+1} = 0,$$

$$\eta''_{m-1} - \eta'_{m-1}\,t + 2\eta_{m-1}\,\eta_m^{-1}\,\eta_{m-1} + (\alpha_0 - \alpha_1 + 2(m-1))\eta_{m-1} = 0.$$

Then

(a) *the functions $f_{0,n}$, $f_{1,n}$, $f_{2,n}$ that satisfy the relations*

$$f_{2,n} = \theta'_{n+1}\,\theta_{n+1}^{-1} + t, \qquad \tfrac{1}{2}f_{1,n}\,f_{2,n} + \tfrac{1}{2}f_{2,n}\,f_{1,n} = \theta_{n+1}\,\theta_n^{-1} - (\alpha_1 - n),$$

$$f_{0,n} = -f_{1,n} - f_{2,n} + t,$$

$$f'_{1,n} = f_{1,n}^2 + f_{1,n}f_{2,n} + f_{2,n}f_{1,n} - t f_{1,n} + (\alpha_1 - n)$$

are solutions of the $\mathrm{P}_4[f_{i,n};n]$ *symmetric form* (16);

(b) *the functions $f_{0,m-1}$, $f_{1,m-1}$, $f_{2,m-1}$ that satisfy the relations*

$$f_{2,m-1} = -\eta_{m-1}^{-1}\,\eta'_{m-1} + t,$$

$$\tfrac{1}{2}f_{0,m-1}\,f_{2,m-1} + \tfrac{1}{2}f_{2,m-1}\,f_{0,m-1} = \eta_m^{-1}\,\eta_{m-1} + (\alpha_0 + m - 1),$$

$$f_{1,m-1} = -f_{0,m-1} - f_{2,m-1} + t,$$

$$f'_{0,m-1} = -f^2_{0,m-1} - f_{0,m-1}f_{2,m-1} - f_{2,m-1}f_{0,m-1} + f_{0,m-1}t + (\alpha_0 + m - 1)$$

are solutions of the $\mathrm{P}_4[f_{i,m-1}; m-1]$ *symmetric form* (16).

Note that the P_4^0 system with scalar parameters is equivalent to (15) with central variable t. To obtain a system with the arbitrary matrix h, it is necessary to reduce the symmetric form to a second-order system and make the shift $t \mapsto t + h$, $h \in R$. After these operations, the element t can be set to be commutative (a detailed discussion is given in Section 4 in the main text). Up to the author knowledge, the remaining two systems, P_4^1 and P_4^2, do not have Bäcklund transformations that define an affine Weyl group of type $A_2^{(1)}$[9]. Therefore, they cannot be solved by the non-commutative Toda equations. The same remark holds for non-commutative systems of the P_4 type, obtained in the paper [CM14] by using the Riemann-Hilbert problem.

The system (16) has an isomonodromic Lax representation, that is equivalent to the Noumi-Yamada pair for the commutative P_4 symmetric form (14) [NY00]. This pair was constructed by using the method suggested in the paper [BS22c] of the non-abelianization of well-known commutative pairs.

Theorem 3.5. *Let* $\mathcal{A}_n(\lambda, t)$ *and* $\mathcal{B}_n(\lambda, t)$ *be* 3×3*-matrices depending on the spectral parameter* λ *as follows*

$$\mathcal{A}_n(\lambda, t) = A_{0,n}(t) + A_{-1,n}(t)\lambda^{-1}, \qquad \mathcal{B}_n(\lambda, t) = B_1\lambda + B_{0,n}(t).$$

Then, for any $n \in \mathbb{Z}$*, system* (16) *has the zero-curvature representation*

$$\partial_t \mathcal{A}_n - \partial_\lambda \mathcal{B}_n = [\mathcal{B}_n, \mathcal{A}_n]$$

where matrices $A_{0,n}$*,* $A_{-1,n}$*,* B_1*, and* $B_{0,n}$ *are given by*

$$A_{0,n} = \begin{pmatrix} 0 & 1 & f_{0,n} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{-1,n} = \begin{pmatrix} \beta_0 & 0 & 0 \\ f_{1,n} & \beta_1 & 0 \\ 1 & f_{2,n} & \beta_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_{0,n} = \begin{pmatrix} -f_{2,n} & 0 & 0 \\ 1 & -f_{0,n} & 0 \\ 0 & 1 & -f_{1,n} \end{pmatrix}.$$

Here scalar parameters β_0*,* β_1*,* β_2 *are related to the* α*'s parameters as*

$$\alpha_0 = 1 + \beta_2 - \beta_0 - n, \qquad \alpha_1 = \beta_0 - \beta_1 + n, \qquad \alpha_2 = \beta_1 - \beta_2.$$

[9] Other Bäcklund transformations can be found in [Adl20].

Note that in the commutative case the Noumi-Yamada pair reduces to the Jimbo-Miwa pair [JM81] (see [JKT07]) with the spectral dependence of the form (11). We prove the same fact in the non-commutative case (see Proposition 3.9 in the main text).

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(1) I. Bobrova and V. Sokolov. On matrix Painlevé-4 equations. Nonlinearity, 35(12):6528, 2022;

(2) I. Bobrova, V. Retakh, V. Rubtsov, and G. Sharygin. A fully non-commutative analog of the Painlevé IV equation and a structure of its solutions. Journal of Physics A: Mathematical and Theoretical, 55(47):475205, 2022.

References

[Adl20] V. E. Adler. Painlevé type reductions for the non-Abelian Volterra lattices. *Journal of Physics A: Mathematical and Theoretical*, 54(3):035204, 2020. arXiv:2010.09021. ← 6, 9, 14

[AK22] V. E. Adler and M. P. Kolesnikov. Non-Abelian Toda lattice and analogs of Painlevé III equation. *J. Math. Phys.*, 63:103504, 2022. arXiv:2203.09977. ← 6

[ARS80] M. J. Ablowitz, A. Ramani, and H. Segur. A connection between nonlinear evolution equations and ordinary differential equations of P-type. II. *Journal of Mathematical Physics*, 21(5):1006–1015, 1980. ← 3

[AS21] V. E. Adler and V. V. Sokolov. On matrix Painlevé II equations. *Theoret. and Math. Phys.*, 207(2):188–201, 2021. arXiv:2012.05639. ← 4, 7, 9, 10

[BCR18] M. Bertola, M. Cafasso, and V. Rubtsov. Noncommutative Painlevé equations and systems of Calogero type. *Communications in Mathematical Physics*, 363(2):503–530, 2018. arXiv:1710.00736. ← 4

[BRRS22] I. Bobrova, V. Retakh, V. Rubtsov, and G. Sharygin. A fully noncommutative analog of the Painlevé IV equation and a structure of its solutions. *Journal of Physics A: Mathematical and Theoretical*, 55(47):475205, 2022. arXiv:2205.05107. ← 12, 13

[BS98] S. P. Balandin and V. V. Sokolov. On the Painlevé test for non-Abelian equations. *Physics letters A*, 246(3-4):267–272, 1998. ← 4, 5, 6, 7

[BS22a] I. Bobrova and V. Sokolov. Classification of Hamiltonian non-abelian Painlevé type systems. *Journal of Nonlinear Mathematical Physics*, pages 1–16, 2022. arXiv:2209.00258. ← 6

[BS22b] I. Bobrova and V. Sokolov. Non-abelian Painlevé systems with generalized Okamoto integral. *arXiv preprint arXiv:2206.10580*, 2022. ← 6

[BS22c] I. A. Bobrova and V. V. Sokolov. On matrix Painlevé-4 equations. *Nonlinearity*, 35(12):6528, nov 2022. arXiv:2107.11680, arXiv:2110.12159. ← 12, 14

[CM08] R. Conte and M. Musette. *The Painlevé Handbook*. Springer, 2008. ← 4

[CM14] M. Cafasso and D. Manuel. Non-commutative Painlevé equations and Hermite-type matrix orthogonal polynomials. *Communications in Mathematical Physics*, 326(2):559–583, 2014. arXiv:1301.2116. ← 4, 6, 14

[CM+18] M. Cafasso, D. Manuel, et al. The Toda and Painlevé systems associated with semiclassical matrix-valued orthogonal polynomials of Laguerre type. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 14:076, 2018. arXiv:1801.08740. ← 4

[DZ04] B. Dubrovin and Y. Zhang. Virasoro symmetries of the extended Toda hierarchy. *Comm. Math. Phys.*, 250:161–193, 2004. ← 3

[DZ05] B. Dubrovin and Y. Zhang. Normal forms of hierarchies of integrable PDEs. *Frobenius manifolds and Gromov-Witten invariants, a new*, 2005. ← 3

[FIN+06] A. S. Fokas, A. R. Its, V. Yu. Novokshenov, A. A. Kapaev, A. I. Kapaev, and V. I. Novokshenov. *Painlevé transcendents: the Riemann-Hilbert approach*. American Mathematical Soc., 2006. ← 4

[Fuc07] R. Fuchs. Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen. *Mathematische Annalen*, 63(3):301–321, 1907. ← 3

[Gam10] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes. *Acta Mathematica*, 33(1):1–55, 1910. ← 3, 10

[Gar12] R. Garnier. Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. In *Annales scientifiques de l'École normale supérieure*, volume 29, pages 1–126, 1912. ← 3

[GR92] I. Gelfand and V. Retakh. A theory of noncommutative determinants and characteristic functions of graphs. *Functional Analysis and Its Applications*, 26(4):231–246, 1992. ← 4, 11

[GR19] I. Yu. Gaiur and V. N. Rubtsov. Dualities for rational multi-particle Painlevé systems: Spectral versus Ruijsenaars. *arXiv preprint arXiv:1912.12588*, 2019. ← 6

[Irf12] M. Irfan. Lax pair representation and Darboux transformation of noncommutative Painlevé's second equation. *Journal of Geometry and Physics*, 62(7):1575–1582, 2012. arXiv:1201.0900. ← 11

[JKM04] N. Joshi, K. Kajiwara, and M. Mazzocco. Generating function associated with the determinant formula for the solutions of the Painlevé II equation. In Loday-Richaud Michèle, editor, *Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes (II)*, number 297 in Astérisque. Société mathématique de France, 2004. arXiv:nlin/0406035. ← 11

[JKM06] N. Joshi, K. Kajiwara, and M. Mazzocco. Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation. *Funkcialaj Ekvacioj*, 49(3):451–468, 2006. arXiv:nlin/0512041. ← 5

[JKT07] N. Joshi, A. V. Kitaev, and P. A. Treharne. On the linearization of the Painlevé III–VI equations and reductions of the three-wave resonant system. *Journal of Mathematical Physics*, 48(10):103512, 2007. arXiv:0706.1750v3. ← 4, 15

[JM81] M. Jimbo and T. Miwa. Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II. *Physica D: Nonlinear Phenomena*, 2(3):407–448, 1981. ← 3, 15

[Kaw15] H. Kawakami. Matrix Painlevé systems. *Journal of Mathematical Physics*, 56(3):033503, 2015. ← 4, 6, 8

[KMN+01] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada. Determinant formulas for the Toda and discrete Toda equations. *Funkcialaj Ekvacioj*, 44:291–307, 2001. arXiv:solv-int/9908007. ← 11

[Kow89] S. Kowalevski. Sur le problème de la rotation d'un corps solide autour d'un point fixe. *Acta Math.*, 12:177–232, 1889. ← 3

[Mal74] B. Malgrange. Intégrales asymptotiques et monodromie. In *Annales scientifiques de l'École normale supérieure*, volume 7, pages 405–430, 1974. ← 3

[NGR+08] H. Nagoya, B. Grammaticos, A. Ramani, et al. Quantum Painlevé equations: from Continuous to discrete. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 4:051, 2008. ← 4, 6, 12

[NY00] M. Noumi and Y. Yamada. Affine Weyl group symmetries in Painlevé type equations. *Citeseer*, 2000. ← 14

[Oka80] K. Okamoto. Polynomial Hamiltonians associated with Painlevé equations, I. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 56(6):264–268, 1980. ← 3

[Oka81] K. Okamoto. On the τ-function of the Painlevé equations. *Physica D: Nonlinear Phenomena*, 2(3):525–535, 1981. ← 3

[Oka86] K. Okamoto. Studies on the Painlevé Equations. III. Second and Fourth Painlevé Equations PII and PIV. *Mathematische Annalen*, 275:221–255, 1986. ← 3

[Oka87a] K. Okamoto. Studies on the Painlevé Equations. I. Sixth Painlevé equation PIV. *Ann. Mat. Pura Appl.*, 146(4):337—381, 1987. ← 3

[Oka87b] K. Okamoto. Studies on the Painlevé equations. II. Fifth Painlevé equation PV. *Japanese journal of mathematics. New series*, 13(1):47–76, 1987. ← 3

[Oka87c] K. Okamoto. Studies on the Painlevé equations. IV. Third Painlevé equation PIII. *Funkcial. Ekvac*, 30(2-3):305–332, 1987. ← 3

[OS98] P. J. Olver and V. V. Sokolov. Integrable evolution equations on associative algebras. *Communications in Mathematical Physics*, 193(2):245–268, 1998. ← 6

[Pai00] P. Painlevé. Mémoire sur les équations différentielles dont l'intégrale générale est uniforme. *Bulletin de la Société Mathématique de France*, 28:201–261, 1900. ← 3

[Pai02] P. Painlevé. Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. *Acta mathematica*, 25:1–85, 1902. ← 3

[RR10] V. S. Retakh and V. N. Rubtsov. Noncommutative Toda Chains, Hankel Quasideterminants and Painlevé II Equation. *Journal of Physics. A, Mathematical and Theoretical*, 43(50):505204, 2010. arXiv:1007.4168. ← 4, 5, 11

[Sch12] L. Schlesinger. Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten. *Journal für die reine und angewandte Mathematik*, pages 96–145, 1912. ← 3

[Tak01] K. Takasaki. Painleve-Calogero correspondence revisited. *J. Math. Phys.*, 42:1443–1473, 2001. ← 4

[VDPS09] M. Van Der Put and M.-H. Saito. Moduli spaces for linear differential equations and the Painlevé equations. In *Annales de l'Institut Fourier*, volume 59, pages 2611–2667, 2009. ← 3

[VS93] A. P. Veselov and A. B. Shabat. Dressing chains and the spectral theory of the Schrödinger operator. *Functional Analysis and Its Applications*, 27(2):81–96, 1993. ← 8, 12

[ZS74] V. E. Zakharov and A. B. Shabat. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. *Functional Analysis and Its Applications*, 8(3):43–53, 1974. ← 3