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Structure of derived categories and geometry of Fano varieties in Grassmannians

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Abstract

This thesis is devoted to the study of the bounded derived categories of coherent sheaves for two Fano varieties in Grassmannians. More precisely, we construct full exceptional collections of vector bundles in the derived categories of these two varieties.

The first variety $\mathrm{IGr}(3,8)$ is the Grassmannian of 3-dimensional isotropic subspaces in an 8-dimensional symplectic vector space. This is the rational homogeneous variety $\mathrm{Sp}(8)/P_3$, where $P_3 \subset \mathrm{Sp}(8)$ is a maximal parabolic subgroup corresponding to the third simple root (we use the Bourbaki indexing of simple roots). The existence of a full exceptional collection on this variety supports the conjecture that says that for a semisimple algebraic group G over an algebraically closed field of characteristic 0 and a parabolic subgroup $P \subset G$ there exists a full exceptional collection of vector bundles on G/P.

The second variety is the so called Cayley Grassmannian $\mathbf{CG}$, the subvariety of the Grassmannian $\mathrm{Gr}(3,7)$ parametrizing 3-dimensional subspaces that are annihilated by a general 4-form. The Cayley Grassmannian $\mathbf{CG}$ is a spherical variety with respect to an action of the exceptional simple Lie group $\mathbb{G}_2$. The geometry and cohomology of $\mathbf{CG}$ were studied in [13] and [2]. In particular, in [2] the semisimplicity of the small quantum cohomology ring of $\mathbf{CG}$ was proved. Thus, the result confirms in this case Dubrovin's conjecture predicting that the semisimplicity of the quantum cohomology ring implies the existence of a full exceptional collection.

The exceptional collections that we construct are *Lefschetz collections*, [6, 8, 7]. A Lefschetz exceptional collection with respect to a line bundle $\mathcal{L}$ is just an exceptional collection which consists of several blocks, each of them is a sub-block of the previous one twisted by $\mathcal{L}$. If all blocks are the same, the collection is called *rectangular*.

In Introduction we give a short overview of the history of investigation of derived categories and state the main theorems.

In Chapter 1 we discuss background material on semiorthogonal decompositions and equivariant vector bundles on Grassmannians, and also recall some basic facts about the algebraic group $\mathbb{G}_2$.

In Chapter 2 we prove Theorem 1.1 for the isotropic Grassmannian $\mathrm{IGr}(3,8)$. In Section 2.1 we prove vanishing lemmas that are essential for the proof of exceptionality and fullness of the constructed collection. In Section 2.2 we collect some important exact sequences and construct the bicomplex that is essential for the proof of Theorem 1.1. Also in this section we prove some important properties of this bicomplex. In Section 2.3 we construct vector bundles F and T and prove exceptionality of the collection of Theorem 1.1. In Section 2.4 we give a proof of fullness of the constructed collection. Finally, in Section 2.5 we provide a couple of applications of our results: compute the residual category of $\mathrm{IGr}(3,8)$ as defined in [11], and construct a pair of (fractional) Calabi–Yau categories related to a half-anticanonical section and anticanonical double covering of $\mathrm{IGr}(3,8)$.

In Chapter 3 we prove Theorem 1.2 for the Cayley Grassmannian. In Section 3.1 we present general constructions with quadric bundles that are essential for the proof of Theorem 1.2. In Section 3.2 we collect some useful results concerning the $\mathbb{G}_2$-action on the Cayley Grassmannian. In Section 3.3 using

the general constructions from Section 3.1 we describe some special quadric bundles on **CG** and prove several self-dualities that are essential for the proof of fullness of the collection. In Section 3.4 we give a proof of fullness of the constructed collection. In Section 3.5 we collect all necessary cohomology computations for **CG** including exceptionality of the the constructed collection, and describe the residual category for the collection. In Section 3.6 we present several geometric constructions for the Cayley Grassmannian: we show that **CG** is isomorphic to the Hilbert scheme of conics on $\mathbb{G}_2^{\mathrm{ad}}$ and describe the Hilbert scheme of lines on **CG**.

1 Notation and conventions

Let $\Bbbk$ be an algebraically closed field of characteristic zero. For any $\Bbbk$-vector space W we denote by $\wedge$ the wedge product operation for skew forms and polyvectors, and by $\lrcorner$ the convolution operation

$$\Lambda^p W \otimes \Lambda^q W^\vee \xrightarrow{\lrcorner} \Lambda^{p-q} W \qquad (\text{if } p \geq q),$$

induced by the natural pairing $W \otimes W^\vee \to \Bbbk$.

If $p = n = \dim(W)$ and $0 \neq \omega \in \Lambda^n W$, the convolution with ω gives an isomorphism

$$\Lambda^q W^\vee \simeq \Lambda^{n-q} W, \quad \xi \mapsto \xi^\vee := \omega \lrcorner \xi.$$

This isomorphism is canonical up to rescaling (since ω is unique up to rescaling). We say that $\xi^\vee$ is the **dual** of ξ.

We say that a q-form $\xi \in \Lambda^q W^\vee$ **annihilates** a k-dimensional subspace $U \subset W$, if $k \leq q$ and $\xi \lrcorner \Lambda^k U = 0$. Analogously, we say that $U \subset W$ is **isotropic** for ξ if $q \leq k$ and $\Lambda^k U \lrcorner \xi = 0$, i.e. $\xi|_U = 0$.

We denote by $\mathrm{Gr}(k, W)$ the Grassmannian of k-dimensional vector subspaces in W. The tautological vector subbundle of rank k on $\mathrm{Gr}(k, W)$ is denoted by $\mathcal{U}_k \subset W \otimes \mathcal{O}_{\mathrm{Gr}(k,W)}$. The quotient bundle is denoted simply by $\mathcal{Q}_k$, and for its dual we use the notation $\mathcal{U}_k^\perp := \mathcal{Q}_k^\vee$. We recall that the line bundle $\det \mathcal{U}_k^\vee \simeq \det \mathcal{Q}_k$ is the ample generator of $\mathrm{Pic}(\mathrm{Gr}(k, W))$; we will denote it by $\mathcal{O}(H_k)$.

Since we will mostly work with $\mathrm{Gr}(3, W)$, we abbreviate its tautological bundle by $\mathcal{U}$, unless this leads to a confusion. The line bundle $\mathcal{O}(H_3)$ will be denoted by $\mathcal{O}(1)$.

For both exceptional collections we will need the following vector bundle on $\mathrm{Gr}(3, W)$

$$\Sigma^{2,1}\mathcal{U}^\vee := (\mathcal{U}^\vee \otimes \Lambda^2\mathcal{U}^\vee)/\Lambda^3\mathcal{U}^\vee. \tag{1}$$

The canonical line bundle of a variety X will be denoted by ω_X.

Chapter 1

In Section 1.1 we recall some well-known facts about semiorthogonal decompositions and exceptional collections. In Section 1.2 we collect some preliminary results from the theory of equivariant vector bundles on Grassmannians: in Subsection 1.2.1 we formulate the Borel–Bott–Weil theorem for the classical Grassmannian and in Subsection 1.2.2 we formulate the Borel–Bott–Weil theorem for the isotropic Grassmannian. In Section 1.3 we recall some basic facts about the simple algebraic group $\mathbb{G}_2$.

Chapter 2

Denote by $\mathfrak{F}$ and $\mathfrak{F}'$ the following collections of vector bundles on $\mathrm{IGr}(3,8)$:

$$\mathfrak{F} := (\qquad\qquad \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee, \Lambda^2\mathcal{U}^\vee, \Sigma^{2,1}\mathcal{U}^\vee), \tag{2}$$

$$\mathfrak{F}' := (\Sigma^{2,1}\mathcal{U}^\vee(-1), \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee, \Lambda^2\mathcal{U}^\vee \qquad\qquad). \tag{3}$$

We will denote by $\mathfrak{F}(i)$ and $\mathfrak{F}'(i)$ the collections consisting of the corresponding five vector bundles twisted by $\mathcal{O}(i)$, and in the same way the subcategories of $\mathrm{D}^b(\mathrm{IGr}(3,8))$ generated by these. We denote by $\mathbb{L}$ and $\mathbb{R}$ the left and right mutation functors. The first main result of this thesis is the following theorem.

Theorem 1.1. *The objects*

$$T := (\mathbb{L}_{\mathfrak{F}}(\Sigma^{3,1}\mathcal{U}^\vee))[-3] \qquad \textit{and} \qquad F := \mathbb{R}_{\Sigma^{2,1}\mathcal{U}^\vee(-1)}(T)$$

are equivariant vector bundles on $\mathrm{IGr}(3,8)$*.*

The collections of 32 *vector bundles on* $\mathrm{IGr}(3,8)$

$$\begin{array}{llllllllll} F, & \mathfrak{F}, & F(1), & \mathfrak{F}(1), & \mathfrak{F}(2), & \mathfrak{F}(3), & \mathfrak{F}(4), & \mathfrak{F}(5), & & \textit{and} \\ T, & \mathfrak{F}', & T(1), & \mathfrak{F}'(1), & \mathfrak{F}'(2), & \mathfrak{F}'(3), & \mathfrak{F}'(4), & \mathfrak{F}'(5) & & \end{array}$$

are full Lefschetz collections with respect to the line bundle $\mathcal{O}(1)$*.*

The collections of 32 *vector bundles on* $\mathrm{IGr}(3,8)$

$$\begin{array}{llllllllll} F, & \mathfrak{F}, & \mathfrak{F}(1), & \mathfrak{F}(2), & F(3), & \mathfrak{F}(3), & \mathfrak{F}(4), & \mathfrak{F}(5), & & \textit{and} \\ T, & \mathfrak{F}', & \mathfrak{F}'(1), & \mathfrak{F}'(2), & T(3), & \mathfrak{F}'(3), & \mathfrak{F}'(4), & \mathfrak{F}'(5) & & \end{array}$$

are full rectangular Lefschetz collections with respect to the line bundle $\mathcal{O}(3)$*.*

A significant part of the proof of Theorem 1.1 is based on the study of a certain interesting bicomplex

$$\begin{array}{cccccccccc} & 0 \to & \Sigma^{3,2}\mathcal{U}^\vee(-3) \to & V_8 \otimes \Sigma^{2,1}\mathcal{U}^\vee(-2) \to & \Lambda^2 V_8 \otimes \mathcal{U}^\vee(-1) \to & \Lambda^4 V_8 \otimes \mathcal{O} \to & \Lambda^2 V_8 \otimes \Lambda^2\mathcal{U}^\vee \to & V_8 \otimes \Sigma^{2,1}\mathcal{U}^\vee \to & \Sigma^{3,1}\mathcal{U}^\vee \to & 0 \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ 0 \to & \Sigma^{3,3}\mathcal{U}^\vee(-4) \to & V_8 \otimes \Sigma^{2,2}\mathcal{U}^\vee(-3) \to & \Lambda^2 V_8 \otimes \Lambda^2\mathcal{U}^\vee(-2) \to & \Lambda^3 V_8 \otimes \mathcal{O}(-1) \to & \Lambda^2 V_8 \otimes \mathcal{O} \to & V_8 \otimes \mathcal{U}^\vee \to & S^2\mathcal{U}^\vee \to & 0, & \end{array}$$

of vector bundles on $\mathrm{IGr}(3,8)$, where V_8 is the tautological 8-dimensional representation of $\mathrm{Sp}(8)$. This bicomplex is $\mathrm{Sp}(8)$-equivariant, its lines are exact and are obtained as the restrictions of the so-called staircase complexes (see [3]) from $\mathrm{Gr}(3,8)$. The vector bundle T is identified with the cohomology of the truncation of this bicomplex, and using the bicomplex we prove an isomorphism

$$\mathbb{L}_{\mathfrak{F}'(1),\mathfrak{F}'(2)}(T(3)) = T(1)[4],$$

which is crucial for the proof of completeness of the above exceptional collections.

To prove the fullness of the exceptional collections in Theorem 1.1 we first prove that some special objects lie in the subcategory $\mathcal{D}$ of $\mathrm{D}^b(\mathrm{IGr}(3,8))$ generated by each of these collections. After that we consider the isotropic flag variety $\mathrm{IFl}(2,3;8)$ with its two projections

$$\begin{array}{ccc} & \mathrm{IFl}(2,3;8) & \\ \swarrow & & \searrow \\ \mathrm{IGr}(2,8) & & \mathrm{IGr}(3,8) \end{array}$$

The first arrow is a $\mathbb{P}^3$-fibration. Using a certain variant of the Lefschetz exceptional collection on $\mathrm{IGr}(2,8)$ from [6] and Orlov's projective bundle formula we construct a very special full exceptional collection on $\mathrm{IFl}(2,3;8)$. The main property of this exceptional collection is that the pushforwards along the second arrow (which is a $\mathbb{P}^2$-fibration) of almost all objects constituting it are contained in the subcategory $\mathcal{D}$, and for the few objects that do not enjoy this property, the pushforwards are contained in the subcategory $\mathfrak{F}(6) \subset \mathrm{D}^b(\mathrm{IGr}(3,8))$. It follows from this that every object of $\mathrm{D}^b(\mathrm{IGr}(3,8))$ contained in the orthogonal ${}^{\perp}\mathcal{D}$ to the subcategory $\mathcal{D}$, belongs to $\mathfrak{F}(6)$. The trivial observation

$${}^{\perp}\mathfrak{F} \cap \mathfrak{F}(6) = 0$$

(that follows immediately from the Serre duality on $\mathrm{IGr}(3,8)$) then shows that ${}^{\perp}\mathcal{D} = 0$, and completes the proof of the fullness of the collections.

Chapter 3

There are several description of the Cayley Grassmannian **CG**.

The first description explains the name. We consider the complexified nonassociative 8-dimensional Cayley algebra $\mathbb{O}$. The Cayley Grassmannian **CG** can be defined as the set of 4-dimensional subalgebras of $\mathbb{O}$; it is a closed subvariety in the Grassmannian $\mathrm{Gr}(4,\mathbb{O}) \simeq \mathrm{Gr}(4,8)$ of 4-dimensional vector subspaces in a 8-dimensional vector space $\mathbb{O}$. All these subalgebras contain the unit element e of $\mathbb{O}$, so we can instead define **CG** as a closed subvariety in $\mathrm{Gr}(3,\mathbb{O}/(\mathbb{C}\cdot e)) \simeq \mathrm{Gr}(3,7)$, that parametrizes the imaginary parts of the four-dimensional subalgebras of $\mathbb{O}$.

The second description makes sense over any algebraically closed field $\Bbbk$ of characteristic 0. We consider the Grassmannian $\mathrm{Gr}(3,V_7)$ of 3-dimensional vector subspaces in a 7-dimensional vector space V_7. Let us fix a general global section of $\mathcal{U}_4^{\perp}(1)$, that is a general skew-symmetric 4-form

$$\lambda \in \Lambda^4 V_7^{\vee}.$$

By definition, the Cayley Grassmannian **CG** is the zero locus of $\lambda \in \mathrm{H}^0(\mathrm{Gr}(3,V_7),\mathcal{U}_4^{\perp}(1))$. Explicitly, **CG** parametrizes 3-dimensional vector subspaces of V_7 annihilated by λ, i.e., $U \subset V_7$ such that $\lambda(u_1,u_2,u_3,-) = 0$ for any $u_1,u_2,u_3 \in U$. From this description we immediately deduce that the Cayley Grassmannian is a smooth Fano eightfold of index 4.

The equivalence of these two descriptions comes from the fact that the stabilizer of a general 4-form on V_7 is isomorphic to the algebraic group $\mathbb{G}_2$, that is the automorphism group of the octonions, see [13] for more details.

Also the Cayley Grassmannian can be described as the Hilbert scheme of conics on the adjoint homogeneous variety $\mathbb{G}_2^{\mathrm{ad}}$.

To describe a full Lefschetz collection on **CG**, we need to make some preparations. The exceptional collection on **CG** that we construct is a Lefschetz collection: it consists of four blocks with respect to the Plücker line bundle $\mathcal{O}(1)$ of $\mathrm{Gr}(3, V_7)$ restricted to **CG**. The common part of these four blocks (the rectangular part of the Lefschetz collection) consists of three vector bundles $(\mathcal{O}, \mathcal{U}^\vee, \Lambda^2\mathcal{U}^\vee)$.

To describe the nonrectangular part of the exceptional collection on **CG** we need to define an additional vector bundle. Note that on **CG** we have an embedding of vector bundles

$$i_\lambda \colon \Lambda^2\mathcal{U} \hookrightarrow \Lambda^2\mathcal{U}_4^\perp$$

given by λ. So on **CG** we can define the quotient bundle $\Lambda^2\mathcal{U}_4^\perp/\Lambda^2\mathcal{U}$. For the exceptional collection we will need its dual

$$\mathcal{R} := (\Lambda^2\mathcal{U}_4^\perp/\Lambda^2\mathcal{U})^\vee.$$

The second main result of the thesis is the following theorem.

Theorem 1.2. *The collection of* 15 *vector bundles on* **CG**

$$\{\underbrace{\mathcal{O}, \mathcal{U}^\vee, \Lambda^2\mathcal{U}^\vee, \mathcal{R}, \Sigma^{2,1}\mathcal{U}^\vee;}_{block\ 1}\ \underbrace{\mathcal{O}(1), \mathcal{U}^\vee(1), \Lambda^2\mathcal{U}^\vee(1), \mathcal{R}(1);}_{block\ 2}\underbrace{\mathcal{O}(2), \mathcal{U}^\vee(2), \Lambda^2\mathcal{U}^\vee(2);}_{block\ 3}\underbrace{\mathcal{O}(3), \mathcal{U}^\vee(3), \Lambda^2\mathcal{U}^\vee(3)}_{block\ 4}\} \quad (4)$$

is a full Lefschetz collection with respect to $\mathcal{O}(1)$.

Let us sketch the idea of the proof of Theorem 1.2. We cover **CG** with the family of subvarieties $\mathbf{CG}_f \overset{i_f}{\hookrightarrow} \mathbf{CG}$ defined as zero loci of sufficiently general global sections $f \in \mathrm{H}^0(\mathbf{CG}, \mathcal{U}^\vee)$. It turns out that $\mathbf{CG}_f$ is isomorphic to the isotropic Grassmannian $\mathrm{IGr}(3, 6)$ and $\mathrm{D}^b(\mathbf{CG}_f)$ posseses a full exceptional collection, so using standard arguments from [6] we reduce the problem to the checking that some objects lie in the subcategory $\mathcal{A} \subset \mathrm{D}^b(\mathbf{CG})$ generated by the collection (4): it is enough to show that $S^2\mathcal{U}^\vee(m) \in \mathcal{A}$ for $m = 0, 1, 2$ and that $\Sigma^{2,1}\mathcal{U}^\vee, \Sigma^{2,1}\mathcal{U}^\vee(1) \in \mathcal{A}$. This check is the most interesting part of the proof, so let us describe it more precisely.

First, we present two general constructions with quadric bundles. Roughly speaking, the first construction shows that we can glue two quadric bundles with isomorphic cokernel sheaves into a quadric bundle that determines self-dual isomorphism; and the second one allows to construct a new quadric bundle from a quadric bundle with the cokernel sheaf supported on a Cartier divisor. Using these contructions we obtain several interesting $\mathbb{G}_2$-equivariant quadric bundles on **CG**. Using again the first contruction we glue the obtained quadric bundles into the following self-dual vector bundles on **CG**

$$\mathcal{E}_{10}(1) \simeq \mathcal{E}_{10}^\vee \quad \text{and} \quad \mathcal{E}_{16}(-1) \simeq \mathcal{E}_{16}^\vee, \qquad (5)$$

where $\mathcal{E}_{10}$ is an extension of $\mathcal{U}_4^\perp$ by $S^2\mathcal{U}$ and $\mathcal{E}_{16}$ is an extension of $\Lambda^2\mathcal{U}_3^\vee \oplus \mathcal{O}(1)$ by $\mathcal{U}_3^\perp \otimes \Lambda^2\mathcal{U}_3^\vee$. Using (5) and some standard exact sequences we prove that $S^2\mathcal{U}^\vee(m) \in \mathcal{A}$ for $m = 0, 1, 2$ and that $\Sigma^{2,1}\mathcal{U}^\vee, \Sigma^{2,1}\mathcal{U}^\vee(1) \in \mathcal{A}$.

Approbation of the results of the dissertation research

The results of this PhD thesis were presented at the following seminars and conferences:

(i) Talk "On the derived category of IGr$(3, 8)$" (Производная категория IGr$(3, 8)$), School and research conference "Algebra and Geometry"(Алгебра и геометрия), (Yaroslavl, Russia), July 2018

(ii) Talk "On the Derived Category of the Cayley Grassmannian" (Производная категория грассманиана Кэли), International Hybrid Conference "Geometry and Homological Mirror Symmetry" (Геометрия и гомологическая зеркальная симметрия), (Sochi, Russia), December 2021

(iii) Talk "On the Derived Category of the Cayley Grassmannian" (Производная категория грассманиана Кэли), International Japanese-Russian conference "Categorical and Analytic Invariants in Algebraic Geometry VIII" (Категорные и аналитические инварианты в алгебраической геометрии VIII), (Moscow, Russia), December 2021

(iv) Talk "On the Derived Category of the Cayley Grassmannian" (Производная категория грассманиана Кэли), conference "Geometry, Algebra and Representation theory" (Геометрия, алгебра и теория представлений), (Dolgoprudny, Russia), July 2022

Publications

The main results of the thesis are presented in 2 papers:

(i) "On the derived category of IGr$(3, 8)$; Mat. Sb. 211 (2020), no. 7, 24–59.

(ii) “On the Derived Category of the Cayley Grassmannian”; Math. Notes, 113:1 (2023), 150–154

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