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Introduction

In the famous paper [1], A. Beilinson defined the motivic cohomology of arbitrary variety as well as the regulator map from these cohomology to the Deligne-Beilinson cohomology. The natural question is: can we decribe the regulator map explicitly? The positive answer to this question was given by A. Goncharov in [9]. It is known [11] that the motivic cohomology can be computed through cubical Bloch's higher Chow group. In loc.cit. A. Goncharov gives the explicit construction of the morphism of complexes, from the cubical Bloch's higher Chow group to the complex computing Deligne-Beilinson cohomology, inducing the Beilinson regulator map.

The simplest non-trivial case of this construction is given by so-called Chow dilogarithm — a non-trivial generalisation of classical Bloch-Wigner dilogarithm [13]. Unlike the classical case this function depends an arbitrary smooth projective curve together with three non-zero rational functions on it. A. Goncharov [9] formulated the conjecture about strong Suslin reciprocity law, which, roughly speaking, says that the Chow dilogarithm can be in a canonical way expressed through Bloch Wigner dilogarithm.

The main result of this dissertation is the full proof of this conjecture. Besides, I prove some new results about functional equations for the elliptic dilogarithm.

The thesis consists of three sections. In Section 1, I give some basic definitions. In Section 2, I formulate results connected with the conjecture of A. Goncharov about strong Suslin reciprocity law. This section is based on results of [4, 5]. In the Section 3, I formulate the results about functional equations for the elliptic dilogarithm obtained in [3].

1 Definitions

Let k be an algebraically closed field of characteristic zero. All algebraic varieties are assumed to be smooth and defined over k. Everywhere we work over $\mathbb{Q}$. So any abelian group is supposed to be tensored by $\mathbb{Q}$. For example, when we write $\Lambda^2 k^\times$ this actually means $(\Lambda^2 k^\times) \otimes \mathbb{Q}$. All exterior powers and tensor products are over $\mathbb{Q}$.

The classical polylogarithm function

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

was defined by L. Euler. This series converges for $|z| < 1$ and can be analytically continued to many-valued meromorphic function on the whole $\mathbb{P}^1(\mathbb{C})$. The definition of single-valued version $\mathcal{L}_n$ of this function can be found in [13]. This function is continuous real-valued function on $\mathbb{P}^1(\mathbb{C})$.

Let F be an arbitrary field. Define the group $\mathbb{Z}[\mathbb{P}^1(F)]_n$ as a free abelian group generated by symbols $\{x\}_n$ where $x \in \mathbb{P}^1(F)$. In the case $F = \mathbb{C}$, the polylogarithm $\mathcal{L}_n$ gives a linear map $\widetilde{\mathcal{L}}_n \colon \mathbb{Z}[\mathbb{P}^1(\mathbb{C})] \to \mathbb{R}$, given by the formula

$$\widetilde{\mathcal{L}}_n(\{z\}_n) = \mathcal{L}_n(z).$$

In [7], A. Goncharov defined some subgroup $\mathcal{R}_n(F)$ of the group $\mathbb{Z}[\mathbb{P}^1(F)]_n$ such that in the case $F = \mathbb{C}$ this group lies in the kernel of the map $\widetilde{\mathcal{L}}_n$. We can think about the subgroup $\mathcal{R}_n(F)$ as the subgroup describing "universal" functional equations for the function $\mathcal{L}_n$.

Definition 1.1 (Higher Bloch group)**.** Define the group $\mathcal{B}_n(F)$ as the following quotient group:

$$\mathcal{B}_n(F) := \mathbb{Z}[\mathbb{P}^1(F)]_n / \mathcal{R}_n(F).$$

This group is called *the n-th Bloch group.*

Remark 1.2. 1. It is not known whether the definitions of the group $\mathcal{R}_n(F)$ from [8] and [7] are equivalent. While it is believed to be the case, this statement relies on the so-called Suslin rigidity conjecture. In this paper we use the later definition, that is the definition from [7].

2. Everywhere in this paper we can replace the complex $\Gamma(F,n)$ with its canonical truncation $\tau_{\geq n-1}\Gamma(F,n)$. Therefore, only the definition of the group $\mathcal{R}_2(F)$ is relevant for us. As it was noted in Section 4.2 of [7] this group is generated by the following elements:

$$\sum_{i=1}^{5}(-1)^i\{c.r.(x_1,\ldots,\widehat{x}_i,\ldots,x_5)\}_2, \{1\}_2, \{0\}_2, \{\infty\}_2.$$

In this formula x_i are five different points on $\mathbb{P}^1$ and $c.r.(\cdot)$ is the cross ratio.

Definition 1.3 (Polylogarithmic complex)**.** Define the complex $\Gamma(F,n)$ as follows:

$$\Gamma(F,n)\colon \mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F)\otimes F^\times \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_n} \mathcal{B}_2(F)\otimes\Lambda^{n-2}F^\times \xrightarrow{\delta_n} \Lambda^n F^\times.$$

This complex is concentrated in degrees $[1,n]$. The differential is defined as follows: $\delta_n(\{x\}_k\otimes y_{k+1}\wedge\cdots\wedge y_n) = \{x\}_{k-1}\otimes x\wedge y_{k+1}\wedge\cdots\wedge y_n$ for $k>2$ and $\delta_n(\{x\}_2\otimes y_3\wedge\ldots y_n) = x\wedge(1-x)\wedge y_3\wedge\cdots\wedge y_n$.

The polylogarithmic complexes was defined by A. Goncharov in [8]. The cohomology of these complexes hypothetically computes motivic cohomology of an arbitrary field.

The proof of the following proposition can be found in [8]:

Proposition 1.4. *Let (F,ν) be a discrete valuation field and $n\geq 3$. There is a unique morphism of complexes $\partial_\nu^{(n)}\colon \Gamma(F,n)\to\Gamma(\overline{F}_\nu,n-1)[-1]$ satisfying the following conditions:*

1. *For any uniformiser π and units $u_2, \dots u_n \in F$ we have $\partial_\nu^{(n)}(\pi \wedge u_2 \wedge \cdots \wedge u_n) = \overline{u_2} \wedge \cdots \wedge \overline{u_n}$.*

2. *For any $a \in F\backslash\{0,1\}$ with $\nu(a) \neq 0$, an integer k satisfying $2 \leq k \leq n$ and any $b \in \Lambda^{n-k}F^\times$ we have $\partial_\nu^{(n)}(\{a\}_k \otimes b) = 0$.*

3. *For any unit u, an integer k satisfying $2 \leq k \leq n$ and $b \in \Lambda^{n-k}F^\times$ we have $\partial_\nu^{(n)}(\{u\}_k \otimes b) = -\{\overline{u}\}_k \otimes \partial_\nu^{(n-k)}(b)$.*

We will call the map $\partial_\nu^{(n)}$ from the previous proposition *the tame symbol map.*

When D is an irreducible divisor on a smooth variety X, we denote by ν_D the corresponding discrete valuation of the field $k(X)$. For any field F denote by $\nu_{\infty,F}$ the discrete valuation of $F(t)$ given by the point $\infty \in \mathbb{P}^1(F)$.

We recall that we have fixed some algebraically closed field k of characteristic zero. Denote by $\mathbf{Fields}_d$ the category of finitely generated extensions of k of transcendence degree d. Any morphism in this category is a finite extension. For $F \in \mathbf{Fields}_d$, denote by $\mathrm{dval}(F)$ the set of discrete valuations given by an irreducible Cartier divisor on some birational model of F. When $F \in \mathbf{Fields}_1$ this set is equal to the set of all 1-dimensional valuations that are trivial on k. In this case, we denote this set simply by $\mathrm{val}(F)$.

Let us define so-called Chow dilogarithm [9]. Let X be a curve over k and f_1, f_2, f_3 be three non-zero rational function on X. Define the following 2-distribution on $X(\mathbb{C})$ (see [9]):

$$r_2(X; f_1, f_2, f_3) = \frac{1}{6} \sum_{\sigma \in S_3} sgn(\sigma) \widetilde{r}_2(X; f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}),$$

$$\widetilde{r}_2(X; g_1, g_2, g_3) = \log|g_1| d\log|g_2| \wedge d\log|g_3| - 3\log|g_1| d\arg(g_2) \wedge d\arg(g_3).$$

The Chow dilogarithm is defined by the formula

$$\mathcal{P}_2(X; f_1, f_2, f_3) = (2\pi i)^{-1} \int_{X(\mathbb{C})} r_2(X; f_1, f_2, f_3).$$

This definition easily implies that Chow dilogarithm vanishes if one of the functions f_i is constant and that for any non-constant map $\varphi\colon X \to Y$, the following formula holds:

$$\mathcal{P}_2(Y; f_1, f_2, f_3) = (\deg \varphi)^{-1}\mathcal{P}_2(X, \varphi^*(f_1), \varphi^*(f_2), \varphi^*(f_3)).$$

Definition 1.5. Let $F \in \mathbf{Fields}_1$. *A lifted reciprocity map* on the field F is a $\mathbb{Q}$-linear map $h\colon \Lambda^3 F^\times \to \mathcal{B}_2(k)$ satisfying the following conditions:

1. The following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{B}_3(F) & \xrightarrow{\delta_3} & \mathcal{B}_2(F) \otimes F^\times & \xrightarrow{\delta_3} & \Lambda^3 F^\times \\ & & \Big\downarrow {\scriptstyle \sum\limits_{\nu\in \mathrm{val}(F)} \partial_\nu^{(3)}} & \swarrow {\scriptstyle h} & \Big\downarrow {\scriptstyle \sum\limits_{\nu\in \mathrm{val}(F)} \partial_\nu^{(3)}} \\ & & \mathcal{B}_2(k) & \xrightarrow{-\delta_2} & \Lambda^2(k^\times). \end{array} \tag{1}$$

2. The map h vanishes on the image of the multiplication map $\Lambda^2 F^\times \otimes k^\times \to \Lambda^3 F^\times$.

2 Lifted reciprocity maps and Chow dilogarithm

Motivated by the analytic properties of Chow dilogarithm, A. Goncharov [9] formulated the following conjecture:

Conjecture 2.1. *For any field $F \in \mathbf{Fields}_1$ one can choose a lifted reciprocity map $\mathcal{H}_F$ on the field F such that for any embedding $j\colon F_1 \to F_2$ we have* $\mathrm{RecMaps}(j)(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$. *Such a collection of lifted reciprocity maps is unique.*

Some partial results towards the solution of this conjecture was obtained in [12]. The main result of this thesis is the proof of the following conjecture:

Theorem 2.2. *Conjecture 2.1 is true.*

The ideas of the proof of this theorem was published in [4]. The detailed proof will be published in [5].

The proof of Theorem 2.2 has the following corollary interesting by its own:

Corollary 2.3. *Let* $L \in \mathbf{Fields}_2$*. For any* $b \in \Lambda^4 L^\times$ *and all but finitely many* $\nu \in \mathrm{dval}(L)$ *we have* $\mathcal{H}_{\overline{L}_\nu}\partial_\nu^{(4)}(b) = 0$*. Moreover, the following sum is equal to zero:*

$$\sum_{\nu \in \mathrm{dval}(L)} \mathcal{H}_{\overline{L}_\nu}\partial_\nu^{(4)}(b) = 0. \tag{2}$$

Applying $\widetilde{\mathcal{L}}_2$ to both sides of (2) we recover the functional equation for Chow dilogarithm proved by A. Goncharov in [9, Section 1.4], see also [6].

3 Elliptic dilogarithm

Let $E = \mathbb{C}/\langle 1, \tau\rangle$ be an elliptic curve over $\mathbb{C}$. The elliptic dilogarithm was defined by Spencer Bloch ([2], see also [13]). The equivalent representation is given by the following formula:

$$D_\tau(\xi) = \sum_{n=-\infty}^{\infty} D(e^{2\pi i\xi + 2\pi i\tau n}).$$

Denote by $\mathbb{Z}[E]$ a free abelian group generated by the points of E. For a point $z \in E$ we denote by $[z]$ the corresponding element in the group $\mathbb{Z}[E]$. The elliptic dilogarithm gives a well-defined map $\widetilde{D}_\tau\colon \mathbb{Z}[E] \to \mathbb{C}$ defined by the formula $\widetilde{D}_\tau([z]) = D_\tau(z)$.

For a rational function g on some smooth algebraic curve, denote by (g) its divisor. Let us formulate the so-called elliptic Bloch relations ([2, Theorem 9.2.1], see also [13]). Let f be a rational function on E of degree n such that

$$(f) = \sum_{i=1}^{n}([\alpha_i] - [\gamma_i]), (1-f) = \sum_{i=1}^{n}([\beta_i] - [\gamma_i]).$$

Define the element $\eta_f \in \mathbb{Z}[E]$ by the following formula:

$$\eta_f = \sum_{i,j=1}^{n} \left([\alpha_i - \beta_j] + [\beta_i - \gamma_j] + [\gamma_i - \alpha_j]\right). \tag{3}$$

The following definition is taken from [10]:

Definition 3.1. Define a subgroup $\mathcal{R}(E)$ of the group $\mathbb{Z}[E]$ generated by the following elements:

1. η_f, where $f \in k(E)$,
2. $[z] + [-z]$, where $z \in E$,
3. $2 \cdot (z - \sum_{2z'=z} [z'])$, where $z \in E$.

The elliptic Bloch group $B_3(E)$ is defined as the quotient $\mathbb{Z}[E]/\mathcal{R}(E)$.

According to [10], the map $\widetilde{D}_\tau$ is zero on $\mathcal{R}(E)$. The main result of [3] is the following theorem:

Theorem 3.2. *Let E be an elliptic curve over $\mathbb{C}$. For any rational function f on E the element η_f can be represented as a linear combination with integer coefficient of the elements of the form η_f with* $\deg f \leq 3$ *and* $[z] + [-z]$.

This implies that when we defining the elliptic Bloch group, we can omit the elements η_f with $\deg f > 3$.

This theorem gives a solution of Conjecture 3.30 from [10].

This statement can be easily deduced from the following result interesting in itself:

Theorem 3.3. *Let E be an elliptic curve over algebraically closed field k of characteristic* 0. *The group $B_2(k(E))$ can be generated by elements of the form $\{f\}_2$, where f is a rational function on E of degree not higher than* 3.

The results of the thesis are published in three articles:

1. V. Bolbachan. Chow dilogarithm and strong suslin reciprocity law. *Journal of algebraic geometry*, 32(3):to appear, 2023
2. V. Bolbachan. Strong suslin reciprocity law and the norm map. *Mathematical Notes*, 112(1):309–312, 2022
3. V. Bolbachan. On functional equations for the elliptic dilogarithm. *European Journal of Mathematics*, 8(2):625–633, 2022

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