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The punctuality stability of the Nash equilibrium: the advantage of a late player in potential and aggregative games

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If all players in a game employ Nash-equilibrium strategies, then no single player benefits from changing their strategy alone. In real games however, some players may intentionally arrive late and get a payoff greater than at the equilibrium. To wit, it sometimes pays to wait for competitors to announce their prices and then set the price for one's own product. The motivation for intentional tardiness is the advantage of making the last move. Can it be arranged so that no player arrives in the game late intentionally? Responding to this challenge, we suggest forming a punctually stable Nash-equilibrium strategy profile. In this study, we investigate whether such a strategy profile exists in potential, aggregative, and symmetric games. What is remarkable about this study is that in some game-theoretical settings all-player punctuality can be achieved without penalizing late arrivals.

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1 Introduction

1.1 Problem description

In normal-form games, each player has to choose a strategy before the instant of getting a payoff. For some players, it is sometimes better to wait until others have chosen their strategies before making their own move. One can gain advantage from being the last to choose a strategy. That is why some players intentionally arrive late for the game. Below are some examples where a player's payoff is augmented by tardiness:

- Coalition formation. Players break into a fixed number of commissions. A chairperson is elected for each commission and each player wants to become one. Suppose a player arrives late. Punctual players partition into commissions without waiting for the tardy player. After the commission member-lists are formed, transitions between commissions are not allowed. The tardy player arrives right before the election of chairpersons. Commissions will not be recast because of one tardy player. The tardy player is invited to choose any of the established commissions. It may prove to be advantageous for a player to wait until the coalition partition is formed and then join the coalition in which their chance of becoming the chairperson is the highest.

- Resource allocation. Consider a situation where any player's strategy is the selection of a resource. We assume that a certain player conceals their presence. The rest of the players form a Nash-equilibrium strategy profile. Before the payoffs on the selected resources are distributed, the covert player shows up. They choose the resource that maximizes their payoff. Other players cannot change their strategy of choice since it is time to divide up the resource. It may prove to be more beneficial to show up late and choose the best resource than to enter the game from the start. This situation can take place in multi-server systems, for example, the resource is a server and the payoff on the resource is the player's service time. The late-arriving player knows the server workloads and can choose the server to minimize their service time.

Similar situations, in which players benefit from arriving late, can be encountered in decision-making on pricing, choosing a college, etc. There is a common feature in all these game situations. At first, punctual players arrive at some equilibrium strategy profile. Then, a tardy player appears and chooses the best strategy knowing the strategies chosen by the punctual players. The punctual players cannot change the strategies they have chosen after that player's late arrival.

The easiest way to avoid intentional tardiness is to impose penalties or to prohibit the arrival of new players. Implementing this in practice, however, is not always possible. Players in the resource allocation process described above have no other rights but the right to use the servers. That is why punctual players cannot prohibit the entry of new players or penalize tardiness. Penalizing is not always welcomed in coalition formation. If a coalition imposes a penalty on its new member for arriving late, then the penalized player may hold a grudge against the coalition, and this cannot end well. Admittedly, there may be different reasons for missing the start of the game. On the one hand, it can be the player's own will, but on the other hand, unforeseen circumstances may prevent a player from coming on time. Punctual players may never find out whether the late arrival was intentional.

How can we make all agents in a game theoretical process be punctual without penalizing them? Having studied this question, we conclude that the answer lies in the game itself. For some games there exists a Nash-equilibrium strategy profile such that no player can augment their payoff by arriving late. We describe such a Nash-equilibrium strategy profile as punctually stable.

The punctuality stability of a Nash equilibrium is defined as follows. Suppose Γ is a normal-form game and N is the set of players. We assume that the strategy profile s^* in the game Γ is a Nash equilibrium. The payoff of player $i, i \in N$ in the game Γ for the profile s^* is $u_i(s^*)$. If player i thinks about their intentional non-appearance by the beginning of the game, they reason as follows: "Players from the set $N \setminus \{i\}$ play a certain game Γ_{-i}. An equilibrium $\hat{s}^*_{-i}$ is formed in the new game. Then, I (player i) appear and choose a strategy s_i". It is disadvantageous for player i to arrive late if $u_i(s^*) \geq u_i(s_i, \hat{s}^*_{-i})\ \forall s_i$. If such inequalities hold for each player, then the equilibrium s^* is punctually stable in the game Γ. Each player abandons the thought of being late for punctually stable equilibrium. A more formal description of the punctuality stability of a Nash equilibrium is given in Section 2.

If nobody is late for the start of the game Γ, then player i gets a payoff of $u_i(s^*)$. If only player i is late, then they get $u_i(s_i, \hat{s}^*_{-i})$, where $\hat{s}^*_{-i}$ is an equilibrium profile of the game Γ_{-i} in which player i did not participate. It is disadvantageous for player i to arrive late if $u_i(s^*) \geq u_i(s_i, \hat{s}^*_{-i})\ \forall s_i$.

In some studies, a player's strategy is the choice of timing for entering and leaving the game, such as dynamic games. We are not interested in finding equilibrium instants of player arrivals in the game. Our objective is the punctuality of all players. To this end, we suggest forming a punctually stable Nash-equilibrium strategy profile.

When investigating the punctuality stability of a Nash equilibrium, a definition should be given not only of the game Γ, but also of the games $\Gamma_{-i}, i \in N$. Players from the set $N \setminus \{i\}$ play the game Γ_{-i} if player i is late. We define the games $\Gamma_{-i}, i \in N$ proceeding from the context of the game Γ. If, for example, the game Γ is a congestion game, then the games $\Gamma_{-i}, i \in N$ are also congestion games.

What adds complexity to the study of the punctuality stability of a Nash equilibrium is that the strategy profiles s^* and $(s_i, \hat{s}^*_{-i})$ may be completely different from one another. Here, different approaches are employed for checking whether a punctually stable Nash equilibrium exists in different game classes.

1.2 Key assumptions regarding tardiness and the privilege of the last move

If there are no tardy players, then players from the set N play the game Γ. If we only have players from the set $N \setminus \{j\}$, then these players play the game $\Gamma_{-j}, j \in N$. The set of punctual players in the game Γ is N, and in the game Γ_{-j} it is $N \setminus \{j\}$. Player j is called tardy. In the following, we formulate five key assumptions regarding player tardiness.

Assumption 1. *Only one player can come late for the start of the game* Γ.

In a Nash equilibrium, no single player can augment their payoff by changing their strategy on their own. The question, however, is whether a player's payoff can be increased if they are the only tardy player enjoying the privilege of the last move. We try to answer this question by assuming there is only one tardy player.

Assumption 2. *When selecting their strategies in the game* Γ_{-j}, *punctual players do not make allowances for the possible arrival of a new player.*

Assumption 2 characterizes a feature of the real-life behavior of players in certain situations. To wit, each agent in coalition formation processes wants, for reasons of caution, to cooperate only with the players present at the moment. If one is guided by the probability that a new player may appear and waits for their arrival, there is a chance of ending up alone.

Similar behavior can be encountered in traffic problems. Drivers use navigation systems to get updated traffic congestion information. Theoretically, drivers can plan their route one step ahead considering that many other drivers can go via uncongested roads. The search for the best strategy requires calculations, but the driver has little time to decide on the route. Because of that, each driver trusts their navigation system and follows the suggested route.

If players from the set $N \setminus \{j\}$ made allowances for the late arrival of player j, they would be considering a two-step game. In the situations mentioned above, however, players take only the current circumstances into account. This behavior is dictated by the players' psychology [25, 26], lack of time for lengthy calculations, short-sightedness, etc. Assumption 2 is also relevant for games with a large number of players. A report that a large game will be joined by only one new individual may not incite much interest or response.

Assumption 3. *The tardy player enters the game after the punctual players have already chosen their strategies and the tardy player chooses the best strategy for themselves.*

Player j learns about the strategies selected by punctual players when entering the game. Knowing the strategies of the punctual players, the tardy player chooses a strategy that will maximize their payoff.

Assumption 4. *The punctual players cannot change their strategies after the arrival of the tardy player.*

In some situations, there is a moment after which the player can no longer change the selected strategy. If, for example, a driver chose a one way street, they cannot turn around. Having invested some funds into a project, the investors do not terminate it if competitors show up. It is after such a moment that a tardy player arrives. Although the punctual players are bound by restrictions, the tardy player enjoys a freedom of choice: a new driver chooses an uncongested road; an investor can evaluate the contributions of competitors and choose the best project.

These assumptions can give an advantage to the tardy player. This player knows the strategies of other players, and punctual players cannot change them. When the tardy player gets the privilege of the last move, then every player wants to be the tardy one. Because of that, players may intentionally arrive late for the game. The question arises of how to prevent intentional tardiness.

Assumption 4 is an obstacle to using the methods applied in dynamic games to tackle intentional tardiness. In a dynamic game, punctual players would be able to respond to the arrival of a tardy player, but in our case it is impossible. Naturally, some of the punctual players would want to change their strategy after the arrival of the new player, but they cannot do so physically or for some other reasons. This article studies how the advantage of the last move for the tardy player can be eliminated without applying the obvious methods used in dynamic games or penalizing tardiness.

Assumption 5. *The tardy player's payoff is determined by the game* Γ.

If nobody is late for the start of the game Γ, then player j gets $u_j(s^*)$. If they arrive late, then punctual players in the game Γ_{-j} form the equilibrium $\hat{s}^*_{-j}$. After arriving in the game, player j chooses strategy s_j and gets $u_j(s_j, \hat{s}^*_{-j})$. Player j does not profit from missing the start of the game if $u_j(s^*) \geq u_j(s_j, \hat{s}^*_{-j})\ \forall s_j \in S_j$. If such an inequality is true for any player j and for any permissible equilibrium $\hat{s}^*_{-j}$, then s^* is called a punctually stable Nash equilibrium.

1.3 The punctuality problem in the literature

The efficiency of interactions between agents in economic and social spheres depends on many factors. Let us mention some studies where delays in elements of the system or the tardiness of agents are detrimental for the process.

Transport. Having vehicles running on time and arriving as scheduled is important for the credibility of a transport company. The authors of [2] suggest a model for the integrated optimization of aircraft holding time to improve flight punctuality and reduce the impact of adverse weather. A model for coordination of the airport operator, airlines, and ground service providers is investigated in [7] to improve the apron's on-time punctuality, without the need for the involved agents to share sensitive information. The paper [8] suggests a mixed integer programming formulation for routing aircraft along a predetermined path. The model takes into account the main punctuality indicators. Many land transport passengers need to get to work, college, or other places on time. Agents' punctuality depends on many factors. Thus, school bell time and bus schedules are coupled. A bell time optimization model is suggested in [29] to ensure students' punctuality. An automated train traffic control system was worked out in [6] to optimize the regular traffic of trains and ensure their punctuality. Deviations from the timetable demand fast rerouting decisions. A linear programming model is applied in [15] to handle this issue.

The punctuality of patients in health care institutions. An approach to sequentially scheduling appointments to provide desirable schedules from the perspective of both patients and medical practices is proposed in [5]. Outpatient appointment scheduling considering tardiness is investigated in [12]. The paper [14] investigates a stochastic patient service model in a clinic with a single doctor. The problem of working out appointment scheduling strategies in [31] is designed to help outpatient clinics to utilize their resources efficiently while limiting patient waiting times.

Customer service. Which appointment scheduling rule is best is still an open question. The appointment scheduling model in [4] takes into account customer tardiness, service interruptions, and delays in session start times. The authors of [30] consider the problem of appointment scheduling on multiple servers. The objective is to minimize the weighted sum of server staffing cost and total expected cost of customer waiting, server idleness, and overtime. The problem studied in [28] is the effect of customer tardiness considering the fact that preventive maintenance depends on when customers return their items to authorized maintenance centers. The application of penalties in scheduling problems is investigated in [24].

Product handling. Punctuality can be addressed in the literature implicitly. Supply chains of perishable products require punctuality. Thus, researchers in [23] study punctuality in such processes. Untimely waste removal can be harmful for the environment. Accordingly, the focus in [13] is on waste treatment.

The results of an experiment where students were recruited to identify the value and provenance of euro coins are described in [1]. The work materials had to be returned by an appointed date. Reportedly, monitoring improved work quality if incentives were harsh but it reduced punctuality. In the absence of monitoring, the percentage of participants arriving on time was much higher.

In the game-theoretical literature, punctuality was investigated in [18, 19]. The player decides whether or not to arrive late and an aspect studied was equilibrium in the given game-theoretical settings.

What distinguishes the present study is that we simulate the punctuality issue in games. Players can intentionally arrive late for the game seeking to augment their payoff. We suggest forming a punctually stable Nash equilibrium in the games to avoid intentional tardiness.

1.4 The contribution of the paper

The main results of the study are:

- The classes of potential games in which the Nash equilibrium which maximizes the potential function is punctually stable are found (Theorem 1).
- The necessary and sufficient conditions for the punctuality stability of the Nash equilibrium in aggregative games with monotone payoff functions are found (Theorem 2).

- The identification of sufficient conditions for the punctuality stability of the Nash equilibrium in symmetric games are found.

The following situation is considered for potential games. There are two scenarios of the game process. In the first scenario, players form a strategy profile that maximizes the potential function. In the second scenario, a player is late and a certain new potential game is formed without the tardy player. In this new potential game, punctual players form a strategy profile which maximizes the new potential function. After that, the tardy player shows up and chooses the best option. The punctual players cannot change their strategies after that player's late arrival. Is it in the first or in the second scenario that the tardy player's payoff is higher? Theorem 1 describes the potential games in which the first scenario is preferable for the players. This means that it is disadvantageous for players to miss the start of the game. The Nash equilibrium which maximizes the potential function is punctually stable in games from Theorem 1. The question of punctuality for transversal value is examined in [10].

If a Nash equilibrium exists in an aggregative game with monotone payoff functions, it is punctually stable iff the inequalities described in Theorem 2 are fulfilled. These inequalities are numerical and they can always be verified for any specific game. A Nash equilibrium in an oligopoly with linear-quadratic functions is shown to be punctually stable. A game that is simultaneously potential and aggregative has been found and it is specified that the Nash equilibrium in this game is not punctually stable in the general case.

Sufficient conditions for symmetric equilibrium punctuality in symmetric games are related to the existence of a saddle point for some special function. The definitions of the punctuality stability of a Nash equilibrium formulated for normal-form and for extensive-form games are different. We prove that a Nash equilibrium of an extensive-form symmetric game is always punctually stable.

Proofs of Theorems and Statements 2-7 are in the Appendix.

2 Punctuality stability

2.1 The definition of the punctuality stability of a Nash equilibrium

We introduce the basic notation. Let

$$\Gamma = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$$

be a game in normal form, where $N = \{1, 2, ..., n\}$ is the finite set of players, S_i is the set of strategies of player $i, S = \prod_{i \in N} S_i = S_1 \times S_2 \times ... \times S_n$, $u_i : S \to \mathbb{R}$ is the payoff function of $i, i \in N$. If S_i is a finite set $\forall i \in N$, then Γ is called a finite game. Denote

$$\Gamma_{-j} = \left\langle N \setminus \{j\}, \{S_i\}_{i \in N \setminus \{j\}}, \{u_i^j\}_{i \in N \setminus \{j\}} \right\rangle$$

as a game in normal form without player j, where $u_i^j : \prod_{k \in N \setminus \{j\}} S_k \to \mathbb{R}$.

The strategy of player i in the games Γ and Γ_{-j} is s_i and s_i^j, respectively. The strategy profile of players in the game Γ is a vector $s = (s_1, s_2, ..., s_n) \in S$, and the strategy profile of players in the game Γ_{-j} is $\hat{s}_{-j} = (s_1^j, s_2^j, ..., s_{j-1}^j, s_{j+1}^j, ..., s_n^j)$, where $s_i^j \in S_i \ \forall i \in N \setminus \{j\}$. The equilibrium profile of strategies in the game Γ is $s^* = (s_1^*, s_2^*, ..., s_n^*)$, and in the game Γ_{-j} is $\hat{s}^*_{-j} = (s_1^{j*}, s_2^{j*}, ..., s_{j-1}^{j*}, s_{j+1}^{j*}, ..., s_n^{j*})$. In order to highlight the strategy of player i in the profile s, we use the notation $s = (s_i, s_{-i})$. We also write $(s_i, \hat{s}_{-i}) = (s_1^i, s_2^i, ..., s_{i-1}^i, s_i, s_{i+1}^i, ..., s_n^i)$, where s_i is the strategy of player i in the game Γ and $\hat{s}_{-i}$ is the profile strategies in the game Γ_{-i}.

We use $NE(\Gamma)$ and $NE(\Gamma_{-j}), j \in N$ to denote the set of the Nash equilibria considered in the games Γ and Γ_{-j}, respectively. In the following, we shall always imply or specify that $NE(\Gamma) \neq \emptyset$ and $NE(\Gamma_{-j}) \neq \emptyset \; \forall j \in N$.

Definition 1. *The Nash equilibrium $s^* = (s_i^*, s_{-i}^*) \in NE(\Gamma)$ is punctually stable if $\forall i \in N$ the following inequalities are fulfilled:*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, \hat{s}_{-i}^*) \; \forall s_i \in S_i \; \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

Let us describe the physical meaning of the punctuality stability of a Nash equilibrium. We have a set of players $N = \{1, 2, ..., n\}$. All players are selfish and seek to maximize their payoff functions. In the process of the game Γ, an equilibrium strategy profile $s^* = (s_i^*, s_{-i}^*) \in NE(\Gamma)$ forms and player i gets $u_i(s^*), i \in N$. Since s^* is an equilibrium, the payoff of player i will not grow if they alter their equilibrium strategy s_i^*.

Next, player i reasons as follows. They consider two situations. In the first situation, player i assumes that no one will be late. In this case, their payoff is $u_i(s^*)$. In the second situation, player i decides to hide themselves. In this situation, players from the set $N \setminus \{i\}$ play Γ_{-i}. A certain equilibrium $\hat{s}_{-i}^*, \hat{s}_{-i}^* \in NE(\Gamma_{-i})$ forms in the game Γ_{-i}. Next, player i shows up. They choose strategy $s_i \in S_i$ and get $u_i(s_i, \hat{s}_{-i}^*)$.

In the first and second situations, player i assumes that the players from the set $N \setminus \{i\}$ play honestly, that is, they do not manipulate their presence. However, in the first case, player i is not late, and in the second case, player i is late intentionally.

If $u_i(s_i^*, s_{-i}^*) < u_i(s_i, \hat{s}_{-i}^*)$, then player i benefits from missing the start of the game Γ. If the Nash equilibrium s^* is punctually stable, then no player can benefit from arriving late or concealing their presence at the start of the game Γ. The assumption in this line of thought is that players from the set $N \setminus \{i\}$ cannot change their strategies upon the arrival of player i. The reason may be that the game Γ_{-i} is coming to an end and players from $N \setminus \{i\}$ have no time to respond to the arrival of the tardy player.

If each player would benefit from missing the start of the game and arriving only close to its end, then the game may never even start. The punctuality stability of a Nash equilibrium implies that all players are interested in being present at the start of the game.

2.2 Simple examples: tardiness in the prisoner's dilemma and in the battle of the sexes game

Let us consider the prisoner's dilemma game,

$$\Gamma = \langle N = \{1, 2\}, S_1 = S_2 = \{a, b\}, \{u_i\}_{i \in N} \rangle,$$

where 1 and 2 are the players, a is the strategy to testify, b is the strategy to remain silent. The players' payoffs are given in Table 1.

Таблица 1: Players' payoffs in the prisoner's dilemma game

		2	
		a	b
1	a	-6;-6	0;-10
	b	-10;0	-2;-2

If both players testify to the crime, then each is sentenced to six years in jail. If one player remains silent while the other testifies, then the one that has kept silent goes free while the one that has testified is sentenced to 10 years. If both remain silent, then both get two years. The prisoner's dilemma has only one pure equilibrium and we want to check it for punctuality, that is, $NE(\Gamma) = \{(a,a)\}$.

We consider the following situation. Players 1 and 2 have committed a crime. One of them decides to surrender voluntarily, and the other one decides to flee. The testimony of the detained player can reduce the time it takes to pin down the player in flight. The investigator proposes a deal to the detained player. If the detained player testifies and conceals nothing from the police, then they get only one year to serve in reward for cooperation. Voluntary surrender coupled with refusal to cooperate results in five years in jail for the detained player. Formally, the games Γ_{-1} and Γ_{-2} have the form

$$\Gamma_{-1} = \left\langle \{2\}, \{a,b\}, u_2^1 \right\rangle, \ \Gamma_{-2} = \left\langle \{1\}, \{a,b\}, u_1^2 \right\rangle,$$
$$u_1^2(a) = u_2^1(a) = -1, \ u_1^2(b) = u_2^1(b) = -5.$$

In the game Γ_{-1} player 1 has fled and player 2 is detained. The situation in the game Γ_{-2} is the opposite. The optimal strategies of players 1 and 2 in the games Γ_{-2} and Γ_{-1} are $s_1^{2*} = a$ and $s_2^{1^*} = a$, respectively.

We have thus defined the games $\Gamma, \Gamma_{-1}, \Gamma_{-2}$. Now, let us check whether the equilibrium (a,a) is punctually stable in Γ. By Definition 1, equilibrium is punctually stable if the following inequalities are fulfilled:

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^{1*}) \ \forall s_1 \in \{a,b\}.$$
$$u_2(s_1^*, s_2^*) \geq u_2(s_1^{2*}, s_2) \ \forall s_2 \in \{a,b\}.$$

Since $(s_1^*, s_2^*) = (a,a), s_1^{2*} = s_2^{1*} = a$, we have

$$u_1(a,a) \geq u_1(a,a), \ u_1(a,a) \geq u_1(b,a),$$
$$u_2(a,a) \geq u_2(a,a), \ u_2(a,a) \geq u_2(a,b).$$

The inequalities hold, so the equilibrium (a,a) is punctually stable in the prisoner's dilemma game. We can say that the rule under which investigators offer a bargain for voluntary surrender is an example of a game design that makes the equilibrium punctually stable. The above reasoning implies that both players will be eventually caught. In this case, each player would want to get caught first to get the bargain from the police.

For a matrix game Γ, the games $\Gamma_{-i} \ \forall i \in N$ are constructed based on the physical meaning of the game.

The next example to consider is the battle of the sexes game,

$$\Gamma = \langle N = \{H, W\}, S_1 = S_2 = \{F, T\}, \{u_i\}_{i \in N} \rangle .$$

The players' payoffs are given in Table 2.

Husband (H) and wife (W) plan to go to a soccer game (F) or to the theater (T). They decide to meet at a given time and buy tickets together. The soccer and theater cash desks are situated close to each other. If one of the spouses misses the appointed time, then the punctual spouse buys a ticket for him or herself and waits for the tardy spouse. The players' payoffs are given in Table 2.

Таблица 2: Players' payoffs in the battle of the sexes game

		W: F	W: T
H	F	2;1	0;0
	T	0;0	1;2

The battle of the sexes game has two pure equilibria. We shall test each one for being punctually stable, that is,

$$NE(\Gamma) = \{(F, F), (T, T)\}.$$

If the player H or W is late, then, formally, two one-player games

$$\Gamma_{-H} = \langle \{W\}, \{F, T\}, u_W^H \rangle \text{ and } \Gamma_{-W} = \langle \{H\}, \{F, T\}, u_H^W \rangle$$

form respectively. It is reasonable to assume that if W missed the appointed time, then H will buy himself a ticket for the soccer game. If H is late for the meeting, then W will buy a theater ticket. This means that the optimal strategies of the players W and H in the games Γ_{-H} and Γ_{-W} are $s_W^{H*} = T, s_H^{W*} = F$. By Definition 1, the strategy profile $s^* = (s_H^*, s_W^*), s^* \in NE(\Gamma)$ is punctually stable if the following inequalities are fulfilled:

$$u_H(s_H^*, s_W^*) \geq u_H(s_H, s_W^{H*}) \; \forall s_H \in \{F, T\},$$
$$u_W(s_H^*, s_W^*) \geq u_W(s_H^{W*}, s_W) \; \forall s_W \in \{F, T\}.$$

If $(s_H^*, s_W^*) = (F, F)$, then the two inequalities written above can be represented as

$$u_H(F, F) \geq u_H(s_H, T) \; \forall s_H \in \{F, T\},$$
$$u_W(F, F) \geq u_W(F, s_W) \; \forall s_W \in \{F, T\}.$$

If $(s_H^*, s_W^*) = (T, T)$, then we have

$$u_H(T, T) \geq u_H(s_H, T) \; \forall s_H \in \{F, T\},$$
$$u_W(T, T) \geq u_W(F, s_W) \; \forall s_W \in \{F, T\}.$$

For each $s^* \in NE(\Gamma)$, the inequalities from Definition 1 hold. The rule under which the punctual spouse buys a ticket for their preference is a game design that makes the Nash equilibrium in the battle of the sexes game punctually stable.

2.3 The necessary condition for punctuality

This section is dedicated to finding the necessary condition for the punctuality stability of a Nash equilibrium in an arbitrary game Γ. The aim is to demonstrate that in the general case the necessary condition is not sufficient.

Statement 1. *Let $NE(\Gamma) \neq \emptyset$ and $NE(\Gamma_{-j}) \neq \emptyset \; \forall j \in N$.*
i) The necessary condition. Let $s^*, s^* \in NE(\Gamma)$ be punctually stable. Then

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i^*, \hat{s}_{-i}^*) \; \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

ii) The sufficient condition. Let the necessary condition be satisfied and $(s_i^*, \hat{s}_{-i}^*)$ be a Nash equilibrium in the game $\Gamma \; \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i})$. Then s^* is punctually stable.

Доказательство. i) Let $s_i = s_i^*$ be satisfied in the inequalities from Definition 1. This proves the first part of the statement.

ii) Since the necessary condition is satisfied, then

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i^*, \hat{s}_{-i}^*) \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

Since $(s_i^*, \hat{s}_{-i}^*)$ is a Nash equilibrium in game Γ $\forall \hat{s}_{-i}^* \in NE(\Gamma_{-i})$, then

$$u_i(s_i^*, \hat{s}_{-i}^*) \geq u_i(s_i, \hat{s}_{-i}^*) \ \forall s_i \in S_i.$$

Therefore,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, \hat{s}_{-i}^*) \ \forall s_i \in S_i \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}),$$

that is, s^* is punctually stable.

□

In the general case, the necessary condition for the punctuality stability of a Nash equilibrium is not sufficient. It is also worth noting that the sufficient condition for a Nash equilibrium in the arbitrary game Γ from Statement 1 to be punctually stable is not constructive. That is why we henceforth consider certain game classes individually and determine whether a punctually stable Nash equilibrium exists in them.

3 Punctuality in potential games

3.1 Tardiness in classical potential games

This subsection investigates the punctuality stability of a Nash equilibrium in the class of potential games [16]. A potential game is a normal-form game for which there exists a function $P : \prod_{i \in N} S_i \to \mathbb{R}$, such that $\forall i \in N$ the following equality is true:

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}) \ \forall s_i, s_i' \in S_i \ \forall s_{-i} \in \prod_{j \in N \setminus \{i\}} S_j.$$

Let Γ and Γ_{-j} be potential games with the potential functions $P : S \to \mathbb{R}$ and $P_{-j} : \prod_{i \in N \setminus \{j\}} S_i \to \mathbb{R} \ \forall j \in N$, respectively. The strategy profile that maximizes the potential function, P, on the set, S, is a pure Nash equilibrium in Γ. Below, we describe three potential games in which the punctuality stability of the equilibrium is checked.

Congestion game [20]. Let M be a finite set of resources and $S_i \subseteq 2^M \ \forall i \in N$. The resource $l, l \in M$ is associated with the payoff function $c_l : \mathbb{R} \to \mathbb{R}$. The congestion game is a normal-form game, Γ, in which the payoff function for player $i, i \in N$ and the potential function have the form

$$u_i(s) = \sum_{l \in s_i} c_l(k_l(s)), \quad P(s) = \sum_{l \in \underset{m \in N}{\cup} s_m} \sum_{k=1}^{k_l(s)} c_l(k),$$

where $k_l(s)$ is the number of players who have chosen the resource l in the profile s.

If the start of the congestion game, Γ, is missed by player j, then a congestion game Γ_{-j} forms, in which the payoff function for player $i, i \in N \setminus \{j\}$ and the potential function are

$$u_i^j(\hat{s}_{-j}) = \sum_{l \in s_i^j} c_l(k_l(\hat{s}_{-j})), \quad P_{-j}(\hat{s}_{-j}) = \sum_{l \in \underset{m \in N \setminus \{j\}}{\cup} s_m^j} \sum_{k=1}^{k_l(\hat{s}_{-j})} c_l(k).$$

The players' payoff functions retain their form if one player is late. The same is valid for the following two potential games.

Bilateral symmetric interaction games (BSI) [27]. Suppose that for any $i, g \in N, i \neq g$ there exists a function $w_{ig} : S_i \times S_g \to \mathbb{R}$ and $h_i : S_i \to \mathbb{R}$. The function w_{ig} shows the influence of player i on player j. The function h_i is the utility ($h_i > 0$) or the costs ($h_i < 0$) of player i using strategy s_i. The equality $w_{ig}(s_i, s_g) = w_{gi}(s_g, s_i)$ $\forall (s_i, s_g) \in S_i \times S_g$ is true for the function w_{ig}. The BSI is a normal-form game, Γ, in which the payoff function for player $i, i \in N$ and the potential function have the form

$$u_i(s) = \sum_{g \in N \setminus \{i\}} w_{ig}(s_i, s_g) - h_i(s_i), \quad P(s) = \sum_{\substack{m<g \\ m,g \in N}} w_{mg}(s_m, s_g) - \sum_{m \in N} h_m(s_m).$$

Γ_{-j} is a BSI game in which the payoff function for player $i, i \in N \setminus \{j\}$ and the potential function have the form

$$u_i^j(\hat{s}_{-j}) = \sum_{g \in N \setminus \{i,j\}} w_{ig}(s_i^j, s_g^j) - h_i(s_i^j), \quad P_{-j}(\hat{s}_{-j}) = \sum_{\substack{m<g \\ m,g \in N \setminus \{j\}}} w_{mg}(s_m^j, s_g^j) - \sum_{m \in N \setminus \{j\}} h_m(s_m^j).$$

The third potential game, **The universal potential game**, is a normal-form game, Γ, in which the payoff function for player i and the potential function have the form

$$u_i(s) = \sum_{\substack{K \subseteq N \\ i \in K}} \Phi_K(s_K), \quad P(s) = \sum_{\substack{K \subseteq N \\ K \neq \emptyset}} \Phi_K(s_K),$$

where $\Phi_K : \prod_{i \in K} S_i \to \mathbb{R}$ $\forall K \subseteq N, K \neq \emptyset$ and $s_K \in \prod_{i \in K} S_i$ [27]. If player j is late for the start of the universal potential game, then a game, Γ_{-j}, forms in which the payoff function for player $i, i \neq j$ and the potential function have the form

$$u_i^j(\hat{s}_{-j}) = \sum_{\substack{K \subseteq N \setminus \{j\} \\ i \in K}} \Phi_K(s_K), \quad P_{-j}(\hat{s}_{-j}) = \sum_{\substack{K \subseteq N \setminus \{j\} \\ K \neq \emptyset}} \Phi_K(\hat{s}_K).$$

We have defined the game Γ_{-j} so that the tardiness of any player does not entail an alteration of the form of the players' payoff functions. If Γ belongs to a certain class of games, then Γ_{-j} also belongs to this class $\forall j \in N$.

3.2 Example of tardiness in a congestion game

This subsection demonstrates that not any equilibrium in a potential game is punctually stable. Let Γ be a congestion game with the set of players $N = \{1, 2, 3, 4\}$ and two resources $M = \{I, II\}$. Each player can choose only one of the resources. The payoff functions c_I and c_{II} are non-monotonic and their values have the form

$$c_I(1) = 3,\ c_I(2) = 5\ c_I(3) = 1,\ c_I(4) = 2.5,$$
$$c_{II}(1) = 2,\ c_{II}(2) = 6,\ c_{II}(3) = 4,\ c_{II}(4) = 1.$$

The strategy profile (I, I, I, I) is a Nash equilibrium. For such a profile, the payoff of each player is $u_i(I, I, I, I) = 2.5$ $\forall i \in N$. For any other equilibrium strategy profile, the number of players selecting the resource I or II is 2 for each. E.g., (I, I, II, II) is also a Nash equilibrium and

$$u_1(I, I, II, II) = u_2(I, I, II, II) = 5, u_3(I, I, II, II) = u_4(I, I, II, II) = 6.$$

Let us check whether the strategy profile (I, I, I, I) is punctually stable. Suppose player $j, j \in N$ arrives late or intentionally conceals their presence from other players. Then, players from the set $N \setminus \{j\}$ play a congestion game, Γ_{-j}, with the same resources. This game has only one equilibrium $\hat{s}^*_{-j} = (II, II, II)$ and the players' payoffs are $u_i^j(II, II, II) = c_{II}(3) = 4 \ \forall i \in N \setminus \{j\}$. After the equilibrium $\hat{s}^*_{-j}$ is established, player j shows up and chooses the best resource for oneself. We have $u_j(I, \hat{s}^*_{-j}) = 3, u_j(II, \hat{s}^*_{-j}) = 1 \ \forall j \in N$. The best answer for player j is to choose I and get 3. This payoff is greater than $u_j(I, I, I, I) = 2.5$, and so the equilibrium (I, I, I, I) is not punctually stable.

Let us now consider the equilibrium (I, I, II, II). For this strategy profile, the payoff for the first and the second players is $c_I(2) = 5$, and the payoff for the third and the fourth players is $c_{II}(2) = 6$. If the player j arrives late, they will get $u_j(I, \hat{s}^*_{-j}) = 3$ after arriving in the game. Since 5 and 6 is more than 3, being tardy is not beneficial for any player. Hence, (I, I, II, II) is a punctually stable Nash equilibrium.

Hence, not any equilibrium profile in the congestion game is punctually stable.

3.3 The theorem of punctuality in potential games

We now test the punctuality stability of the equilibrium strategy profiles that maximize the potential function:

$$NE(\Gamma) = \{s^* | s^* \in \underset{s \in S}{\operatorname{argmax}}\ P(s)\},$$

$$NE(\Gamma_{-j}) = \{\hat{s}^*_{-j} | \hat{s}^*_{-j} \in \underset{\hat{s}_{-j} \in \prod_{i \in N \setminus \{j\}} S_i}{\operatorname{argmax}} P_{-j}(\hat{s}_{-j})\}.$$

Theorem 1. *The following two assertions are true:*

1. Let the payoff function for player $i, i \in N$ in the game Γ have the form

$$u_i(s) = P(s) - P_{-i}(s_{-i}),$$

where P and P_{-i} are the potential functions in the games Γ and Γ_{-i}, respectively. In this case, any strategy profile from $NE(\Gamma)$ is punctually stable.

2. If Γ is a congestion game or a BSI game or a universal potential game, then any profile from $NE(\Gamma)$ is punctually stable.

In the games from Theorem 1, players can be asked to form a strategy profile that maximizes the potential function. In this case, we get an equilibrium and no player would want to arrive late or conceal their presence before the start of the game.

As demonstrated in [27], a normal-form game is a potential game iff the normal-form game is a universal potential game. Hence, for an arbitrary potential game it is always possible to find an array of functions $\{\Phi_K\}_{\substack{K \subseteq N \\ K \neq \emptyset}}$ to express the players' payoffs. If $u_i^j \ \forall i \in N \ \forall j \in N \setminus \{i\}$ can also be expressed through the array $\{\Phi_K\}_{\substack{K \subseteq N \\ K \neq \emptyset}}$ as stated in Section 2.1, then, according to Theorem 1, any profile from $NE(\Gamma)$ is punctually stable.

It is known that the payoff functions for players in an arbitrary potential game can be written in the form

$$u_i(s) = P(s) + Q_i(s_{-i}),$$

where P is the potential function and $Q_i : \prod_{k \in N \setminus \{i\}} S_k \to \mathbb{R}$. If $Q_i = -P_{-i} \ \forall i \in N$, then we get the players' payoff functions described in the first point of Theorem 1.

Suppose we are checking the punctuality stability in a potential game from a certain class of potential games. If any player is late, we will have a potential game from the same class. Will the equilibrium that maximizes the potential function be punctually stable in this case? The answer to this question is negative. The game described in Section 4.3. is a counterexample.

3.4 Punctuality in the marginal game

Let $\Pi(N)$ be the set of all coalition partitions of the set N,

$$\Pi(N) = \big\{\{B_1, B_2, ..., B_l\} | B_j \cap B_g = \emptyset, 1 \le j < g \le l, \cup_{j=1}^{l} B_j = N\big\}.$$

The coalition structure $\pi = \{B_1, B_2, ..., B_l\}$ is an element of $\Pi(N)$. We denote by $B(i)$ the coalition in the partition π which contains the player i, that is, $i \in B(i) \in \pi$. The coalition partition game is a pair (N, H) where $H : \Pi(N) \to \mathbb{R}^n$. In the game (N, H), the question of the existence of a stable partition is interesting. In a Nash-stable partition, it is not profitable for any player to move from their coalition to another. For a permutation-stable partition, the sum of players' payoffs from different coalitions will decrease if they switch places. The strategy of the player in the coalition partition game follows from the type of stability.

The marginal game is a coalition partition game (N, H) in which the players' payoff functions have the form

$$H_i(\pi) = v(B(i)) - v(B(i) \setminus \{i\}) \ \forall i \in N,$$

where $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$. In a marginal game, as demonstrated in [9], there always exists a coalition structure, π^*, which is simultaneously Nash-stable and permutation-stable. To find π^* it suffices to solve the discrete optimization problem

$$\pi^* \in \operatorname*{argmax}_{\pi \in \Pi(N)} \sum_{B \in \pi} v(B),$$

where $P(\pi) = \sum_{B\in\pi} v(B)$ is a simultaneously potential and permutation-potential function. Denote

$$\pi^* \in \operatorname*{argmax}_{\pi \in \Pi(N)} P(\pi), \quad \psi^*_{-i} \in \operatorname*{argmax}_{\psi_{-i} \in \Pi(N \setminus \{i\})} P(\psi_{-i}),$$

$$Y(\psi^*_{-i}) = \Big\{\{A \cup \{i\}, \psi^*_{-i,A}\} | \forall A \in \psi^*_{-i}\Big\}.$$

The coalition structure $\psi, \psi \in Y(\psi^*_{-i})$ is obtained from ψ^*_{-i} by joining player i to some coalition. For example, if $\psi^*_{-1} = \{\{2\}, \{3, 4\}\}$, then

$$Y(\psi^*_{-1}) = \Big\{\big\{\{1\}, \{2\}, \{3, 4\}\big\}, \big\{\{1, 2\}, \{3, 4\}\big\}, \big\{\{2\}, \{1, 3, 4\}\big\}\Big\}.$$

The coalition structure, π^*, which maximizes the potential function of the game (N, H), is punctually stable if

$$H_i(\pi^*) \ge H_i(\psi) \ \forall i \in N, \forall \psi \in Y(\psi^*_{-i}).$$

Statement 2. *The partition, π^*, is punctually stable in the marginal game.*

It is worth making the following remark regarding the marginal game and a certain type of stability. Partition π is a utilitarian order (not strictly) in the cooperative game, (N, v), if $\forall A, B \in \pi \ \forall i \in A$ the following inequalities are fulfilled:

$$\sum_{A \in \pi} v(A) \geq \sum_{A \in \rho} v(A) \ \forall \rho \in \Pi(N).$$

Total social welfare $\sum_{A\in\pi} v(A)$ achieved in π is not less than in ρ. This type of stability is often studied in the literature on telecommunications [3, 21, 22]. $P(\pi) = \sum_{A\in\pi} v(A)$, however, is a potential function for a marginal game. Hence, the utilitarian order is the same as a Nash-stable partition in a marginal game. Now, considering Statement 2 and the result from [9], we can conclude that, in a marginal game, there exists a coalition structure which is simultaneously a utilitarian order, Nash-stable, permutation-stable, and punctually stable.

3.5 Punctuality in singleton congestion games

The singleton congestion game, Γ, is a congestion game in which each player can choose only one resource from the set of resources, M. The payoff function for player $i, i \in N$ has the form

$$u_i(s) = c_{s_i}(k_{s_i}(s)).$$

As a reminder, s_i is the resource chosen by player i, and $k_{s_i}(s)$ is the number of players who have chosen the resource s_i in the profile s. We assume in this section that $c_l \ \forall l \in M$ is a monotone decreasing function.

The profile s^* is a Nash equilibrium in a singleton congestion game if the following inequalities are fulfilled:

$$c_{s_i^*}(k_{s_i^*}(s^*)) \geq c_{s_i}(k_{s_i}(s^*) + 1) \ \forall i \in N \ \forall s_i \in M.$$

We are concerned with the punctuality of the equilibrium profile formed in the following manner. Let $NE(\Gamma) = \{s^* = (s_1^*, s_2^*, ..., s_n^*)\}$, where

$$s_1^* = \underset{s_1 \in M}{\operatorname{argmax}}\, c_{s_1}(1),$$
$$s_i^* = \underset{s_i \in M}{\operatorname{argmax}}\, c_{s_i}(k_{s_i}(s_1^*, ..., s_{i-1}^*) + 1), \ \forall i \in \{2, 3, ..., n\}.$$

Players, one after another, choose the resources that represent their best answers. The best resource is first chosen by player 1, then by player 2, and so forth. The last one to find the best answer is player n. It is easy to see that, after such a sequence of best answers we get an equilibrium strategy profile. The resultant equilibrium is of interest because it is obtained within n steps. Although new players arrive at the resources, old players do not benefit from switching to another resource. This equilibrium is also practical. An equilibrium strategy profile can be obtained simply by asking players to choose, one after another, the resource which best answers their needs. We assume that each player's best response is found uniquely. As a result, we get a unique equilibrium at the end of the sequence of best choices.

Suppose player j is late. Punctual players play the singleton congestion game, Γ_{-j}, with the payoff functions

$$u_i^j(\hat{s}_{-j}) = c_{s_i^j}(k_{s_i^j}(\hat{s}_{-j})).$$

For Γ_{-j}, we have $NE(\Gamma_{-j}) = \{\hat{s}^*_{-j} = (s_1^{j*}, s_2^{j*}, ..., s_{j-1}^{j*}, s_{j+1}^{j*}, ..., s_n^{j*})\}$, where

$$s_1^{j*} = \underset{s_1^j \in M}{\operatorname{argmax}}\, c_{s_1^j}(1),$$

$$s_l^{j*} = \underset{s_l^j \in M}{\text{argmax}}\, c_{s_l^j}(k_{s_l^j}(s_1^{j*}, ..., s_{l-1}^{j*}) + 1),\ \forall l \in \{2, 3, ..., j-1, j+1, ..., n\}.$$

In the absence of player j, punctual players find their best answers one after another and player j finds their best answer last. The next statement demonstrates that the player's tardiness does not increase their payoff.

Statement 3. *The strategy profile $s^* \in NE(\Gamma)$ is punctually stable.*

To achieve punctuality in a congestion game, the players only need to agree to form a strategy profile that maximizes the potential function. This is a corollary from Theorem 1. If, however, we have a singleton game with monotone decreasing functions, the punctuality problem can be approached in a different way. When a player appears in the game, they choose the best resource for themselves. After the last player has made their move, we get a Nash equilibrium. It follows from Statement 3 that being the last to choose is not beneficial for any player. Hence, there is no interest for the players to intentionally miss the start of the game.

Let us consider an example. Let $N = \{1, 2, ..., 6\}, M = \{I, II, III\}$. The functions of payoff on resources are

$$c_I(k) = \frac{16}{k}, c_{II}(k) = \frac{11}{k}, c_{III}(k) = \frac{9}{k}.$$

Players make moves one after another. Player 1 is the first to make a move, player 2 is the second, and so forth. The distribution of players among resources will take the form:

I	II	III
1	2	3
4	5	
6		

Players 1, 4, 6 get $\frac{16}{3}$ as payoff. Players 2, 5 get $\frac{11}{2}$. The payoff of player 3 is 9. Note that player 6 gets the smallest payoff although the resource size is 16.

Suppose the second player has missed the start of the game. Then players 1, 3 — 6 find their best answers one after another and player 2 makes their move last. In this case, the distribution among resources is

I	II	III
1	3	4
5	6	
2		

The sequence of the resources chosen does not change in the case of any player's tardiness. So, the tardy player takes the position of player 6. Since the payoff of player 6 is the smallest (in the game without tardiness), nobody wants to be in this position. Hence, players do not want to be late.

4 Punctuality in aggregative games

4.1 Theorem of punctuality in aggregative games

The aggregative game is a normal-form game, Γ, for which $S_i \subseteq \mathbb{R}^+\ \forall i \in N$ and there exists a function $f_i : S_i \times \mathbb{R}^+ \to \mathbb{R}$ such that the payoff of player i has the form

$$u_i(s) = f_i(s_i, \sum_{k \in N} s_k).$$

The strategy profile $s^*, s^* \in S$ is a Nash equilibrium in the aggregative game, Γ, if $\forall i \in N$ the following inequalities are fulfilled:

$$f_i(s_i^*, s_i^* + \sum_{k \in N \setminus \{i\}} s_k^*) \geq f_i(s_i, s_i + \sum_{k \in N \setminus \{i\}} s_k^*) \ \forall s_i \in S_i.$$

If player $j, j \in N$ has missed the start of the aggregative game, Γ, then an aggregative game, Γ_{-j} forms, in which the payoff of player $i, i \in N \setminus \{j\}$ is

$$u_i^j(\hat{s}_{-j}) = f_i(s_i^j, \sum_{k \in N \setminus \{j\}} s_k^j).$$

Sufficient conditions for an equilibrium to exist in an aggregative game were found in [11]; in the general case, an equilibrium may not exist.

In this section, the sets $NE(\Gamma)$ and $NE(\Gamma_{-j})$ $\forall j \in N$ consist of all Nash equilibria of the games Γ and Γ_{-j}, respectively.

Theorem 2. *Let $NE(\Gamma)$ and $NE(\Gamma_{-j})$ be non-empty sets $\forall j \in N$ and $f_i(s_i, y)$ be a monotonically increasing (decreasing) function by y $\forall i \in N$. Then, the profile $s^*, s^* \in NE(\Gamma)$ is punctually stable iff*

$$\sum_{k \in N \setminus \{i\}} s_k^* \geq (\leq) \sum_{k \in N \setminus \{i\}} s_k^{i*} \ \forall i \in N \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

It follows from Statement 1 that for an arbitrary normal-form game the necessary condition for punctuality is not sufficient. Theorem 2, however, finds a simultaneously necessary and sufficient condition for the punctuality stability of a Nash equilibrium in an aggregative game.

The algorithm for applying the outcome of Theorem 2 to an aggregative game is the following. First, we find an equilibrium in the games Γ and Γ_{-j}. Next, the inequalities from Theorem 2 are checked. If the inequalities are true, then the equilibrium is punctually stable. If the inequalities are not true, then the equilibrium is not punctually stable and players can be found who will benefit from missing the start of the game. The next subsections demonstrate the application of Theorem 2 to aggregative games.

4.2 Punctuality in a Cournot oligopoly

Let the game Γ be a Cournot oligopoly in which the players' payoff functions have the form

$$u_i(s) = s_i \cdot L\left(\sum_{k \in N} s_k\right) - C_i(s_i),$$

where s_i is the product output of player i, $L : \mathbb{R} \to \mathbb{R}$ is a decreasing inverse demand function, and $C_i : S_i \to \mathbb{R}$ is the cost of player i. The oligopoly is an example of an aggregative game for which

$$f_i(s_i, y) = s_i \cdot L(y) - C_i(s_i), u_i(s) = f_i(s_i, \sum_{k \in N} s_k).$$

We assume that $NE(\Gamma) \neq \emptyset, NE(\Gamma_{-j}) \neq \emptyset \ \forall j \in N$. Since $f_i(s_i, y)$ decreases by y, Theorem 2 can be applied. The profile $s^* \in NE(\Gamma)$ is punctually stable iff

$$\sum_{k\in N\setminus\{i\}} s_k^* \leq \sum_{k\in N\setminus\{i\}} s_k^{i*} \ \forall i\in N\ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

The value $\sum_{k\in N\setminus\{i\}} s_k^*$ is the supply of the product in the market in a game with the set of players, N, without the release of the product from player i. The value $\sum_{k\in N\setminus\{i\}} s_k^{i*}$ is the supply of the product in a game with the set of players $N\setminus\{i\}$. Hence, the equilibrium s^* is punctually stable in a Cournot oligopoly if players from the set $N\setminus\{i\}$ in the game Γ_{-i} together generate a product volume greater or equal to that in the game Γ $\forall i\in N$.

Statement 4. *In an oligopoly with linear-quadratic payoff functions*

$$u_i(s) = s_i \cdot \left(p - \sum_{k\in N} s_k\right) - c\cdot s_i,\ p>0, c>0, p-c>0.$$

the Nash equilibrium is punctually stable.

The proof of Statement 4 is based on Theorem 2. For the game from Statement 4, however, Theorem 1 can be applied. The potential function for an oligopoly with linear-quadratic payoff functions has the form

$$P(s) = (p-c)\cdot \sum_{k\in N} s_k - \sum_{k\in N}\sum_{m\in N} s_k \cdot s_m.$$

The potential function for an oligopoly without player j is

$$P_{-j}(\hat{s}_{-j}) = (p-c)\cdot \sum_{k\in N\setminus\{j\}} s_k^j - \sum_{k\in N\setminus\{j\}}\sum_{m\in N\setminus\{j\}} s_k^j \cdot s_m^j.$$

It is easy to show that the payoff functions for players in a Cournot oligopoly with linear-quadratic payoff functions meet the first point of Theorem 1. Hence, the Nash equilibrium that maximizes the potential function is punctually stable.

Suppose a certain game is simultaneously a potential one and an aggregative one. In this case, the outcome of Theorem 2 is wider for this game than the outcome of Theorem 1. Theorem 1 is concerned with the punctuality of the strategy profile which maximizes the potential function, whereas Theorem 2 can be used to test the punctuality of any Nash equilibrium. The next section deals with a potential aggregative game to which only Theorem 2 can be applied.

4.3 Regarding punctuality in a potential aggregative game: data communication in a multichannel system

In this subsection, we simulate the process of data communication in a multichannel system and check the punctuality of the equilibrium.

Let us consider the game $\Gamma = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ in which S_i is the interval $[0, A_i]$ and a player's payoff is

$$u_i(s) = h_i(A_i - s_i) + h(\sum_{k\in N} s_k),$$

where $s_i \in [0; A_i], h:\mathbb{R}\to\mathbb{R}, h_i:\mathbb{R}\to\mathbb{R}$ and h, h_i are monotonically increasing functions $\forall i\in N$.

Let us illustrate the physical meaning of the game, Γ. The number A_i is the amount of the data of player i. There is one common and n private channels for data communication. A certain player is matched bijectively to each private channel. The number s_i is the data of player i which is communicated via the common channel. An $A_i - s_i$ amount of data is communicated via the private channel.

The data communication system is arranged as follows. First, data are communicated via the common channel. After data communication via the common channel is over, private channels are engaged. If $s_i = A_i$, then there is no need to engage the private channel for player i. If $s_i \neq A_i$, then player i sends an $A_i - s_i$ amount of data via the private channel. The availability of the common channel helps lower the load on the data transmission system and reduce the number of the private channels engaged.

The numbers $h(\sum_{k\in N} s_k)$ and $h_i(A_i - s_i)$ are the period of data communication via the common and via the private channel of player i, respectively. The function $u_i(s)$ is the service time for player i. Player i is interested in minimizing $u_i(s)$.

The game, Γ, is a potential game with the potential function

$$P(s) = h(\sum_{k\in N} s_k) + \sum_{k\in N} h_k(A_k - s_k),$$

hence, a Nash equilibrium exists. If player j is late for the start of the game, then the game Γ_{-j} forms, in which the payoff of player $i, i \in N \setminus \{j\}$ is

$$u_i^j(\hat{s}_{-j}) = h_i(A_i - s_i^j) + h(\sum_{k\in N\setminus\{j\}} s_k^j).$$

The game Γ_{-j} is a potential game $\forall j \in N$, for which the potential function has the form

$$P_{-j}(\hat{s}_{-j}) = h(\sum_{k\in N\setminus\{j\}} s_k^j) + \sum_{k\in N\setminus\{j\}} h_k(A_k - s_k^j).$$

Let us check whether the payoff functions for players in Γ fulfill the conditions set out in the first point of Theorem 1,

$$P(s) - P_{-i}(s_{-i}) = u_i(s) - h(\sum_{k\in N\setminus\{i\}} s_k) \neq u_i(s).$$

Hence, Theorem 1 is not applicable to the game in question. However, Γ is an aggregative game,

$$u_i(s) = f_i(s_i, \sum_{k\in N} s_k), f_i(s_i, y) = h_i(A - s_i) + h(y).$$

Statement 5. *Let $h_i = h \ \forall i \in N$ and h is a convex function. Then the following is true:*
1) If

$$\frac{A_i}{\sum_{k\in N} A_k} \geq \frac{1}{n+1} \text{ and } \frac{A_i}{\sum_{k\in N\setminus\{j\}} A_k} \geq \frac{1}{n} \ \forall i, j \in N, i \neq j,$$

then the equilibrium strategies of players in the games Γ and Γ_{-j} are

$$s_k^* = A_k - \frac{\sum_{l\in N} A_l}{n+1}, s_k^{j*} = A_k - \frac{\sum_{l\in N\setminus\{j\}} A_l}{n}.$$

2) The equilibrium of the game Γ from the first point is punctually stable iff

$$\frac{A_i}{\sum_{l\in N\setminus\{i\}} A_l} \geq \frac{1}{n} \ \forall i \in N.$$

The equilibrium in the game Γ does not possess the punctuality property in the general case. Statement 5 finds the sufficient condition for punctuality of the equilibrium with some constraints.

Let $A_1 \geq A_2 \geq ... \geq A_n$. Then,

$$\frac{A_1}{\sum_{k\in N\setminus\{1\}} A_k} \geq \frac{A_2}{\sum_{k\in N\setminus\{2\}} A_k} \geq ... \geq \frac{A_n}{\sum_{k\in N\setminus\{n\}} A_k}.$$

Hence, if the last fraction is not less than $\frac{1}{n}$, then all other fractions are also not less than $\frac{1}{n}$. We can therefore observe the following regarding Statement 5. If the player with the smallest amount of data is disadvantaged by missing the start of the game, then being tardy is disadvantageous for other players as well.

5 Punctuality in symmetric games

5.1 Symmetric mixed equilibrium

Let the strategy sets of players in the game Γ coincide, i.e., $S_1 = S_2 = ... = S_n$. Then, Γ is called symmetric if for any permutation of players σ and for any strategy profile $s = (s_1, s_2, ..., s_n)$ the following equalities hold:

$$u_i(s_1, s_2, ..., s_n) = u_{\sigma(i)}(s_{\sigma(1)}, s_{\sigma(2)}, ..., s_{\sigma(n)})\ \forall i \in N.$$

We assume that $\{1, 2, ..., m\}$ is a finite set of strategies of each player. We denote by $\overline{S}$ the set of mixed strategies of the players,

$$\overline{S} = \left\{ (x_1, x_2, ..., x_m) | \sum_{j=1}^{m} x_j = 1, x_j \geq 0, \forall j \in \{1, 2, ..., m\} \right\}.$$

As has been proven that any finite symmetric game has a symmetric mixed equilibrium $(x^*, x^*, ..., x^*)$,
$x^* \in \overline{S}$ [17], that is, the following inequalities are fulfilled:

$$u_i(x^*, ..., x^*, ..., x^*) \geq u_i(x^*, ..., x, ..., x^*)\ \forall i \in N\ \forall x \in \overline{S}.$$

The aim of this subsection is to obtain the sufficient condition for a mixed symmetric equilibrium to be punctually stable.

Let $u(x, y) = u_1(x, y, ..., y)\ \forall x, y \in \overline{S}$. The pair (x^*, y^*) is the saddle point of the function $u(x, y)$ if the following inequalities are true:

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y)\ \forall x, y \in \overline{S}.$$

Statement 6. *Let Γ_{-j} be a symmetric game $\forall j \in N$ and in games $\Gamma_{-1}, \Gamma_{-2}, ..., \Gamma_{-n}$ there is the same symmetric mixed equilibrium $(y^*, y^*, ..., y^*) \in \mathbb{R}^{n-1}, y^* \in \overline{S}$. Then the symmetric mixed equilibrium x^* of the symmetric game Γ is punctually stable if (x^*, y^*) is the saddle point of the function $u(x, y)$.*

The symmetric equilibrium of a normal-form symmetric game is rarely punctually stable. The next section, however, demonstrates that a pure equilibrium in an extensive-form symmetric game is always punctually stable.

5.2 Games with a move order

A strategic game with a move order is a tuple $E = \langle N, \sigma, \{S\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, where N is the set of players, σ is a permutation of the set N, and S is the set of strategies at the player's position in σ. The function $u_i : S^n \to R$ is the payoff of player i.

We assume that players select their strategies sequentially. The first one to choose a strategy is the player occupying the first position in the permutation σ. The second move is made by the player occupying the second position, and so on. A move order exists in extensive form games, but in such games the order is fixed. In our case, the move order can change. Let $\sigma_1 = (2, 3, ..., n, 1), \sigma_n = (1, 2, ..., n), \sigma_i = (1, 2, ..., i-1, i+1, ..., n, i), i \in \{2, ..., n-1\}$. Denote E_i as a strategic game with the permutation σ_i and $s^*(E_i)$ as a perfect Nash equilibrium.

Definition 2. *The equilibrium $s^*(E_n)$ is punctually stable in the game E_n if*

$$u_i(s^*(E_n)) \geq u_i(s^*(E_i)) \; \forall i \in N.$$

In a punctually stable equilibrium $s^*(E_n)$, no single player benefits from moving from their position in the permutation $\sigma_n = (1, 2, ..., n)$ to the last position. There is a significant difference in Definition 1 and Definition 2. The difference is that in the first case we introduce new games $\Gamma_{-i}, i \in N$ without player i, whereas in the second case the new games include player i.

For the game E with the permutation $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$, we plot an oriented graph G in the following way. Any graph vertex is labeled with the player's number, and edges are labeled by strategies. The root vertex of G is labeled as σ_1. There are $|S|$ edges (σ_1, σ_2) running from the vertex labeled as σ_1, and each of the edges is labeled by the corresponding strategy from S. Each vertex labeled as σ_2 has $|S|$ edges (σ_2, σ_3) labeled as $s, s \in S$, respectively, running from it. There is a total of $|S|^2$ vertices labeled as σ_3. We continue adding more edges until we have added the edges (σ_{n-1}, σ_n), labeled with players' strategies. Next, we add the edges (σ_n, x), labeled by strategies from S, where x is the corresponding vector of players' payoffs. We denote by G_i the graph of the game $E_i, \forall i \in N$.

Let us consider an example. Let $N = \{1, 2, 3\}, S = \{a, b\}$. The graph G_3 of the game E_3 and the payoffs of players are shown in Fig. 1.

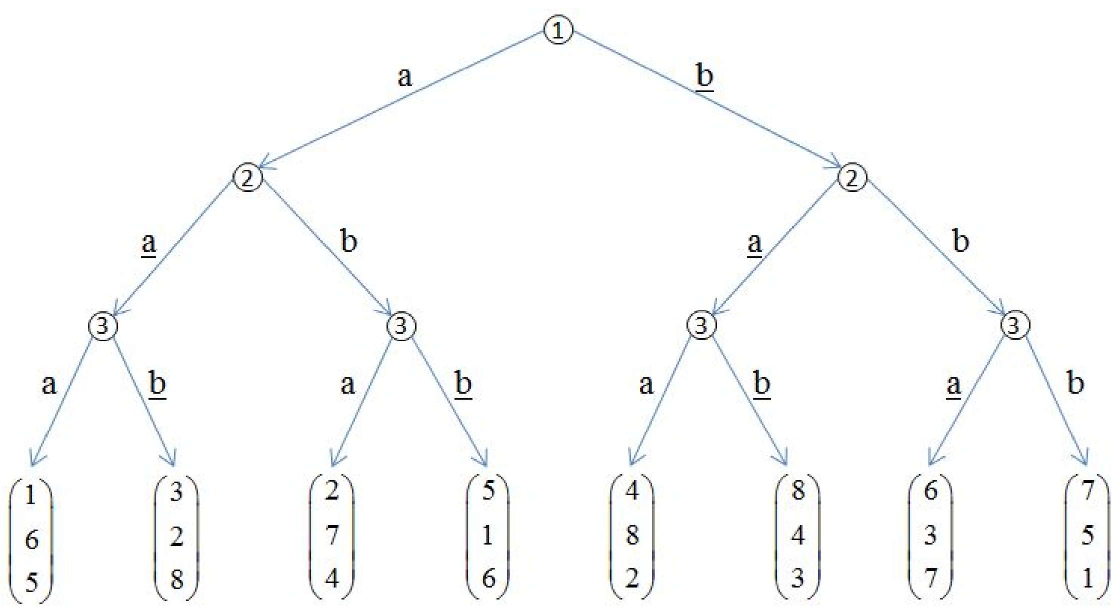

Fig. 1. The graph of the game E_3.

According to extensive form games, player 1 has 2 strategies, a and b. Player 2 has 4 strategies, aa, ab, ba, bb, and player 3 has 8 strategies. In further reasoning, it is enough for us to know only the player's action relative to their position, so we do not use the classical notation. We only need to know the player's choice (a or b) regarding their position.

It follows from Fig. 1 that $u_1(a, a, a) = 1, u_2(a, a, a) = 6, u_3(a, a, a) = 5$. Let us find $s^*(E_3)$. At first, the third player finds their optimal strategy. Then, the second player, knowing the optimal strategy of player 3, finds their best answer. Player 1 then acts in the same manner. The players' best answers are underlined in Fig 1. We have $s^*(E_3) = (b, a, b)$. The payoffs of the players are $u_1(s^*(E_3)) = 8, u_2(s^*(E_3)) = 4, u_3(s^*(E_3)) = 3$.

Let us check whether the equilibrium $s^*(E)$ is punctually stable. Suppose that only player 2 is late. After player 2 shows up, there forms the game E_2, the graph of which is shown in Fig. 2.

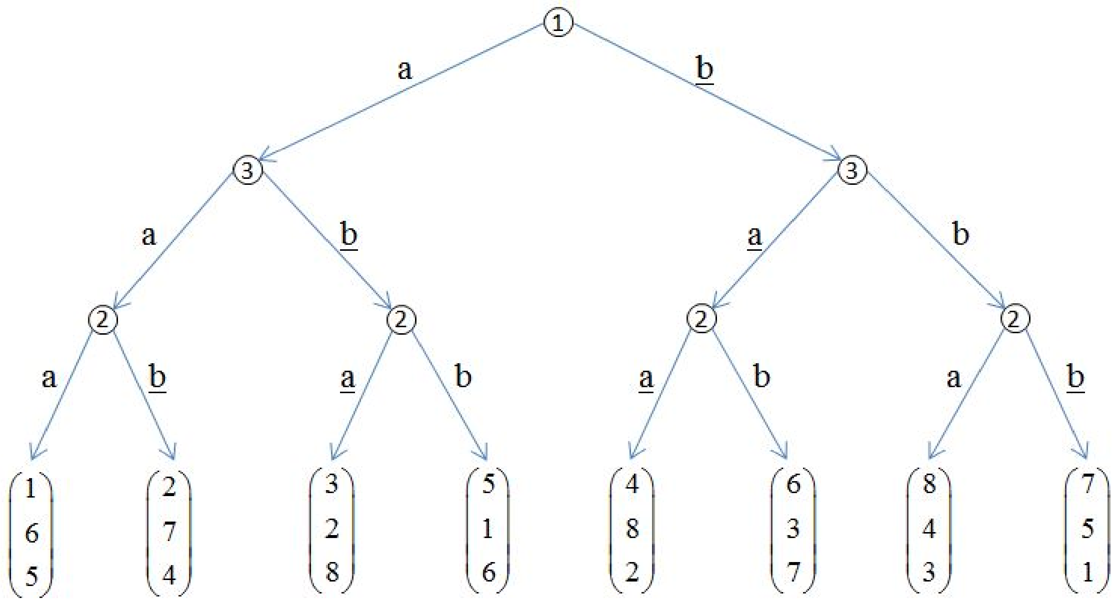

Fig. 2. The graph of the game E_2.

We have $s^*(E_2) = (b, a, a)$. The payoffs of the players have the form $u_1(s^*(E_2)) = 4, u_2(s^*(E_2)) = 8, u_3(s^*(E_2)) = 2$. Since $u_2(s^*(E_2)) > u_2(s^*(E))$, player 2 benefits from being tardy in the game E. Hence, the equilibrium $s^*(E)$ is not punctually stable.

Let us now consider the case where player 1 arrives late. The tree of the game E_1 is shown in Fig. 3.

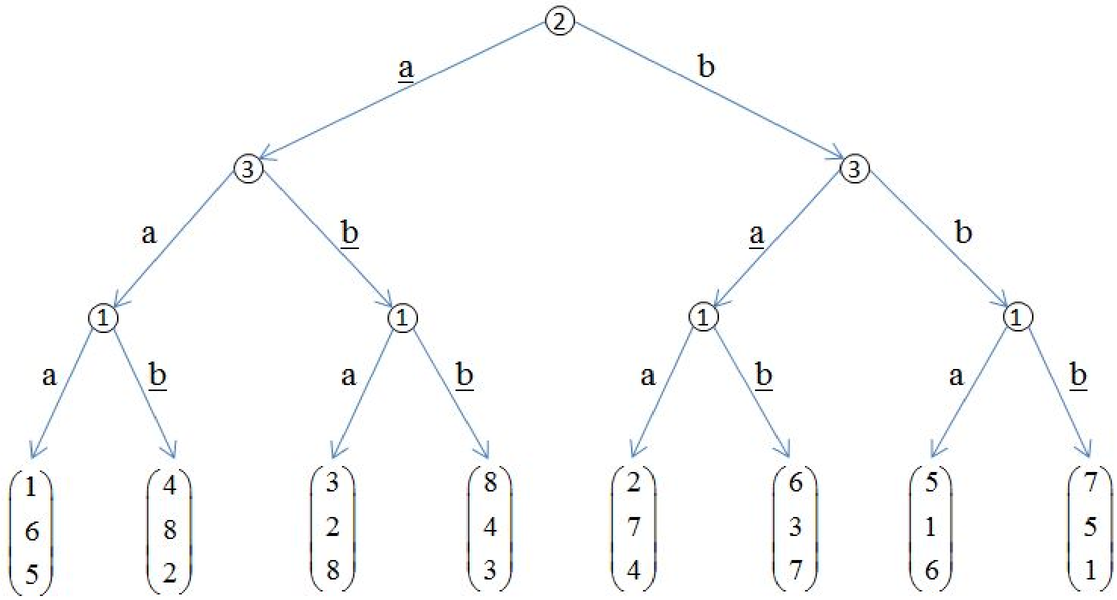

Fig. 3. The graph of the game E_1.

We have $s^*(E_1) = (b, a, b)$, which coincides with $s^*(E_3)$. Hence, being late is disadvantageous for player 1.

In the general case, the profile $s^*(E_3)$ is not punctually stable. For symmetric games, however, the following result was obtained.

Statement 7. *For a symmetric game E_n the profile $s^*(E_n)$ is punctually stable.*

The proof is based on the fact that as a result of the tardiness of player $i, i \in N$ the permutation σ_n is replaced by the permutation σ_i. The trees G_n and G_i are superposed and it is demonstrated that the corresponding players' best answers are also superposed. Using the symmetry property, we then conclude that being late is disadvantageous for the players.

6 Conclusions

To avoid the intentional tardiness of individual players in games, we recommend that the players form a Nash equilibrium that would be punctually stable. In the potential games described in Theorem 1, players need to agree on forming a strategy profile that maximizes the potential function. In this case, no single player would benefit from missing the start of the game alone. Mind that there may exist an equilibrium in the potential games under consideration that does not maximize the potential function. Such an equilibrium cannot be punctually stable.

A solution for singleton congestion games with decreasing functions of payoff on resources can be the queuing of players. The first player to arrive chooses the best resource for oneself. The last one to arrive in the game is the last one to choose a resource. After the last player's move, we get an equilibrium. Such an equilibrium is punctually stable.

A symmetric mixed equilibrium in an arbitrary symmetric game may prove not to be punctually stable. Yet, a pure Nash equilibrium of an extensive-form symmetric game is always punctually stable.

A Nash equilibrium in aggregative games with monotone payoff functions may also not be punctually stable. However, the necessary and sufficient conditions from Theorem 2 can help to identify which player can benefit from missing the start of the game.

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Appendix

Proof of Theorem 1. 1. Let $s^* = (s_i^*, s_{-i}^*) \in NE(\Gamma), \hat{s}_{-i}^* \in NE(\Gamma_{-i})$. The following sequence of equalities is true,

$$u_i(s_i^*, s_{-i}^*) - u_i(s_i, \hat{s}_{-i}^*) = \big(P(s_i^*, s_{-i}^*) - P_{-i}(s_{-i}^*)\big) - \big(P(s_i, \hat{s}_{-i}^*) - P_{-i}(\hat{s}_{-i}^*)\big)$$
$$= \big(P(s_i^*, s_{-i}^*) - P(s_i, \hat{s}_{-i}^*)\big) + \big(P_{-i}(\hat{s}_{-i}^*) - P_{-i}(s_{-i}^*)\big).$$

Since $s^* \in NE(\Gamma)$ and $\hat{s}_{-i}^* \in NE(\Gamma_{-i})$, then $s^* \in \operatorname*{argmax}_{s\in S} P(s)$ and $\hat{s}_{-i}^* \in \operatorname*{argmax}_{\hat{s}\in\prod_{j\in N\setminus\{i\}} S_j} P_{-i}(\hat{s})$. Therefore, $P(s_i^*, s_{-i}^*) - P(s_i, \hat{s}_{-i}^*) \geq 0 \ \forall s_i \in S_i \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i})$ and $P_{-i}(\hat{s}_{-i}^*) - P_{-i}(s_{-i}^*) \geq 0 \ \forall s_{-i}^* \in \prod_{j\in N\setminus\{i\}} S_j$. Hence, $u_i(s_i^*, s_{-i}^*) - u_i(s_i, \hat{s}_{-i}^*) \geq 0 \ \forall s_i \in S_i \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i})$. By definition, s^* is punctually stable

2. We show that the games in question are special cases of the game from point 1.

Let Γ be a congestion game. Then,

$$P(s) - P_{-i}(s_{-i}) = \sum_{l\in \underset{m\in N}{\cup} s_m} \sum_{k=0}^{k_l(s)} c_l(k) - \sum_{l\in \underset{m\in N\setminus\{i\}}{\cup} s_m} \sum_{k=0}^{k_l(s_{-i})} c_l(k)$$

$$= \left(\sum_{l\in s_i}\sum_{k=0}^{k_l(s)} c_l(k) + \sum_{\substack{l\in\cup_{j\in N\setminus\{i\}} s_j \\ l\notin s_i}} \sum_{k=0}^{k_l(s)} c_l(k)\right) - \left(\sum_{l\in s_i}\sum_{k=0}^{k_l(s_{-i})} c_l(k) + \sum_{\substack{l\in\cup_{j\in N\setminus\{i\}} s_j \\ l\notin s_i}} \sum_{k=0}^{k_l(s_{-i})} c_l(k)\right)$$

$$= \sum_{l\in s_i}\left(\sum_{k=0}^{k_l(s)} c_l(k) - \sum_{k=0}^{k_l(s_{-i})} c_l(k)\right) + \sum_{\substack{l\in\cup_{j\in N\setminus\{i\}} s_j \\ l\notin s_i}} \left(\sum_{k=0}^{k_l(s)} c_l(k) - \sum_{k=0}^{k_l(s_{-i})} c_l(k)\right).$$

If $l \in \cup_{j\in N\setminus\{i\}} s_j$ and $l \notin s_i$, then $k_l(s) = k_l(s_{-i})$. This means that the number of players will not change on the resource l, if the player i has not chosen l. In the case of $l \in s_i$ we have $k_l(s) - 1 = k_l(s_{-i})$. Then,

$$P(s) - P_{-i}(s_{-i}) = \sum_{l\in s_i}\left(\sum_{k=0}^{k_l(s)} c_l(k) - \sum_{k=0}^{k_l(s)-1} c_l(k)\right) = \sum_{l\in s_i} c_l(k_l(s)) \equiv u_i(s),$$

where $u_i(s)$ is the payoff function of the player $i, i \in N$ in the congestion game.

Let Γ be a BSI game, then

$$P(s)-P_{-i}(s_{-i}) = \left(\sum_{\substack{m<g \\ m,g\in N}} w_{mg}(s_m, s_g) - \sum_{m\in N} h_m(s_m)\right) - \left(\sum_{\substack{m<g \\ m,g\in N\setminus\{i\}}} w_{mg}(s_m, s_g) - \sum_{m\in N\setminus\{i\}} h_m(s_m)\right)$$

$$= \left(\sum_{\substack{m<g \\ m,g \in N}} w_{mg}(s_m, s_g) - \sum_{\substack{m<g \\ m,g \in N\setminus\{i\}}} w_{mg}(s_m, s_g) \right) - h_i(s_i)$$

$$= \sum_{g<i} w_{gi}(s_g, s_i) + \sum_{g>i} w_{ig}(s_i, s_g) - h_i(s_i) = \sum_{g<i} w_{ig}(s_i, s_g) + \sum_{g>i} w_{ig}(s_i, s_g) - h_i(s_i)$$

$$= \sum_{g \in N\setminus\{i\}} w_{ig}(s_i, s_g) - h_i(s_i) \equiv u_i(s),$$

where $u_i(s)$ is the payoff function of the player i in the BSI game.

Let Γ be a universal potential game. Simplify the difference of potential functions,

$$P(s) - P_{-i}(s_{-i}) = \sum_{\substack{K\subseteq N \\ K\neq\emptyset}} \Phi_K(s_K) - \sum_{\substack{K\subseteq N\setminus\{i\} \\ K\neq\emptyset}} \Phi_K(s_K) = \sum_{\substack{K\subseteq N \\ i\in K}} \Phi_K(s_K) \equiv u_i(s),$$

where $u_i(s)$ is the payoff function of the player i in a potential game.

Thus, the payoff functions of the players in the considered games satisfy the equality from the first point of the theorem. Therefore, the profile of the strategy maximizing the potential function is punctually stable.

$\square$

Proof of Statement 2. The idea of proving the Statement 2 is similar to the proof of Theorem 1.

Let π_{-i} be a partition of π without a player i. Then,

$$P(\pi) - P(\pi_{-i}) = \sum_{B\in\pi} v(B) - \sum_{B\in\pi_{-i}} v(B)$$

$$= \left(v(B(i)) + \sum_{\substack{B\in\pi \\ B\neq B(i)}} v(B) \right) - \left(v(B(i)\setminus\{i\}) + \sum_{\substack{B\in\pi \\ B\neq B(i)}} v(B) \right)$$

$$= v(B(i)) - v(B(i)\setminus\{i\}) = H_i(\pi).$$

Therefore, $\forall i \in N\ \forall \pi \in \Pi(N)$ the following equality is true

$$H_i(\pi) = P(\pi) - P(\pi_{-i}).$$

Let $\psi \in Y(\psi^*_{-i})$, then

$$H_i(\pi^*) - H_i(\psi) = (P(\pi^*) - P(\pi^*_{-i})) - (P(\psi) - P(\psi^*_{-i}))$$
$$= (P(\pi^*) - P(\psi)) + (P(\psi^*_{-i}) - P(\pi^*_{-i}))$$

The partitions π^* and ψ^*_{-i} maximize the potential function on the sets $\Pi(N)$ and $\Pi(N\setminus\{i\})$, respectively. Then $P(\pi^*) - P(\psi) \geq 0, P(\psi^*_{-i}) - P(\pi^*_{-i}) \geq 0$. Hence, $H_i(\pi^*) - H_i(\psi) \geq 0\ \forall i \in N\ \forall \psi \in Y(\psi^*_{-i})$. Therefore, π^* is punctually stable. $\square$

Proof of Statement 3. We investigate the punctuality of the strategy profile, which can be written as follows,

$$s^* = (s_1^*, s_2^*, ..., s_{j-1}^*, s_j^*, s_{j+1}^*, s_{j+2}^*, ..., s_{n-2}^*, s_{n-1}^*, s_n^*).$$

If the player $j, j \in N$ is late, then the players from $N \setminus \{j\}$ find their best answers, and the player j finds their best answer last. The sequence of selected resources will not change if one player is late. Then we will get the following strategy profile

$$s = (s_1^*, s_2^*, ..., s_{j-1}^*, s_n^*, s_j^*, s_{j+1}^*, ..., s_{n-3}^*, s_{n-2}^*, s_{n-1}^*).$$

Therefore, the equilibrium profile is punctually stable if the payoff of player n is smallest.

The strategy profile $(s_1^*, ..., s_{n-1}^*)$ is a Nash equilibrium, hence

$$c_{s_i^*}(k_{s_i^*}(s_1^*, ..., s_{n-1}^*)) \geq c_l(k_l(s_1^*, ..., s_{n-1}^*) + 1) \ \forall i \in \{1, 2, ..., n-1\} \ \forall l \in M.$$

Player n finds the best answer and their payoff is

$$c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*, s_n^*)) = c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*) + 1).$$

The payoff of player i in the profile s^* can be represented as follows,

$$c_{s_i^*}(k_{s_i^*}(s^*)) = \begin{cases} c_{s_i^*}(k_{s_i^*}(s_1^*, ..., s_{n-1}^*)), & s_i^* \neq s_n^*; \\ c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*) + 1), & s_i^* = s_n^*. \end{cases}$$

If $s_i^* = s_n^*$, then

$$c_{s_i^*}(k_{s_i^*}(s^*)) = c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*) + 1) = c_{s_n^*}(k_{s_n^*}(s^*))$$

If $s_i^* \neq s_n^*$, then we put in the above inequality $l = s_n^*$. We get that

$$\begin{aligned} c_{s_i^*}(k_{s_i^*}(s^*)) &= c_{s_i^*}(k_{s_i^*}(s_1^*, ..., s_{n-1}^*)) \geq c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*) + 1) \\ &= c_{s_n^*}(k_{s_n^*}(s_1^*, ..., s_{n-1}^*, s_n^*)) = c_{s_n^*}(k_{s_n^*}(s^*)) \ \forall i \in \{1, 2, ..., n-1\}. \end{aligned}$$

Thus, the payoff of player n is no greater than any other player' payoff. Therefore, it is not profitable for each player to be late. Hence, s^* is punctually stable. □

Proof of Theorem 2. Necessity. Let s^* is punctuality stable Nash equilibrium. We show that the inequalities are satisfied

$$\sum_{k \in N \setminus \{i\}} s_k^* \geq (\leq) \sum_{k \in N \setminus \{i\}} s_k^{i*} \ \forall i \in N \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

The profile s^* is punctually stable, so $\forall i \in N$ the following inequalities are true,

$$f_i(s_i^*, s_i^* + \sum_{k \in N \setminus \{i\}} s_k^*) \geq f_i(s_i, s_i + \sum_{k \in N \setminus \{i\}} s_k^{i*}) \ \forall s_i \in S_i \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

Let it $s_i = s_i^*$. Since $f_i(s_i, y)$ monotonically increases (decreases) by $y \ \forall i \in N$, then

$$f_i(s_i^*, s_i^* + \sum_{k \in N \setminus \{i\}} s_k^*) \geq f_i(s_i^*, s_i^* + \sum_{k \in N \setminus \{i\}} s_k^{i*}) \Rightarrow s_i^* + \sum_{k \in N \setminus \{i\}} s_k^* \geq (\leq) \ s_i^* + \sum_{k \in N \setminus \{i\}} s_k^{i*}$$

$$\Rightarrow \sum_{k \in N \setminus \{i\}} s_k^* \geq (\leq) \sum_{k \in N \setminus \{i\}} s_k^{i*} \ \ \forall i \in N \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

Sufficiency. Let

$$\sum_{k\in N\setminus\{i\}} s_k^* \geq (\leq) \sum_{k\in N\setminus\{i\}} s_k^{i*} \; \forall i \in N \; \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

We show that s^* is punctually stable

The profile s^* is a Nash equilibrium, so $\forall i \in N$ we have

$$f_i(s_i^*, s_i^* + \sum_{k\in N\setminus\{i\}} s_k^*) \geq f_i(s_i, s_i + \sum_{k\in N\setminus\{i\}} s_k^*) \; \forall s_i \in S_i.$$

Given the increase (decrease) of the function $f_i(s_i, y)$ by y and the sufficient condition of the theorem, the right part of the above inequality can be evaluated as follows,

$$f_i(s_i, s_i + \sum_{k\in N\setminus\{i\}} s_k^*) \geq (\geq)\; f_i(s_i, s_i + \sum_{k\in N\setminus\{i\}} s_k^{i*}) \; \forall s_i \in S_i \; \forall \hat{s}_{-i}^* \in NE(\Gamma).$$

Therefore,

$$f_i(s_i^*, s_i^* + \sum_{k\in N\setminus\{i\}} s_k^*) \geq (\geq)\; f_i(s_i, s_i + \sum_{k\in N\setminus\{i\}} s_k^{i*}) \; \forall s_i \in S_i \; \forall \hat{s}_{-i}^* \in NE(\Gamma),$$

that is, s^* is punctually stable. □

Proof of Statement 4. In case the player i is late, we have the game Γ_{-i}, in which the payoff of $j, j \in N \setminus \{i\}$ is

$$u_j^i(\hat{s}_{-i}) = s_j^i \cdot \left(p - \sum_{k\in N\setminus\{i\}} s_k^i\right) - c \cdot s_j^i.$$

The equilibrium in Γ is $s^* = (s_1^*, s_2^*, ..., s_n^*), s_k^* = \frac{p-c}{n+1} \; \forall k \in N$. In an oligopoly without a player i the equilibrium strategy of the player k is $s_k^{i*} = \frac{p-c}{n}$. Let's check the necessary and sufficient conditions of Theorem 2,

$$\sum_{k\in N\setminus\{i\}} s_k^* \leq \sum_{k\in N\setminus\{i\}} s_k^{i*} \Leftrightarrow (n-1)\cdot\frac{p-c}{n+1} \leq (n-1)\cdot\frac{p-c}{n} \Leftrightarrow \frac{1}{n+1} \leq \frac{1}{n}.$$

Since the sufficient condition of Theorem 2 is fulfilled, the equilibrium of the game Γ is punctually stable. □

Proof of Statement 5. 1) Write down the necessary conditions,

$$\frac{\partial u_i}{\partial s_i} = 0 \; \forall i \in N \Rightarrow h(A_i - s_i) = h(\sum_{k\in N} s_k) \; \forall i \in N \Rightarrow s_i^* = A_i - \frac{\sum_{l\in N} A_l}{n+1} \; \forall i \in N.$$

Because of the conditions of the first point, $s_i^* \in [0; A_i] \; \forall i \in N$. is satisfied. Since the function h is convex, it is not difficult to show that $u_i(s)$ is convex by s_i. Therefore, the necessary conditions are sufficient. Similarly, there is an equilibrium in the game $\Gamma_{-j} \; \forall j \in N$.

2) In Theorem 2 and in the game under consideration, the player maximizes and minimizes the payoff, respectively. Since the player minimizes their waiting time and the function h monotonically increases, the equilibrium is punctually stable if and only if

$$\sum_{k\in N\setminus\{i\}} s_k^* \leq \sum_{k\in N\setminus\{i\}} s_k^{i*} \ \forall i \in N \ \forall \hat{s}_{-i}^* \in NE(\Gamma_{-i}).$$

We assume that $NE(\Gamma_{-i})$ consists of a single strategy profile, which is found in the first point. Substitute numeric values in the above inequality, we have

$$\sum_{k\in N\setminus\{i\}} s_k^* \leq \sum_{k\in N\setminus\{i\}} s_k^{i*} \Rightarrow -(n-1)\cdot\frac{\sum_{l\in N} A_l}{n+1} \leq -(n-1)\cdot\frac{\sum_{l\in N\setminus\{i\}} A_l}{n}$$

$$\Rightarrow \frac{\sum_{l\in N} A_l}{\sum_{l\in N\setminus\{i\}} A_l} \geq \frac{n+1}{n} \Rightarrow \frac{A_i}{\sum_{l\in N\setminus\{i\}} A_l} \geq \frac{1}{n} \ \forall i \in N.$$

□

Proof of Statement 6. Since $(y^*, y^*, ..., y^*)$ is a symmetric mixed equilibrium in games $\Gamma_{-j}, j \in \{1, 2, ..., n\}$, then x^* is punctually stable if $\forall x \in \overline{S}$ inequalities

$$u_1(x^*, x^*, ..., x^*) \geq u_1(x, y^*, ..., y^*),$$
$$u_2(x^*, x^*, ..., x^*) \geq u_2(y^*, x, ..., y^*),$$
$$...$$
$$u_n(x^*, x^*, ..., x^*) \geq u_n(y^*, y^*, ..., y^*)$$

are true. Using the symmetry properties of the players' payoff functions, we can swap players 1 and i in places, $i \in N\setminus\{1\}$. Then the above written inequalities are equivalent to the inequality

$$u_1(x^*, x^*, ..., x^*) \geq u_1(x, y^*, ..., y^*) \ \forall x \in \overline{S},$$

which can be converted to the form

$$u(x^*, x^*) \geq u(x, y^*) \ \forall x \in \overline{S}.$$

By condition, (x^*, y^*) is the saddle point of the function $u(x, y)$, that is, the inequalities

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y) \ \forall x, y \in \overline{S}.$$

are true. Therefore,

$$u(x^*, y) \geq u(x, y^*) \ \forall x, y \in \overline{S}.$$

Let $y = x^*$, then

$$u(x^*, x^*) \geq u(x, y^*) \ \forall x \in \overline{S}.$$

Hence, the mixed symmetric equilibrium x^* is punctually stable.

□

Proof of Statement 7. Let E_n be a symmetric game. If the player i is late, then the game E_n is transformed into the game E_i

We overlay the trees G_n and G_i so that the root and leaf vertices coincide with each other, respectively. The corresponding edges of the graphs G_n and G_i, marked with the same strategy, also overlap each other. Due to the overlap of trees, each vertex has a double label. The labels of the vertices of the first level, the second, etc. are $(1,1),(2,2),...,(i-1,i-1),(i,i+1),...,(n-1,n),(n,i)$, respectively.

After overlaying the trees, we find the best answers of the players, starting with leaf vertices. All leaf vertices have the label (n,i). Let for some $s_j, j \neq i$ the best answer of the player n in the game E_n is s_n, that is

$$u_n(s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) \geq u_n(s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s'_n) \; \forall s'_n \in S.$$

Since

$$u_n(s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) = u_i(s_1, s_2, ..., s_{i-1}, s_n, s_i, ..., s_{n-1}),$$
$$u_n(s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s'_n) = u_i(s_1, s_2, ..., s_{i-1}, s'_n, s_i, ..., s_{n-1}),$$

then

$$u_i(s_1, s_2, ..., s_{i-1}, s_n, s_i, ..., s_{n-1}) \geq u_i(s_1, s_2, ..., s_{i-1}, s'_n, s_i, ..., s_{n-1}) \; \forall s'_n.$$

Therefore, the best answer of the player n in the game E_n is the best answer of the player i in the game E_i. Due to the symmetry of the game E_n, a similar property holds for the remaining pairs $(1,1),(2,2),...,(i-1,i-1),(i,i+1),...,(n-1,n)$. As a result of overlapping trees, the best answers of the game E_n are superimposed on the best answers of the game E_i. Means, if $(s_1^*, s_2^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*)$ is the equilibrium of the game E, then $(s_1^*, s_2^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*)$ is the equilibrium in the game E_i. The payoffs of the late player in the equilibrium profiles are the same due to the symmetry of the game E. Therefore, the equilibrium of the game E_n is punctually stable.

□

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