National Research University Higher School of Economics

as a manuscript

Shirokov Dmitry Sergeevich

Some algebro-geometric methods in field theory and other applications

DISSERTATION SUMMARY

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Relevance and degree of the problem development

1) At present, the laws of particle physics are described by quantum gauge theories[1]. The Yang–Mills theory[2] describes three of the four fundamental interactions in nature (electromagnetic, electroweak, and strong interactions). The electromagnetic interaction is described by the Maxwell equations, which are a particular case of the Yang–Mills equations with the gauge (Abelian) Lie group U(1). The electroweak interaction is described by the Yang–Mills equations with the gauge (non-Abelian) Lie group $\mathrm{U}(1)\times\mathrm{SU}(2)$, the strong interaction is described by the gauge (non-Abelian) Lie group SU(3). Problems related to the Yang–Mills equations are in the focus of attention of specialists; it is hoped that the solution of these problems in the future may lead to answers to such fundamental problems of mathematical physics as the problem of the mass defect, the mass spectrum, and an understanding of the confinement mechanism.

Exact solutions of the Yang–Mills equations are important for the development of gauge theory (in particular, for describing the vacuum structure of the theory[3][4] and a more complete understanding of the gauge theory[5]). The complexity of studying the Yang–Mills equations is explained by the non-linearity of these equations. Through the efforts of a number of researchers, some non-trivial classes of particular solutions of the Yang–Mills equations were found: monopoles[6][7][8], instantons[9][10], merons[11], and others. Let us note the well-known ADHM-construction[12], which allows one to completely describe the moduli space of instantons using algebraic and geometric methods. Various particular classes of solutions of the Yang–Mills equations with SU(2) gauge group are presented in the review[13]. This review contains references to a number of other works on exact solutions of the Yang–Mills equations.

Constant (which do not depend on the point x of the Euclidean $\mathbb{R}^n$ or pseudo-Euclidean space $\mathbb{R}^{p,q}$)

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[1] Faddeev L. D., Slavnov A. A., *Gauge field: Introduction to Quantum Theory*. Benjamin/Cummings, Advanced Book Program, 1980.

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solutions of the Yang–Mills equations with zero current are considered in the works of R. Schimming and E. Mundt[14] [15], where the authors write: *"The following problems concerning constant Yang–Mills fields are actual ones in our opinion: Is there a gauge- and coordinate-invariant characterization of those Yang–Mills fields which admit constant potentials with respect to some gauge and some coordinate system? Find as many as possible (in the ideal case: all) constant Yang–Mills fields and classify them!"*. In our work, we give a complete answer to the above problem in the case of the Lie group SU(2). Our results for an arbitrary current are consistent with the results of the mentioned papers for zero current. In particular, it is proved in [14,15] that in the case of zero current $J = 0$ the Yang–Mills field strength is zero $F = 0$ for all constant potentials A satisfying the Yang–Mills equations in the case of Euclidean and Lorentzian signatures. This fact is explicitly confirmed in our work for all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry, besides, in our paper, we present solutions with non-zero field strength $F \neq 0$ and zero current $J = 0$ in all other cases $p \geq 2$ and $q \geq 2$.

Note that the constant solutions of the Yang–Mills equations are essentially non-linear solutions and, from this point of view, are especially important for applications.

Almost all known classes of solutions of the Yang–Mills equations are considered for zero current and, most often, only for the particular cases of Euclidean or Minkowski spaces. Instantons are solutions of the Yang–Mills equations in Euclidean space-time (with imaginary time).

The advantage of this work is that all constant solutions are presented not only for zero current, but also for an arbitrary nonzero current. One of the main results of this work is the representation of all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry for an arbitrary non-Abelian current in an arbitrary pseudo-Euclidean (and Euclidean) n-dimensional space. Using algebraic and geometric methods, we present a general solution of special-type algebraic systems of $3n$ cubic equations with $3n$ unknowns and $3n$ parameters. This problem is solved using the singular value decomposition (SVD) method in the case of Euclidean spaces and the hyperbolic singular value decomposition (HSVD) method in the case of pseudo-Euclidean spaces. Using the invariance of the Yang–Mills equations with respect to (pseudo-)orthogonal coordinate changes and the gauge symmetry, we choose a specific coordinate system and a specific gauge fixing for each constant solution and obtain all constant solutions of the Yang–Mills equations in this coordinate system with this gauge fixing, and then in the original coordinate system with the original gauge fixing. The proposed approach essentially uses the two-sheeted covering of the orthogonal group SO(3) by the spin group $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

[14]Schimming R., *On constant solutions of the Yang–Mills equations* // Arch. Math., 24:2 (1988).

[15]Schimming R., Mundt E., *Constant potential solutions of the Yang–Mills equation* // J. Math. Phys., 33 (1992) 4250.

Some classes of particular solutions of the Yang–Mills–Dirac equations are known[16] [17] [18] [19] [20] [21] [22]. In this work, we present all constant solutions of this system of equations in the Minkowski space using the methods of hyperbolic singular value decomposition and the two-sheeted covering of the orthogonal group by the spin group. We also use the gauge symmetry of the Dirac equation with respect to the pseudo-unitary group $\mathrm{SU}(2,2)$[23].

The Proca equation[24] is a generalization of Maxwell's equations. It is not gauge invariant and describes massive particles with spin 1. The Yang–Mills–Proca equations are considered, for example, in the paper[25]. These equations are at the same time a generalization of the Yang–Mills equations and the Proca equation, they are also not gauge invariant. We present all constant solutions of the system of Yang–Mills–Proca equations in the case of the Lie group $\mathrm{SU}(2)$ in Euclidean and pseudo-Euclidean spaces of arbitrary dimension and signature.

Plane wave solutions of the Yang–Mills equations are considered in the papers[26] [27] [28] [29] [30] [31] [32] [33]. We present all plane wave solutions of the Yang–Mills equations with $\mathrm{SU}(2)$ gauge symmetry and zero current in Euclidean and pseudo-Euclidean spaces of arbitrary finite dimension and signature.

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2) The method of singular value decomposition[34 35] (SVD) was independently proposed by E. Beltrami[36] and C. Jordan[37 38] in 1873 and 1874 respectively. This method is widely used in various applications – computer science, engineering, signal and image processing, process control, least squares fitting of data, etc.

The hyperbolic singular value decomposition (HSVD) method was first proposed by R. Onn, A. O. Steinhardt, and A. W. Bojanczyk in 1989[39] for the special case of complex matrices $A_{n\times N}$ with $n \geq N$, $\mathrm{rank}(A\eta A^\dagger) = \mathrm{rank}(A) = N$ (here and below, the notation corresponds to Theorem 3)[40]. In this particular case, we have $d = 0$, and the matrix Σ is diagonal with all positive diagonal entries. In the next paper[41], the same three authors formulated the statement for the slightly more general case of arbitrary n and N, $\mathrm{rank}(A\eta A^\dagger) = \mathrm{rank}(A) = \min(n, N)$. The third paper by these authors[42] presents a generalization of HSVD to the case $\mathrm{rank}(A\eta A^\dagger) < \mathrm{rank}(A)$. In this generalization, some elements of the matrix Σ are complex. H. Zha in his paper[43] pointed out that this generalization looked unnatural and proposed another generalization using only the matrix Σ with real entries. B. C. Levy[44] presented Zha's statement in a different form, using a different proof. At the same time, Levy's result is weaker: there are additional arbitrary diagonal blocks instead of the identical blocks I_d in the Σ matrix; the explicit form of the matrix $\hat{\eta}$ is not presented; only the case $n \geq N$ is considered. We also note the interesting results of S. Hassi [45], B. N. Parlett[46], and V. Šego[47 48] on other generalizations of SVD to

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[39]Onn R., Steinhardt A. O., Bojanczyk A. W., *The hyperbolic singular value decomposition and applications* // Proceedings of the 32nd Midwest Symposium on Circuits and Systems. 1989.

[40]For consistency throughout this work, we denote the Hermitian conjugate of the matrix A by $A^\dagger$, as is customary in the theory of Yang–Mills equations and other physical applications; mathematicians also denote the Hermitian conjugate by A^* or A^H.

[41]Onn R., Steinhardt A. O., Bojanczyk A. W., *The hyperbolic singular value decomposition and applications* // IEEE Trans. Signal Proc., 39 (1991).

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the hyperbolic case, as well as multilinear singular value decomposition[49]. The hyperbolic singular value decomposition is used in signal and image processing[50], engineering[51], computer science[52 53], physics[54], and others.

In this work, we present a new version of HSVD for an arbitrary complex (or real) matrix. The advantage of the new version of HSVD over the previous versions (of which the most complete version is given by H. Zha) is that it does not use hyperexchange matrices, which do not form a group. Instead of hyperexchange matrices, we use matrices from pseudo-unitary and pseudo-orthogonal groups, which are more natural from the theoretical and practical points of view. Another advantage of the new version is that it contains only three invariant parameters (d, x, and y) and does not contain the other redundant parameters (k and s) from the Zha's result. Also, the new version of HSVD naturally includes the usual SVD as a special case and, thus, is a more general mathematical apparatus. The need to use HSVD instead of SVD arises when we can use only one orthogonal and one pseudo-orthogonal transformation (instead of two orthogonal ones), as, for example, in the case of the Yang–Mills equations in pseudo-Euclidean spaces. Another result of our work is to present the relationship between HSVD and the generalized eigenvalue problem. The new version of HSVD allows us to reduce the problem of computing HSVD to calculating eigenvalues, eigenvectors, and generalized eigenvectors of some auxiliary matrices in the general case. These results generalize known results about the relationship between SVD and the eigenvalue problem.

3) In this work, we actively use and develop methods related to Clifford algebras (or geometric algebras). Clifford algebras were proposed in 1878 by W. K. Clifford[55] as a generalization of Hamilton's quaternions[56] and the exterior Grassmann's algebra[57]. Currently, Clifford algebras are widely used in various sciences – physics, field theory, mechanics, space dynamics, geometry, engineering, robotics, computer science, computer vision, signal and image processing, chemistry, etc. Clifford

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algebras play a special role in the study of the Dirac equation[58 59], which includes the so-called Dirac γ-matrices generating the Clifford algebra of signature $(1,3)$. Currently, major international conferences on applications of Clifford algebras in various sciences are regularly held – International Conference on Clifford Algebras and their Applications in Mathematical Physics (the last conferences were held in 2020, 2017, 2014, 2011), International Conference on Applied Geometric Algebras in Computer Science and Engineering (2021, 2018, 2015, 2012), Alterman Conference on Geometric Algebra and Summer School on Kähler Calculus (2019, 2018, 2017, 2016), Empowering Novel Geometric Algebra for Graphics & Engineering Worksop at the International Conference Computer Graphics International (2022, 2021, 2020, 2019, 2018, 2017), International Conference of Advanced Computational Applications of Geometric Algebra (2022), and others. Note recent surveys[60 61] on modern applications of Clifford algebras in various sciences, in which four articles of the author are discussed [4, 5, 7, 11].

The real Clifford algebras $\mathcal{C}\ell_{p,q}$ are isomorphic to matrix algebras over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{H}$ or $\mathbb{H}\oplus\mathbb{H}$ depending on $p-q \mod 8$ (the so-called Cartan periodicity), the complexified Clifford algebras $\mathbb{C}\otimes\mathcal{C}\ell_{p,q}$ are isomorphic to matrix algebras over $\mathbb{C}$ or $\mathbb{C}\oplus\mathbb{C}$ depending on $n \mod 2$. The advantage of using Clifford algebras in applications instead of the corresponding matrix algebras lies in the richer mathematical apparatus, which allows us to naturally realize various algebraic and geometric structures, spin groups[62 63], spinors[64 65 66 67 68], and others. In this connection, the problem arises of transposing the known matrix methods into the Clifford algebras formalism[69 70 71].

The problem of computing the inverse in Clifford algebras has been studied in many papers in the

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case of low dimensions[72] [73] [74] [75]. The characteristic polynomial in Clifford algebras was considered in the paper[76]. We have proposed explicit formulas for all coefficients of the characteristic polynomial in Clifford algebras in the case of an arbitrary dimension and signature. In particular, formulas for the determinant are obtained that make it possible to compute the inverse in Clifford algebras of arbitrary dimension and signature. Our results are already actively used by other scientists in symbolic computing[77] [78]. We have applied these results to obtain an explicit solution of the Sylvester equation[79] and the Lyapunov equation in Clifford algebras. The Sylvester equation and its special case, the Lyapunov equation, are widely used in control theory, stability theory, image and signal processing, and mathematical modeling.

There is a known geometric analogue (or generalization) of Clifford algebras – Atiyah–Kähler algebras[80] [81] [82] [83] [84] [85]. We use a generalization of the Atiyah–Kähler algebras and the algebra of differential forms, which is called the algebra of h-forms. Instead of the differentials dx^μ, the Clifford field vectors $h^\mu = h^\mu(x)$ are used, which satisfy the anticommutative relations of the Clifford algebra at each point $x \in \mathbb{R}^{p,q}$ of (pseudo-)Euclidean space. This technique is used by us in studying spin connection, presenting a new class of particular solutions of the Yang–Mills equations, and proving the local Pauli's theorem on the connection of two sets of anticommutative elements in Euclidean space. Note that the spin connection[86] is widely used in the theory of the Dirac equation on curved

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pseudo-Riemannian manifolds of signature $(1,3)$.

In this work, various Lie groups and Lie algebras in Clifford algebras are studied. Note the papers[87] [88] on the connection between classical matrix groups and Clifford algebras and a number of other works, including the application of unitary, symplectic, and pseudo-unitary groups in the formalism of Clifford algebras in field theory and physics[89] [90] [91] [92]. We generalize the Hestenes method[93] (which works only in the case of dimension 4) for computing elements of spin groups, using the corresponding elements of orthogonal groups under a two-sheeted covering, to the case of an arbitrary dimension and signature. The method of averaging in Clifford algebras, which was developed in the previous papers of the author, is used.

Aim and objectives of the research

The aim of the work is to develop new algebraic and geometric methods related to the singular value decomposition, hyperbolic singular value decomposition, Clifford algebras and their generalizations, Lie groups and Lie algebras, and their application in the study of various applied problems related to the Yang–Mills, Yang–Mills–Dirac, Yang–Mills–Proca equations, Sylvester and Lyapunov equations, spin groups, spin connection, Pauli's theorem, etc.

Research objectives are:

1. Find all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry with an arbitrary non-Abelian current in an arbitrary Euclidean space $\mathbb{R}^n$.
2. Generalize the hyperbolic singular value decomposition (HSVD) method to an arbitrary case using pseudo-orthogonal and pseudo-unitary matrices. Find a method of computing the HSVD in the general case.
3. Find all constant solutions of the system of Yang–Mills–Proca equations in the case of the Lie group SU(2) in Euclidean and pseudo-Euclidean spaces of arbitrary dimension and signature.
4. Find all plane wave solutions of the Yang–Mills equations with SU(2) gauge symmetry and zero current in Euclidean and pseudo-Euclidean spaces of arbitrary dimension and signature.
5. Solve the problem of computing the inverse, determinant, and other coefficients of the characteristic polynomial in Clifford algebras of arbitrary dimension. Find a basis-free solution of the Sylvester and Lyapunov equations in Clifford algebras of arbitrary dimension.

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6. Find a method for computing elements of spin groups (using the corresponding elements of orthogonal groups under the two-sheeted covering) in the case of an arbitrary dimension and signature.
7. Find an expression for the spin connection of a general form. Based on this expression, present a new class of solutions of the Yang–Mills equations. Generalize Pauli's theorem to the local case when two sets of anticommutative elements depend smoothly on a point of Euclidean space.
8. Give a classification of all Lie groups and Lie algebras of specific type (Lie algebras are direct sums of subspaces of quaternion types) in Clifford algebras; find isomorphisms to classical matrix Lie groups and Lie algebras in the case of arbitrary dimension and signature.
9. Give a complete classification of Lie groups that define inner automorphisms that leave invariant fundamental subspaces of Clifford algebras determined by the reversion and grade involution.

Main results to be defended

1. A classification and an explicit form of all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry with an arbitrary non-Abelian current in an arbitrary Euclidean space $\mathbb{R}^n$ are presented [9] .

2. The hyperbolic singular value decomposition (HSVD) is formulated [6] for the case of an arbitrary complex or real matrix without using hyperexchange matrices and using only pseudo-unitary or pseudo-orthogonal matrices. The computing HSVD is reduced to the calculation of eigenvalues, eigenvectors, and generalized eigenvectors of some auxiliary matrices.

3. All constant solutions of the system of Yang–Mills–Proca equations are presented [1] in the case of the Lie group SU(2) in Euclidean and pseudo-Euclidean spaces of arbitrary dimension and signature.

4. An explicit form of all plane wave solutions of the Yang–Mills equations with SU(2) gauge symmetry and zero current in Euclidean and pseudo-Euclidean spaces of arbitrary dimension and signature is presented [8].

5. The problem of computing the inverse, determinant, and other coefficients of the characteristic polynomial in Clifford algebras of arbitrary dimension is solved [5, 3]. On the basis of these results, a basis-free solution of the Sylvester and Lyapunov equations in the Clifford algebra of arbitrary dimension is presented [4].

6. Based on the method of averaging in Clifford algebras [14], a generalization of the Hestenes method for computing elements of spin groups (using the corresponding elements of orthogonal groups under the two-sheeted covering) is given [11] for the case of arbitrary dimension and signature.

7. An expression for the spin connection of a general form is found [17]. Based on this expression, a new class of solutions of the Yang–Mills equations is presented [13], and a generalization of the Pauli's theorem on the connection of two sets of anticommutative elements is given [10] in the local case, when both sets smoothly depend on a point of Euclidean space.

8. A complete classification of Lie groups and Lie algebras of specific type (Lie algebras are direct sums of subspaces of quaternion types) in Clifford algebras is given [12, 16, 18]; isomorphisms to classical matrix Lie groups and Lie algebras are proved in the case of arbitrary dimension and signature.

9. A complete classification of Lie groups is given [7, 2] that define inner automorphisms that leave invariant fundamental subspaces of Clifford algebras determined by the reversion and grade involution.

Scientific novelty

All the main results of the dissertation listed above were obtained personally by the author and are new.

Research methods

The dissertation uses various methods of algebra, geometry, mathematical physics, computational mathematics, differential geometry, representation theory, theory of Lie groups and Lie algebras. In particular, the following methods are used: the methods of singular value decomposition and hyperbolic singular value decomposition of an arbitrary real or complex matrix, two-sheeted coverings of orthogonal groups by spin groups in the case of arbitrary dimension and signature, the method of averaging from the representation theory of finite groups, the Faddeev – LeVerrier method and the method of Bell polynomials for computing coefficients of characteristic polynomial, etc.

Theoretical and practical significance

The dissertation has theoretical and practical significance. The practical significance of the work is manifested in the use of the results in such applied fields as physics, engineering, computer science, robotics, control theory, stability theory, signal and image processing, mathematical modeling, symbolic computation. The results are used in the study of the problems related to the Yang–Mills equations, Yang–Mills–Dirac equations, Yang–Mills–Proca equations, spin groups, spin connection, the Sylvester and Lyapunov equations, the Pauli's theorem, and others.

Reliability of the obtained results

Reliability of the results of the dissertation is confirmed by the given rigorous mathematical proofs of the corresponding statements.

Approbation of the obtained results

The main results of the dissertation were reported at the following international conferences and symposiums:

1. International Conference "Computer Graphics International 2022", Empowering Novel Geometric Algebra for Graphics & Engineering Workshop (2022, Geneva, Switzerland, online), talk "On Noncommutative Vieta Theorem in Geometric Algebras";
2. The 8th Conference on Applied Geometric Algebras in Computer Science and Engineering (2021, Brno, Czech Republic, online), talk "On Lie groups defining inner automorphisms that leave invariant fundamental subspaces of geometric algebra";
3. International Conference "Marchuk Scientific Readings 2021" (2021, Academgorodok, Novosibirsk, Russia, online), talk "Hyperbolic SVD for obtaining solutions of SU(2) Yang–Mills equations";
4. International Conference "Mathematical Physics, Dynamical Systems and Infinite-Dimensional Analysis 2021" (2021, Dolgoprudny, Russia, online), talk "On constant solutions of the Yang–Mills–Dirac equations";
5. International Conference "Computer Graphics International 2020", Empowering Novel Geometric Algebra for Graphics & Engineering Workshop (2020, Geneva, Switzerland, online), talk "On basis-free solution to Sylvester equation in geometric algebra";
6. International Conference on Mathematical Physics in Memory of Academic V. S. Vladimirov (2020, Moscow, Russia, online), talk "On some equations modeling the Yang–Mills equations";
7. The 12th International Conference on Clifford Algebras and Their Applications in Mathematical Physics (2020, Hefei, China, online), talk "On determinant, other characteristic polynomial coefficients, and inverses in Clifford algebras";
8. 9th International Conference on Mathematical Modeling (2020, Yakutsk, Russia, online), talk "On determinant and inverses in Clifford algebras";
9. International Bogolyubov Conference "Problems of theoretical and mathematical physics" (2019, Moscow — Dubna, Russia), talk "On constant solutions of SU(2) Yang–Mills equations";
10. IX-th International Conference "Solitons, Collapses and Turbulence: Achievements, Developments and Perspectives" (SCT–19) in honor of Vladimir Zakharov's 80th birthday (2019, Yaroslavl, Russia), talk "Classification of all constant solutions of SU(2) Yang–Mills equations with arbitrary current";
11. 4th Alterman Conference on Computational and Geometric Algebra-cum-Workshop on Kähler Calculus (2019, Manipal, India), plenary talk "Method of averaging in Clifford algebras and applications";
12. International Conference "Mathematical Physics, Dynamical Systems and Infinite-Dimensional Analysis" (2019, Dolgoprudny, Russia), talk "On constant solutions of SU(2) Yang–Mills equations";

13. The 2nd JNMP Conference on Nonlinear Mathematical Physics (2019, Santiago, Chile), talk "On constant solutions of SU(2) Yang–Mills equations";
14. International Symposium on Wen-Tsun Wu's Academic Thought and Mathematics Mechanization (2019, Beijing, China), talk "SVD and hyperbolic SVD for obtaining solutions of SU(2) Yang–Mills equations";
15. International Conference on Mathematical Methods in Physics (2019, Marrakesh, Morocco), talk "Method of averaging in Clifford algebras and applications";
16. International Conference "Modern Mathematical Physics. Vladimirov-95" (2018, Moscow, Russia), talk "On some solutions of Yang–Mills equations with SU(2) gauge symmetry";
17. The 7th Conference on Applied Geometric Algebras in Computer Science and Engineering (2018, Campinas, Brazil), talk "Calculation of elements of spin groups using method of averaging in Clifford's geometric algebra";
18. Operators, Functions, and Systems of Mathematical Physics Conference (2018, Baku, Azerbaijan), talk "On some solutions of Yang–Mills equations with SU(2) gauge symmetry";
19. The 11th International Conference on Clifford Algebras and Their Applications in Mathematical Physics (2017, Ghent, Belgium), talk "Yang–Mills equations and Clifford algebras";
20. International Conference on Mathematical Modeling (2017, Yakutsk, Russia), talk "Local generalized Pauli's theorem and one field equation";
21. The 2nd French-Russian Conference "Random Geometry and Physics" (2016, Paris, France), talk "On connection between two sets of higher-dimensional gamma matrices and a primitive field equation";
22. International Conference "New trends in Mathematical and Theoretical Physics" (2016, Moscow, Russia), talk "Covariantly constant solutions of the Yang–Mills equations";
23. VI Russian-Armenian Conference on Mathematical Analysis, Mathematical Physics and Analytical Mechanics (2016, Rostov-on-Don, Russia), talk "Covariantly constant solutions of the Yang–Mills equations";
24. Alterman Conference on Geometric Algebra and Summer School on Kähler Calculus (2016, Brasov, Romania), talk "On some Lie groups containing Spin groups in Clifford algebra";
25. Physical and Mathematical Problems of Advanced Technology Development, devoted to the 50th Anniversary of the Scientific and Educational Division "Fundamental Sciences" of the Bauman Moscow State Technical University (2014, Moscow, Russia), talk "New class of gauge invariant solutions of Yang–Mills equations";
26. The Fourth International Conference on Mathematical Physics and Its Applications (2014, Samara, Russia), talk "Method of contractions in Clifford algebras with applications to the field theory equations";

27. The 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (2014, Tartu, Estonia), talk “The method of contractions in Clifford algebras”.

In addition, the main results of the dissertation were presented at the following seminars

1. Seminar of the Department of Mathematical Physics, Steklov Mathematical Institute, Russian Academy of Sciences (2021, Moscow, chairman: Corr. memb. of RAS I. V. Volovich);
2. Seminar “Infinite dimensional analysis and mathematical physics”, Department of Function theory and functional analysis , Faculty of Mechanics and Mathematics, Lomonosov Moscow State University (2021, 2023, Moscow, chairmen: Prof. O. G. Smolyanov, Prof. E. T. Shavgulidze);
3. V. P. Mikhailov seminar, Steklov Mathematical Institute, Russian Academy of Sciences (2018, Moscow, chairmen: Prof. A. K. Gushchin, Prof. V. V. Zharinov);
4. Seminar “Supercomputer simulations in science and engineering”, MIEM HSE (2023, Moscow, chairman: Prof. L. N. Shchur);
5. Seminar “Perspective Mathematical Technologies”, Laboratory of Mathematical Methods in Natural Science, HSE University (2023, Moscow, chairman: Prof. V. G. Danilov);
6. Seminar “Quantum mathematical physics”, Research and Educational Center of Steklov Mathematical Institute, Russian Academy of Sciences (2013–2015, Moscow, chairmen: Academician V. V. Kozlov, Corr. memb. of RAS I. V. Volovich, Prof. S. V. Kozyrev, Prof. A. S. Trushechkin);
7. Seminar of Mathematics Mechanization Research Center, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (2019, Beijing, China, chairman: Prof. Hongbo Li);
8. Spectral Theory and PDE Seminar, Pontificia Universidad Catolica de Chile (2019, Santiago, Chile, chairman: Prof. Georgi Raikov).

Based on the results of the dissertation, the author delivered the following special courses:

1. semi-annual course at the Steklov Mathematical Institute of Russian Academy of Sciences “Foundations of the theory of Clifford algebras and spinors” (spring 2021);
2. semi-annual course at the Steklov Mathematical Institute of Russian Academy of Sciences “Clifford algebras and field theory equations” (autumn 2014);
3. open optional course at the HSE University “Foundations of the theory of Clifford algebras and spinors” (autumn 2020);
4. elective course for master students at the HSE University (magolego) “Foundations of the theory of Clifford algebras” (spring 2020, spring 2022);
5. lecture course “Introduction to the Theory of Clifford Algebras” at the International Summer School “Hypercomplex Numbers, Lie Groups, and Applications”, Varna, Bulgaria (summer 2017).

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3. grant 18-71-00010 of the Russian Science Foundation "Algebraic and geometric methods in the theory of nonlinear equations of mathematical physics", 2018–2020, head of project;
4. grant 20-11-00009 of the Russian Foundation for Basic Research "Theory of Clifford algebras and spinors", book publishing, 2020, head of project;
5. grant 20-01-003 of the HSE Academic Fund Programme "Computational problems in Clifford algebra theory" (Individual Research Project), 2020–2021, head of project;
6. grant MK-404.2020.1 of the President of the Russian Federation "Some problems of the theory of Clifford algebras arising in mathematical physics", 2020–2021, head of project;
7. grant 21-71-00043 of the Russian Science Foundation "Algebraic and geometric methods in theory of Yang–Mills equations", 2021–2023, head of project;
8. grant 22-00-001 of the HSE Academic Fund Programme "Clifford algebras and applications" (Research and Study Group), 2022, head of project, website of Research and Study Group https://economics.hse.ru/clifford, website of scientific seminar https://economics.hse.ru/clifford/seminar.

Publications

The main results of the dissertation is published in 20 articles [1] – [20] in peer-reviewed scientific journals (all indexed in WoS/Scopus; 12 papers in Q1 – Q2; 14 papers without co-authors)[94]. There is also a peer-reviewed[95] book [21] and 5 peer-reviewed articles [22] – [26] in conference proceedings (all indexed in WoS/Scopus).[96]

The author's publications in peer-reviewed scientific journals included in the international citation system WoS/Scopus:

[1] Shirokov D. S., *Hyperbolic Singular Value Decomposition in the Study of Yang–Mills and Yang–Mills–Proca Equations* // Computational Mathematics and Mathematical Physics, 62:6 (2022), 1007–1019, **Scopus Q2**, https://doi.org/10.1134/S0965542522060136

[94] Requirements of the HSE Dissertation Council: at least 10 articles (WoS/Scopus); at least 4 articles in Q1 – Q2; at least 3 articles without co-authors (or the applicant is the main co-author).

[95] There are two anonymous reviews by RFBR experts.

[96] All the papers [1] – [26] were published after 2015, while the PhD thesis was defended in 2013, so the main results of these papers are not used twice to obtain a degree.

[2] Filimoshina E. R., Shirokov D. S., *On generalization of Lipschitz groups and spin groups* // Mathematical Methods in the Applied Sciences, 26 pp. (2022), **WoS Q1**, https://doi.org/10.1002/mma.8530

[3] Abdulkhaev K. S., Shirokov D. S., *Basis-free Formulas for Characteristic Polynomial Coefficients in Geometric Algebras* // Advances in Applied Clifford Algebras, 32 (2022), 57, 27 pp., **Scopus Q3**, https://doi.org/10.1007/s00006-022-01232-0

[4] Shirokov D. S., *Basis-free solution to Sylvester equation in Clifford algebras of arbitrary dimension* // Advances in Applied Clifford Algebras, 31 (2021), 70, 19 pp., **Scopus Q2**, https://doi.org/10.1007/s00006-021-01173-0

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Peer-reviewed monograph:

[21] Machuk N. G., Shirokov D. S., *Theory of Clifford algebras and spinors* [In Russian], Krasand, Moscow, 2020 (1st edition, RFBF) and 2021 (2nd edition), 560 pp., ISBN 978-5-396-01014-7, http://urss.ru/cgi-bin/db.pl?lang=Ru&blang=ru&page=Book&id=263794

The author's publications in peer-reviewed conference proceedings included in the international citation system WoS/Scopus:

[22] Shirokov D. S., *Clifford algebras and their applications to Lie groups and spinors* // Proceedings of the Nineteenth International Conference on Geometry, Integrability and Quantization (Varna, Bulgaria, June 2017), eds. I. Mladenov and A. Yoshioka, Avangard Prima, Sofia, Bulgaria, 2018, 11–53, **Scopus**, https://doi.org/10.7546/giq-19-2018-11-53

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Personal contribution of the author

All results presented in the dissertation and submitted for defense were obtained by the author personally.

Structure and volume of the dissertation

The dissertation consists of an introduction, 3 chapters, appendices, a conclusion, and a bibliography. The volume of the dissertation without appendices is 249 pages, the bibliography includes 202 titles.

The main content of the work

In **Chapter 1**, we study the Yang–Mills equations with SU(2) gauge symmetry using the methods of singular value decomposition (SVD) and hyperbolic singular value decomposition (HSVD). The systems of Yang–Mills–Dirac and Yang–Mills–Proca equations are also studied.

In **Section 1.1**, the Yang–Mills equations are considered in the pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or, as a special case, the Euclidean space $\mathbb{R}^n$):

$$\partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] =: F_{\mu\nu}, \tag{1}$$

$$\partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] = J^\nu, \tag{2}$$

where $A_\mu \in \mathfrak{g}\mathrm{T}_1$, $J^\nu \in \mathfrak{g}\mathrm{T}^1$, $F_{\mu\nu} = -F_{\nu\mu} \in \mathfrak{g}\mathrm{T}_2$ are tensor fields (potential, current, and strength of the Yang–Mills filed, respectively) with values in the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ (the case of this Lie algebra is considered below), ρ is a real constant (coupling constant). The metric tensor of $\mathbb{R}^{p,q}$ is given by the diagonal matrix

$$\eta = (\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}), \qquad p + q = n. \tag{3}$$

We can represent the potential and current of the Yang–Mills field in the form

$$A^\mu = A^\mu_{\ a}\tau^a, \qquad J^\mu = J^\mu_{\ a}\tau^a, \qquad A^\mu_{\ a}, J^\mu_{\ a} \in \mathbb{R}, \tag{4}$$

using the basis $\tau^a = \frac{\sigma^a}{2i}$, $a = 1, 2, 3$ of the Lie algebra $\mathfrak{su}(2)$, where σ^a, $a = 1, 2, 3$, are the Pauli matrices. From (1), (2), we get

$$\partial_\mu(\partial^\mu A^\nu{}_k - \partial^\nu A^\mu{}_k) - \rho\epsilon^{ab}{}_k(\partial_\mu(A^\mu{}_a A^\nu{}_b) + \eta_{\mu\alpha}A^\alpha{}_a(\partial^\mu A^\nu{}_b - \partial^\nu A^\mu{}_b)) + \rho^2\eta_{\mu\alpha}A^\alpha{}_c A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu{}_k. \quad (5)$$

We can consider (5) as the system of equations for elements of two matrices $A = (A^\mu{}_k)$ and $J = (J^\nu{}_k)$ of size $n \times 3$. Further, we assume that the matrix of current J is given or depends on the unknown matrix A in some given way (for example, in the case of the Yang–Mills–Proca equations, we have $J = -m^2 A$). Using the invariance of the Yang–Mills equations (5) with respect to pseudo-orthogonal transformations of coordinates from the group $\mathrm{O}(p, q)$, the gauge invariance of these equations with respect to transformations from the Lie group $\mathrm{SU}(2)$, and the two-sheeted covering of the special orthogonal group $\mathrm{SO}(3)$ by the spin group $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, we obtain the following theorem.

Theorem 1 *The system of equations* (5) *is invariant under the transformations*

$$A \to \hat{A} = QA, \qquad J \to \hat{J} = QJ, \qquad Q \in \mathrm{O}(p, q), \quad (6)$$

and the transformations

$$A \to \acute{A} = AP + \Omega, \qquad J \to \acute{J} = JP, \qquad P = (p^a_b) \in \mathrm{SO}(3), \quad (7)$$

where

$$\Omega = \Omega(P) = (\omega^\mu{}_d), \qquad \omega^\mu{}_d = \frac{1}{8}\delta_{ac}\epsilon^{bk}{}_d(p^c_k \partial^\mu p^a_b - p^a_k \partial^\mu p^c_b).$$

Combining these two transformations, we conclude that the system (5) is invariant under the transformation

$$A \to QAP + \Omega, \quad J \to QJP, \qquad Q \in \mathrm{O}(p, q), \quad P \in \mathrm{SO}(3), \quad \Omega = \Omega(P). \quad (8)$$

The system of Yang–Mills equations for constant (independent of $x \in \mathbb{R}^{p,q}$) solutions takes the form

$$\rho^2\eta_{\mu\alpha}A^\alpha_c A^\mu_a A^\nu_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu_k \quad (9)$$

with the global symmetry

$$A \to QAP, \qquad J \to QJP, \qquad Q \in \mathrm{O}(p, q), \qquad P \in \mathrm{SO}(3). \quad (10)$$

Multiplying a matrix on the left by a pseudo-orthogonal matrix and on the right by an orthogonal matrix allows you to transform it to a canonical form with a large number of zeros. In the case of the Euclidean space $\mathbb{R}^n$, we use the singular value decomposition, and in the case of the pseudo-Euclidean space $\mathbb{R}^{p,q}$, $p \neq 0$, $q \neq 0$, we use the hyperbolic singular value decomposition. Further, for convenience, we set the coupling constant equal to $\rho = 1$.

In **Section 1.2**, we present all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry in an arbitrary Euclidean space $\mathbb{R}^n$. Using the invariance of the Yang–Mills equations under the orthogonal transformations of coordinates and gauge invariance, we choose a specific system of coordinates and a specific gauge fixing for each constant current and obtain all constant solutions of the Yang–Mills equations in this system of coordinates with this gauge fixing, and then in the original system of coordinates with the original gauge fixing. We use the singular value decomposition method and the method of two-sheeted covering of the orthogonal group by the spin group.

We use the singular value decomposition, namely, for an arbitrary real matrix $A \in \mathbb{R}^{n\times N}$ there exist orthogonal matrices $L \in \mathrm{O}(n)$ and $R \in \mathrm{O}(N)$ such that $L^{\mathrm{T}}AR = D$, where $D = \mathrm{diag}(\mu_1, \dots, \mu_s) \in \mathbb{R}^{n\times N}$, $s = \min(n, N)$, where singular values can always be arranged in descending order $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s \geq 0$.

Theorem 2 *Let $A = (A^\nu{}_k)$, $J = (J^\nu{}_k)$ satisfy the system of $3n$ cubic equations in $\mathbb{R}^n$*

$$A_{\mu c}A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu{}_k, \qquad \nu = 1, \dots, n, \qquad k = 1, 2, 3. \tag{11}$$

Then there exist matrices $P \in \mathrm{SO}(3)$ and $Q \in \mathrm{O}(n)$ such that QAP is diagonal. For all such matrices P and Q, the matrix QJP is also diagonal, and the system (11) takes the following form under the transformation (10):

$$-a_1((a_2)^2 + (a_3)^2) = j_1, \qquad -a_2((a_1)^2 + (a_3)^2) = j_2, \qquad -a_3((a_1)^2 + (a_2)^2) = j_3 \tag{12}$$

in the case $n \geq 3$, and

$$-a_1(a_2)^2 = j_1, \qquad -a_2(a_1)^2 = j_2 \tag{13}$$

in the case $n = 2$. We denote the diagonal elements of the matrix QAP by a_1, a_2, a_3 (or a_1, a_2) and the diagonal elements of the matrix QJP by j_1, j_2, j_3 (or j_1, j_2).

The system (12) has the following symmetry.

Lemma 1 *If the system (12) has a solution (a_1, a_2, a_3), where $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$, then this system also has a solution $(\frac{K}{a_1}, \frac{K}{a_2}, \frac{K}{a_3})$, where $K = (a_1a_2a_3)^{\frac{2}{3}}$.*

Further in the dissertation, the general solution of the written systems of equations is presented in terms of the potential A, the strength F, and the invariant $F^2 := F_{\mu\nu}F^{\mu\nu}$ of the Yang–Mills field. All these expressions depend only on the singular values j_1, j_2, j_3 of the matrix of current J. Thus, a complete classification of all constant solutions of the Yang–Mills equations with SU(2) gauge symmetry in an arbitrary Euclidean space $\mathbb{R}^n$ is given. The number of non-zero solutions depending on the matrix of current for the case of an arbitrary Euclidean space $\mathbb{R}^n$, $n \geq 2$ is given in Table 1;

Table 1: All constant solutions of the Yang–Mills equations with SU(2) gauge symmetry in $\mathbb{R}^n$.

n	rank(J)	additional conditions	rank(A)	A	F	F^2
$n \geq 2$	0		0	$A = 0$	$F = 0$	$F^2 = 0$
$n \geq 2$	0		1	∞ solutions	$F = 0$	$F^2 = 0$
$n \geq 2$	1			$\varnothing$	$\varnothing$	$\varnothing$
$n \geq 2$	2		2	1 solution	$F \neq 0$	$F^2 \neq 0$
$n \geq 3$	3	$j_1 = j_2 = j_3$	3	1 solution	$F \neq 0$	$F^2 \neq 0$
$n \geq 3$	3	$j_1 = j_2 > j_3$	3	2 solutions	$F \neq 0$	two $F^2 \neq 0$
$n \geq 3$	3	$j_3 > j_1 = j_2$	3	2 solutions	$F \neq 0$	one $F^2 \neq 0$
$n \geq 3$	3	all different j_1, j_2, j_3	3	2 solutions	$F \neq 0$	two $F^2 \neq 0$

explicit formulas for A, F, and F^2 in all cases are given in the dissertation. It is shown that the number (0, 1 or 2) of constant solutions of the Yang–Mills equations in terms of the strength of the Yang–Mills field depends on the singular values of the matrix of current.

In **Section 1.3**, we present a new formulation of the hyperbolic singular value decomposition (HSVD) for an arbitrary complex (or real) matrix without using hyperexchange matrices, which do not form a group. In our formulation, we use only matrices from pseudo-unitary (or pseudo-orthogonal) groups. We show that the computing the HSVD in the general case reduces to the calculation of eigenvalues, eigenvectors, and generalized eigenvectors of some auxiliary matrices. The new formulation is more natural and useful for applications. It naturally includes the ordinary singular value decomposition (SVD).

Let us give a formulation for the complex case. To obtain a real analogue, it suffices to replace the Hermitian conjugation $\dagger$ with the transposition T, the unitary $\mathrm{U}(N)$ and pseudo-unitary groups $\mathrm{U}(p,q)$ with the corresponding orthogonal $\mathrm{O}(N)$ and pseudo-orthogonal groups $\mathrm{O}(p,q)$.

Theorem 3 *Assume η (3), $p+q=n$. For an arbitrary matrix $A \in \mathbb{C}^{n\times N}$, there exist $R \in \mathrm{U}(N)$ and $L \in \mathrm{U}(p,q)$ such that*

$$L^\dagger A R = \Sigma, \qquad \Sigma = \left(\begin{array}{cccc} X_x & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & I_d & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & Y_y & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & I_d & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c} X \\ O \\ O \end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c} X \\ O \\ O \end{array}}\right\} q \end{array} \in \mathbb{R}^{n\times N}, \tag{14}$$

where the first block has p rows and the second block has q rows, X_x and Y_y are diagonal matrices of the corresponding sizes x and y with all positive, uniquely defined diagonal entries (up to permutation).

Moreover, choosing R, one can swap columns of the matrix Σ. Choosing L, one can swap rows in individual blocks but not across blocks. Thus, we can always arrange the diagonal elements of the matrices X_x and Y_y in decreasing (or increasing) order.

Here we have

$$d = \operatorname{rank}(A) - \operatorname{rank}(A^\dagger \eta A), \qquad x + y = \operatorname{rank}(A^\dagger \eta A),$$

and x is the number of positive eigenvalues of the matrix $A^\dagger \eta A$, y is the number of negative eigenvalues of the matrix $A^\dagger \eta A$.

The diagonal elements of the matrices X, Y are called hyperbolic singular values.

Theorem 4 *For the matrices A, R, L, and Σ from Theorem 3, we have the following equations:*

$$(A^\dagger \eta A) R = R(\Sigma^{\mathrm{T}} \eta \Sigma), \qquad (\eta A A^\dagger) L = L(\eta \Sigma \Sigma^{\mathrm{T}}). \tag{15}$$

The hyperbolic singular values of the matrix A are the square roots of the modules of the eigenvalues of the matrix $A^\dagger \eta A$. The columns of the matrix R are eigenvectors of the matrix $A^\dagger \eta A$. The columns of the matrix L are eigenvectors of the matrix $\eta A A^\dagger$ (in the case $d = 0$) or eigenvectors and generalized eigenvectors of the matrix $\eta A A^\dagger$ (in the case $d \neq 0$).

In **Section 1.4**, we present all constant solutions of the Yang–Mills equations with $\mathrm{SU}(2)$ gauge symmetry in pseudo-Euclidean space $\mathbb{R}^{p,q}$ of arbitrary dimension and signature.

Using the global symmetry (10) and the hyperbolic singular value decomposition (Theorem 3) for the real case, one of the matrices $A = (A^\nu{}_k)$, $J = (J^\nu{}_k)$ can be reduced to canonical form; the second matrix will have a specific form due to the equations (9). For each constant current, a specific system of coordinates and a specific gauge fixing are selected; the general solution of the corresponding systems of equations is given in terms of the potential A, the strength F, and the invariant F^2 of the Yang–Mills field. All these expressions depend only on the hyperbolic singular values of the matrix of current J and the parameters x_J, y_J, d_J. Thus, a complete classification of all constant solutions of the Yang–Mills equations with $\mathrm{SU}(2)$ gauge symmetry in an arbitrary pseudo-Euclidean space $\mathbb{R}^{p,q}$ is given.

In **Section 1.5**, we present a complete classification of all constant solutions of the Yang–Mills–Dirac equations with $\mathrm{SU}(2)$ gauge symmetry in Minkowski space $\mathbb{R}^{1,3}$. An explicit form of all solutions is presented. We use our results on the hyperbolic singular value decomposition for two different cases (for the real matrix $A \in \mathbb{R}^{4\times 3}$ of the potential of the Yang–Mills field and for the complex matrix $\Psi \in \mathbb{C}^{4\times 2}$).

The system of Yang–Mills–Dirac equations with $\mathrm{SU}(2)$ gauge symmetry has the form

$$\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] =: F_{\mu\nu}, \tag{16}$$

$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = J^\nu := i\Psi^\dagger\gamma^0\gamma^\nu\Psi - \frac{1}{2}\mathrm{tr}(i\Psi^\dagger\gamma^0\gamma^\nu\Psi)I_2, \tag{17}$$

$$i\gamma^\mu(\partial_\mu\Psi + \Psi A_\mu) - m\Psi = 0, \qquad m \geq 0, \tag{18}$$

for unknown $\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^{4\times 2}$ and $A^\mu : \mathbb{R}^{1,3} \to \mathfrak{su}(2)$. The system for constant solutions ($\Psi \in \mathbb{C}^{4\times 2}, A \in \mathbb{R}^{4\times 3}$) of the system (16), (17), (18) takes the form

$$[A_\mu, [A^\mu, A^\nu]] = J^\nu := i\Psi^\dagger\gamma^0\gamma^\nu\Psi - \frac{1}{2}\mathrm{tr}(i\Psi^\dagger\gamma^0\gamma^\nu\Psi)I_2, \tag{19}$$

$$i\gamma^\mu\Psi A_\mu - m\Psi = 0, \qquad m \geq 0. \tag{20}$$

Using the basis of the Lie algebra $\mathfrak{su}(2)$, we write the current and potential of the Yang–Mills field in the form

$$A^\mu = A^\mu_{\ a}\tau^a, \qquad J^\mu = J^\mu_{\ a}\tau^a, \qquad A^\mu_{\ a}, J^\mu_{\ a} : \mathbb{R}^{1,3} \to \mathbb{R}, \qquad A := (A^\mu_{\ a}), \quad J := (J^\mu_{\ a}).$$

We prove that the system (19), (20) is invariant under the global transformation

$$\Psi \to \Psi S, \qquad A \to QAP, \qquad J \to QJP, \qquad S \in \mathrm{SU}(2), \qquad Q \in \mathrm{O}(1,3), \qquad P \in \mathrm{SO}(3), \tag{21}$$

where P and S are related as the two-sheeted covering

$$S^{-1}\tau^a S = p^a_b\tau^b, \qquad P = (p^a_b) \in \mathrm{SO}(3), \qquad \pm S \in \mathrm{SU}(2). \tag{22}$$

Using the hyperbolic singular value decomposition for the matrix A, we obtain the explicit form of all solutions (Ψ, A) of the system of equations (19), (20). We also give explicit formulas for the corresponding current $J = (J^\mu_{\ a})$ and the invariant $F^2 = F_{\mu\nu}F^{\mu\nu}$. Some solutions are found using the pseudo-unitary symmetry of the Dirac equation, namely, the system of equations (19), (20) is invariant with respect to the transformation

$$\Psi \to W^{-1}\Psi, \qquad \gamma^\mu \to W^{-1}\gamma^\mu W, \qquad W \in \mathrm{SU}(2,2). \tag{23}$$

Non-constant solutions of the Yang–Mills–Dirac equations are considered in the form of perturbation theory series, where constant solutions are taken as the zero approximation.

In **Section 1.6**, we present an explicit form of all constant solutions of the system of Yang–Mills–Proca equations in the case of the Lie group $\mathrm{SU}(2)$ in an arbitrary pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or Euclidean space $\mathbb{R}^n$). The Yang–Mills–Proca equations have the form

$$\partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] =: F_{\mu\nu}, \tag{24}$$

$$\partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] + m^2 A^\nu = 0, \tag{25}$$

where $A_\mu \in \mathfrak{g}\mathrm{T}_1$, $J^\nu \in \mathfrak{g}\mathrm{T}^1$, $F_{\mu\nu} = -F_{\nu\mu} \in \mathfrak{g}\mathrm{T}_2$, $\rho \in \mathbb{R}$. These equations differ from the Yang–Mills equations (1), (2) by the presence of the term m^2A^ν with mass $m \in \mathbb{R}$. If the mass is zero $m = 0$, then the equations (24), (25) coincide with the Yang–Mills equations (1), (2) with zero current $J^\nu = 0$. The case $m \neq 0$ is considered below. For constant solutions of the Yang–Mills–Proca equations, we obtain the system of equations

$$\eta_{\mu\alpha} A^{\alpha}{}_{c} A^{\mu}{}_{a} A^{\nu}{}_{b} \epsilon^{ab}{}_{d} \epsilon^{cd}{}_{k} = -\lambda A^{\nu}{}_{k}, \qquad \lambda = \frac{m^2}{\rho^2} > 0. \tag{26}$$

which can be interpreted as the system of Yang–Mills equations for constant solutions with the current $J^\nu = -\lambda A^\nu$ depending on the potential A^ν.

We use the invariance of the system of equations (26) with respect to the global transformation

$$A \to QAP, \qquad Q \in \mathrm{O}(p,q), \qquad P \in \mathrm{SO}(3), \tag{27}$$

and the hyperbolic singular value decomposition for the matrix A. A classification of all solutions of the system of equations (26) in terms of A, F, and F^2 is given.

Non-constant solutions of the Yang–Mills–Proca equations are considered in the form of series of perturbation theory, where the constant solutions of the Yang–Mills–Proca equations are taken as the zero approximation. For the first approximation, systems of linear partial differential equations with constant coefficients are written out, which can be further investigated using well-known numerical methods and methods of the theory of linear partial differential equations.

In **Section 1.7**, solutions of the system of Yang–Mills equations (1), (2) in the form of plane waves are considered:

$$A_\mu = a_\mu e^\rho, \quad \text{where} \quad \rho = \xi_\mu x^\mu, \tag{28}$$

and a_μ are components of a constant covector field with values in the Lie algebra $\mathfrak{g}$. In the case of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, zero current $J = 0$, and (pseudo-)Euclidean space $\mathbb{R}^{p,q}$, $p + q = n$ of arbitrary finite dimension n, we get the system of equations

$$\xi_\mu \xi^\mu a^\nu - \xi^\nu \xi_\mu a^\mu = 0, \tag{29}$$

$$-3\xi_\mu [a^\mu, a^\nu] = 0, \tag{30}$$

$$[a_\mu, [a^\mu, a^\nu]] = 0. \tag{31}$$

An explicit form of all solutions $\{a^\mu, \xi_\nu\}$ of this system of equations is given, the solutions are written out with an appropriate choice of the coordinate system and gauge (any solution of the system of equations under study can be reduced using (pseudo-)orthogonal change of coordinates and gauge fixing to those solutions).

Thus, all plane wave solutions of the system of Yang–Mills equations with SU(2) gauge symmetry in an arbitrary pseudo-Euclidean (or Euclidean) space with zero current are presented. Solutions of the Yang–Mills equations in the form of a sum of waves are also discussed. Three systems of equations are proposed that model the Yang–Mills equations, which may be of interest for further research.

In **Chapter 2**, a number of applied problems of the theory of Clifford algebras are solved.

In **Section 2.1**, we consider the notion of the real Clifford algebra (or geometric algebra) $\mathcal{C}\!\ell_{p,q,r}$. The generators of the algebra $\mathcal{C}\!\ell_{p,q,r}$ satisfy the relations $e_a e_b + e_b e_a = 2\eta_{ab} e$, where

$$\eta = (\eta_{ab}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{0, \ldots, 0}_{r}), \qquad p+q+r=n. \tag{32}$$

An arbitrary element of the real Clifford algebra $U \in \mathcal{C}\!\ell_{p,q,r}$ can be written as

$$U = ue + \sum_{a=1}^{n} u_a e_a + \sum_{a<b} u_{ab} e_{ab} + \cdots + u_{1\ldots n} e_{1\ldots n} = \sum_{A} u_A e_A, \tag{33}$$

where $u, u_a, u_{ab}, \ldots, u_{1\ldots n} \in \mathbb{R}$ are real numbers, e is the identity element, $e_A = e_{a_1 \ldots a_k} = e_{a_1} \cdots e_{a_k}$ are the basis elements, $1 \leq a_1 < \cdots < a_k \leq n$. Here and below, by A we denote an arbitrary ordered multi-index of length from 0 to n.

In the particular case $r = 0$, we get *non-degenerate real Clifford algebra* $\mathcal{C}\!\ell_{p,q} := \mathcal{C}\!\ell_{p,q,0}$. In the particular case $p = q = 0$, $r = n$, we obtain *Grassmann algebra (or exterior algebra)* $\Lambda_n := \mathcal{C}\!\ell_{0,0,n}$. We also consider the *complexified Clifford algebra* $\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q,r}$. An arbitrary element $U \in \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q,r}$ has the form (33), where u, u_a, u_{ab}, $\ldots$, $u_{1\ldots n} \in \mathbb{C}$ are complex numbers. We also consider the *complex Clifford algebra*, for which the matrix (32) can been considered diagonal with p ones and r zeroes on the diagonal, $p + r = n$, $q = 0$. In the case when the reasoning is valid for a real or complex Clifford algebra of arbitrary signature (including degenerate cases), we simply write $\mathcal{C}\!\ell$, thus emphasizing that the reasoning does not depend on the signature.

The Clifford algebra $\mathcal{C}\!\ell$ can be represented as the direct sum $\mathcal{C}\!\ell = \bigoplus_{k=0}^{n} \mathcal{C}\!\ell^k$, where subspaces $\mathcal{C}\!\ell^k := \{\sum_{A:|A|=k} u_A e_A\}$ are linear spans of the basis elements e_A with multi-indices of length $|A| = k$ and are called subspaces of grade k. We consider projection operations onto these subspaces $\langle U \rangle_k = \sum_{A:|A|=k} u_A e_A \in \mathcal{C}\!\ell^k$. We consider three classical operations of conjugation: grade involution $\widehat{\ }$, reversion (anti-involution) $\widetilde{\ }$, and superposition of these two operations $\widehat{\widetilde{\ }}$, which is called the Clifford conjugation (anti-involution):

$$\widehat{U} = \sum_{k=0}^{n} (-1)^k \langle U \rangle_k, \qquad \widetilde{U} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle U \rangle_k, \qquad \widehat{\widetilde{U}} = \sum_{k=0}^{n} (-1)^{\frac{k(k+1)}{2}} \langle U \rangle_k, \tag{34}$$

$$\widehat{UV} = \widehat{U}\widehat{V}, \qquad \widetilde{UV} = \widetilde{V}\widetilde{U}, \qquad \widehat{\widetilde{UV}} = \widehat{\widetilde{V}}\widehat{\widetilde{U}}, \qquad \forall U, V \in \mathcal{C}\!\ell. \tag{35}$$

The Clifford algebra $\mathcal{C}\!\ell$ is a Z_2-graded algebra (or, using physical terminology, a *superalgebra*), namely, it can be represented as a direct sum of even and odd subspaces

$$\mathcal{C}\!\ell = \mathcal{C}\!\ell^{(0)} \oplus \mathcal{C}\!\ell^{(1)}, \quad \mathcal{C}\!\ell^{(j)} := \bigoplus_{k=j \mod 2} \mathcal{C}\!\ell^k = \{U \in \mathcal{C}\!\ell \,|\, \widehat{U} = (-1)^j U\}, \quad \mathcal{C}\!\ell^{(i)} \mathcal{C}\!\ell^{(j)} \subset \mathcal{C}\!\ell^{(i+j) \mod 2}, \quad i, j = 0, 1.$$

The Clifford algebra $\mathcal{C}\!\ell$ can be represented as a direct sum of four subspaces $\mathcal{C}\!\ell = \mathcal{C}\!\ell^{\overline{0}} \oplus \mathcal{C}\!\ell^{\overline{1}} \oplus \mathcal{C}\!\ell^{\overline{2}} \oplus \mathcal{C}\!\ell^{\overline{3}}$, which are defined as

$$\mathcal{C}\!\ell^{\overline{j}} := \bigoplus_{k=j \mod 4} \mathcal{C}\!\ell^k = \{U \in \mathcal{C}\!\ell \,|\, \widehat{U} = (-1)^j U, \quad \widetilde{U} = (-1)^{\frac{j(j-1)}{2}} U\}, \qquad j = 0, 1, 2, 3,$$

and are called *subspaces of quaternion types 0, 1, 2, 3*. In the case of the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$, we denote these four subspaces by $\overline{\mathbf{0}}$, $\overline{\mathbf{1}}$, $\overline{\mathbf{2}}$, and $\overline{\mathbf{3}}$. We have the properties

$$[\overline{\mathbf{k}}, \overline{\mathbf{k}}] \subseteq \overline{\mathbf{2}}, \qquad [\overline{\mathbf{k}}, \overline{\mathbf{2}}] \subseteq \overline{\mathbf{k}}, \qquad k = 0, 1, 2, 3, \qquad [\overline{\mathbf{0}}, \overline{\mathbf{1}}] \subseteq \overline{\mathbf{3}}, \quad [\overline{\mathbf{0}}, \overline{\mathbf{3}}] \subseteq \overline{\mathbf{1}}, \quad [\overline{\mathbf{1}}, \overline{\mathbf{3}}] \subseteq \overline{\mathbf{0}}; \tag{36}$$

$$\{\overline{\mathbf{k}}, \overline{\mathbf{k}}\} \subseteq \overline{\mathbf{0}}, \qquad \{\overline{\mathbf{k}}, \overline{\mathbf{0}}\} \subseteq \overline{\mathbf{k}}, \qquad k = 0, 1, 2, 3, \qquad \{\overline{\mathbf{1}}, \overline{\mathbf{2}}\} \subseteq \overline{\mathbf{3}}, \quad \{\overline{\mathbf{1}}, \overline{\mathbf{3}}\} \subseteq \overline{\mathbf{2}}, \quad \{\overline{\mathbf{2}}, \overline{\mathbf{3}}\} \subseteq \overline{\mathbf{1}} \tag{37}$$

with respect to the operations of commutator $[U, V] := UV - VU$ and anticommutator $\{U, V\} := UV + VU$.

In **Section 2.2** , we solve the problem of computing the inverse in Clifford algebras of arbitrary dimension. We present basis-free formulas of different types (explicit and recursive) for the determinant and all other coefficients of the characteristic polynomial, adjugate, and inverse in real Clifford algebras of arbitrary dimension and signature. The formulas involve only operations of multiplication, summation, and conjugation operations and do not use the corresponding matrix representations. We use methods from the matrix theory and computational methods (the Faddeev–LeVerrier method; the method for computing the coefficients of the characteristic polynomial using Bell polynomials).

Let us consider the complexified Clifford algebra and isomorphisms to matrix algebras

$$\beta : \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q} \to M_{p,q} := \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C}), & \text{if } n \text{ is even}, \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } n \text{ is odd}. \end{cases} \tag{38}$$

We have the exact representation β of the complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$ of the corresponding (minimum) dimension over $\mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C}$ depending on $n \mod 2$. We have $\mathcal{C}\!\ell_{p,q} \subset \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$, and $\mathcal{C}\!\ell_{p,q}$ is isomorphic to some subalgebra of $M_{p,q}$. Then we can consider the representation (of non-minimal dimension)

$$\beta : \mathcal{C}\!\ell_{p,q} \to \beta(\mathcal{C}\!\ell_{p,q}) \subset M_{p,q}. \tag{39}$$

Let us denote the dimension of the representation (39) by $N := 2^{[\frac{n+1}{2}]}$.

Lemma 2 *For the matrix representation β (39), we have*

$$\frac{1}{N} \mathrm{tr}(\beta(U)) = \langle U \rangle_0 \in \mathcal{C}\!\ell^0_{p,q}.$$

Let us introduce the concept of *determinant*

$$\mathrm{Det}(U) := \det(\beta(U)) \in \mathcal{C}\!\ell^0_{p,q} \equiv \mathbb{R}, \qquad U \in \mathcal{C}\!\ell_{p,q} \tag{40}$$

in the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$ using the representation β (39).

Lemma 3 *The determinant (40) is well defined, i.e. does not depend on the choice of the representation β (39).*

We call the *characteristic polynomial* of the element $U \in \mathcal{C}\!\ell_{p,q}$

$$\begin{aligned} \varphi_U(\lambda) &:= \mathrm{Det}(\lambda e - U) = \det(\beta(\lambda e - U)) = \det(\lambda I_N - \beta(U)) \\ &= \lambda^N - C_{(1)}\lambda^{N-1} - \cdots - C_{(N-1)}\lambda - C_{(N)} \in \mathcal{C}\!\ell^0_{p,q}, \end{aligned} \tag{41}$$

where the coefficients of the characteristic polynomial $C_{(j)} = C_{(j)}(U) \in \mathcal{C}\!\ell^0_{p,q} \equiv \mathbb{R}$, $j = 1, \ldots, N$ can be interpreted as scalars or as elements of grade 0. We have $C_{(j)}(U) = c_{(j)}(\beta(U))$, where $c_{(j)}(\beta(U))$ are the coefficients of the characteristic polynomial of the matrix $\beta(U)$. In particular, we have $C_{(N)} = -\mathrm{Det}(U)$ and $C_{(1)} = \mathrm{tr}(\beta(U)) = N\langle U\rangle_0$.

Let us call the *adjugate* of an element $U \in \mathcal{C}\!\ell_{p,q}$ the element $\mathrm{Adj}(U) \in \mathcal{C}\!\ell_{p,q}$ such that $\mathrm{Adj}(U)U = U\mathrm{Adj}(U) = \mathrm{Det}(U)$. The inverse exists $U^{-1} = \frac{\mathrm{Adj}(U)}{\mathrm{Det}(U)}$ if and only if $\mathrm{Det}(U) \neq 0$. The expression $\mathrm{Adj}(U)$ is an analogue of the adjugate of matrix, namely, we have $\mathrm{Adj}(U) = \mathrm{adj}(\beta(U))$.

Theorem 5 *Let us consider an arbitrary element of the Clifford algebra $U \in \mathcal{C}\!\ell_{p,q}$, $n = p+q$. Let us denote $N := 2^{[\frac{n+1}{2}]}$. Consider the following set of elements of the Clifford algebra $U_{(k)}$, $k = 1, \ldots, N$, and the set of scalars $C_{(k)} \in \mathcal{C}\!\ell^0_{p,q} \equiv \mathbb{R}$, $k = 1, \ldots, N$:*

$$U_{(1)} := U, \qquad U_{(k+1)} := U(U_{(k)} - C_{(k)}), \qquad C_{(k)} = \frac{N}{k}\langle U_{(k)}\rangle_0 \in \mathcal{C}\!\ell^0_{p,q} \equiv \mathbb{R}. \tag{42}$$

Then $C_{(k)}$ are the coefficients of the characteristic polynomial,

$$\mathrm{Det}(U) = -U_{(N)} = -C_{(N)} = U(C_{(N-1)} - U_{(N-1)}) \in \mathcal{C}\!\ell^0_{p,q} \equiv \mathbb{R} \tag{43}$$

is the determinant of the element U, and

$$\mathrm{Adj}(U) = C_{(N-1)} - U_{(N-1)} \in \mathcal{C}\!\ell_{p,q} \tag{44}$$

is the adjoint element for U.

Alternatively, using the set of scalars

$$S_{(k)} := (-1)^{k-1}N(k-1)!\langle U^k\rangle_0 \in \mathcal{C}\!\ell^0_{p,q} = \mathbb{R}, \qquad k = 1, \ldots, N, \tag{45}$$

we have

$$C_{(k)} = \frac{(-1)^{k+1}}{k!}B_k(S_{(1)}, S_{(2)}, S_{(3)}, \ldots, S_{(k)}), \qquad k = 1, \ldots, N, \tag{46}$$

$$\mathrm{Det}(U) = -C_{(N)} = \frac{1}{N!}B_N(S_{(1)}, S_{(2)}, S_{(3)}, \ldots, S_{(N)}), \tag{47}$$

$$\mathrm{Adj}(U) = \sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!}U^{N-k-1}B_k(S_{(1)}, S_{(2)}, S_{(3)}, \ldots, S_{(k)}), \tag{48}$$

where we use the complete Bell polynomials with the following two equivalent definitions

$$\begin{aligned} B_k(x_1,\ldots,x_k) \quad &:= \quad \sum_{i=1}^{k}\sum \frac{k!}{j_1!j_2!\cdots j_{k-i+1}!}(\frac{x_1}{1!})^{j_1}(\frac{x_2}{2!})^{j_2}\cdots(\frac{x_{k-i+1}}{(k-i+1)!})^{j_{k-i+1}} \\ &= \quad \det\begin{pmatrix} x_1 & C_{k-1}^1 x_2 & C_{k-1}^2 x_3 & \cdots & x_k \\ -1 & x_1 & C_{k-2}^1 x_2 & \cdots & x_{k-1} \\ 0 & -1 & x_1 & \cdots & x_{k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x_1 \end{pmatrix}, \end{aligned}$$

where the second sum is taken over all sequences $j_1, j_2, \ldots, j_{k-i+1}$ of nonnegative integers satisfying the conditions $j_1 + j_2 + \cdots + j_{k-i+1} = i$ and $j_1 + 2j_2 + 3j_3 + \cdots + (k-i+1)j_{k-i+1} = k$.

Theorem 6 *In the case $n = 1$, we have*

$$C_{(1)} = U + \widehat{U} \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Det}(U) = -C_{(2)} = U\widehat{U} \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Adj}(U) = \widehat{U}, \qquad U^{-1} = \frac{\widehat{U}}{\mathrm{Det}(U)}.$$

In the case $n = 2$, we have

$$C_{(1)} = U + \widehat{\widetilde{U}} \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Det}(U) = -C_{(2)} = U\widehat{\widetilde{U}} \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Adj}(U) = \widehat{\widetilde{U}}, \qquad U^{-1} = \frac{\widehat{\widetilde{U}}}{\mathrm{Det}(U)}.$$

In the case $n = 3$, we have

$$\begin{aligned} &C_{(1)} = U + \widehat{U} + \widetilde{U} + \widehat{\widetilde{U}} \in \mathcal{C}\ell^0_{p,q}, \qquad C_{(2)} = -(U\widetilde{U} + U\widehat{U} + U\widehat{\widetilde{U}} + \widehat{U}\widehat{\widetilde{U}} + \widetilde{U}\widehat{\widetilde{U}} + \widehat{U}\widetilde{U}) \in \mathcal{C}\ell^0_{p,q}, \\ &C_{(3)} = U\widehat{U}\widehat{\widetilde{U}} + U\widetilde{U}\widehat{\widetilde{U}} + U\widehat{U}\widetilde{U} + \widehat{U}\widetilde{U}\widehat{\widetilde{U}} \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Det}(U) = -C_{(4)} = U\widehat{U}\widetilde{U}\widehat{\widetilde{U}} \in \mathcal{C}\ell^0_{p,q}, \\ &\mathrm{Adj}(U) = \widehat{U}\widetilde{U}\widehat{\widetilde{U}}, \qquad U^{-1} = \frac{\widehat{U}\widetilde{U}\widehat{\widetilde{U}}}{\mathrm{Det}(U)}. \end{aligned}$$

In the case $n = 4$, we have

$$\begin{aligned} &C_{(1)} = U + \widehat{\widetilde{U}} + \widehat{U}^\triangle + \widetilde{U}^\triangle \in \mathcal{C}\ell^0_{p,q},\ C_{(2)} = -(U\widehat{\widetilde{U}} + U\widehat{U}^\triangle + U\widetilde{U}^\triangle + \widehat{\widetilde{U}}\widehat{U}^\triangle + \widehat{\widetilde{U}}\widetilde{U}^\triangle + (\widehat{U}\widetilde{U})^\triangle) \in \mathcal{C}\ell^0_{p,q}, \\ &C_{(3)} = U\widehat{\widetilde{U}}\widehat{U}^\triangle + U\widehat{\widetilde{U}}\widetilde{U}^\triangle + U(\widehat{U}\widetilde{U})^\triangle + \widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^\triangle \in \mathcal{C}\ell^0_{p,q}, \qquad \mathrm{Det}(U) = -C_{(4)} = U\widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^\triangle \in \mathcal{C}\ell^0_{p,q}, \\ &\mathrm{Adj}(U) = \widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^\triangle, \quad U^{-1} = \frac{\widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^\triangle}{\mathrm{Det}(U)}. \end{aligned}$$

Here and below, we use the additional operation of conjugation

$$U^\triangle = \sum_{k=0}^{n}(-1)^{C_k^4}\langle U\rangle_k = \sum_{k=0,1,2,3 \mod 8}\langle U\rangle_k - \sum_{k=4,5,6,7 \mod 8}\langle U\rangle_k, \qquad n \geq 4.$$

In **Section 2.3**, we consider the Sylvester equation, a linear equation of the form $AX - XB = C$ for known A, B, C and unknown X. The Sylvester equation and its special case, the Lyapunov equation (with $B=-A^{\dagger}$), are widely used in various applications – image processing, control theory, stability theory, signal processing, mathematical modeling, etc. We study the Sylvester equation in the Clifford algebras $\mathcal{C}\!\ell_{p,q}$ and present a basis-free solution of this equation in the case of arbitrary dimension $n=p+q$.

Let us first present statements for the particular cases $n=4,5$ with explicit formulas, and then a statement for arbitrary n with recursive formulas for the solution.

Theorem 7 *Let us consider the Sylvester equation in the algebra $\mathcal{C}\!\ell_{p,q}$, $p+q=4$*

$$AX-XB=C, \tag{49}$$

for known $A,B,C\in\mathcal{C}\!\ell_{p,q}$ and uknown $X\in\mathcal{C}\!\ell_{p,q}$.

If $Q:=D\widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^{\triangle}\neq 0$, then

$$X=\frac{\widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^{\triangle}F}{Q}, \tag{50}$$

where

$$\begin{aligned}
D &:= A^4-A^3(B+\widehat{\widetilde{B}}+\widehat{B}^{\triangle}+\widetilde{B}^{\triangle})+A^2(B\widehat{\widetilde{B}}+B\widehat{B}^{\triangle}+B\widetilde{B}^{\triangle}+\widehat{\widetilde{B}}\widehat{B}^{\triangle}+\widehat{\widetilde{B}}\widetilde{B}^{\triangle}+(\widehat{B}\widetilde{B})^{\triangle}) && (51)\\
&- A(B\widehat{\widetilde{B}}\widehat{B}^{\triangle}+B\widehat{\widetilde{B}}\widetilde{B}^{\triangle}+B(\widehat{B}\widetilde{B})^{\triangle}+\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle})+B\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle},\\
F &:= A^3C-A^2C(\widehat{\widetilde{B}}+\widehat{B}^{\triangle}+\widetilde{B}^{\triangle})+AC(\widehat{\widetilde{B}}\widehat{B}^{\triangle}+\widehat{\widetilde{B}}\widetilde{B}^{\triangle}+(\widehat{B}\widetilde{B})^{\triangle})-C\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle}. && (52)
\end{aligned}$$

Theorem 8 *Let us consider the Sylvester equation in the algebra $\mathcal{C}\!\ell_{p,q}$, $p+q=5$,*

$$AX-XB=C \tag{53}$$

for known $A,B,C\in\mathcal{C}\!\ell_{p,q}$ and unknown $X\in\mathcal{C}\!\ell_{p,q}$.

If $Q:=D\widetilde{D}(\widehat{D}\widehat{\widetilde{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widehat{\widetilde{D}})^{\triangle})^{\triangle}\neq 0$, then

$$X=\frac{\widetilde{D}(\widehat{D}\widehat{\widetilde{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widehat{\widetilde{D}})^{\triangle})^{\triangle}F}{Q}, \tag{54}$$

where

$$\begin{aligned}
D &:= A^4-A^3(B+\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})+A^2(B\widetilde{B}+B\widehat{B}^{\triangle}+B\widetilde{\widehat{B}}^{\triangle}+\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle}) && (55)\\
&- A(B\widetilde{B}\widehat{B}^{\triangle}+B\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+B(\widehat{B}\widetilde{\widehat{B}})^{\triangle}+\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle})+B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle},\\
F &:= A^3C-A^2C(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})+AC(\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle})-C\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}. && (56)
\end{aligned}$$

Theorem 9 *Let us consider the Sylvester equation in the algebra $\mathcal{C}\!\ell_{p,q}$, $p+q=n$,*

$$AX-XB=C \tag{57}$$

for known $A, B, C \in \mathcal{C}\!\ell_{p,q}$ and unknown $X \in \mathcal{C}\!\ell_{p,q}$.

Let us denote $N := 2^{[\frac{n+1}{2}]}$. If $Q := d_{(N)} \neq 0$, then

$$X = \frac{(D_{(N-1)} - d_{(N-1)})F}{Q}, \qquad \textit{where} \quad D := -\sum_{j=0}^{N} A^{N-j} b_{(j)}, \quad F := \sum_{j=1}^{N} A^{N-j} C(B_{(j-1)} - b_{(j-1)}), \tag{58}$$

and the following expressions are defined recursively ($k = 1, \ldots, N$):

$$b_{(k)} = \frac{N}{k}\langle B_{(k)}\rangle_0, \quad B_{(k+1)} = B(B_{(k)} - b_{(k)}), \quad B_{(0)} := 0, \quad b_{(0)} := -1,$$
$$d_{(k)} = \frac{N}{k}\langle D_{(k)}\rangle_0, \quad D_{(k+1)} = D(D_{(k)} - d_{(k)}), \quad D_{(0)} := 0, \quad d_{(0)} := -1.$$

Note that D (58) is the characteristic polynomial of the element B with the substitution of A.

In **Section 2.4**, we present a method for computing elements of spin groups in the case of arbitrary dimension using the corresponding elements of orthogonal groups under a two-sheeted covering. This method generalizes the Hestenes method, which works only in the case of dimension 4. We use the method of averaging in Clifford algebras proposed earlier by the author.

Let us consider the pseudo-orthogonal group $\mathrm{O}(p,q)$, $p+q=n$:

$$\mathrm{O}(p,q) := \{P \in \mathrm{Mat}(n,\mathbb{R}) \,|\, P^{\mathrm{T}}\eta P = \eta\}. \tag{59}$$

Denote by $p^A_B = p^{a_1\ldots a_k}_{b_1\ldots b_k}$, $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$, minors of the matrix $P = (p^a_b)$. In the case of empty multi-indices A and B, the corresponding minor is equal to 1 by definition. The group $\mathrm{O}(p,q)$ has the following subgroups:

$$\mathrm{O}_+(p,q) := \{P \in \mathrm{O}(p,q) \,|\, p^{1\ldots p}_{1\ldots p} \geq 1\}, \qquad \mathrm{O}_-(p,q) := \{P \in \mathrm{O}(p,q) \,|\, p^{p+1\ldots n}_{p+1\ldots n} \geq 1\},$$
$$\mathrm{SO}(p,q) := \{P \in \mathrm{O}(p,q) \,|\, \det P = 1\}, \qquad \mathrm{SO}_+(p,q) := \{P \in \mathrm{SO}(p,q) \,|\, p^{1\ldots p}_{1\ldots p} \geq 1\}.$$

For example, in the particular case of Minkowski space we have the following groups: Lorentz group $\mathrm{O}(1,3)$, special (or proper) Lorentz group $\mathrm{SO}(1,3)$, orthochronous Lorentz group $\mathrm{O}_+(1,3)$, orthochorous Lorentz group $\mathrm{O}_-(1,3)$, special orthochronous Lorentz group $\mathrm{SO}_+(1,3)$.

The subset of all invertible elements of any set M is denoted by $M^\times$. Let us consider the Lipschitz group

$$\Gamma^\pm_{p,q} := \{S \in \mathcal{C}\!\ell^{(0)\times}_{p,q} \cup \mathcal{C}\!\ell^{(1)\times}_{p,q} \,|\, S\mathcal{C}\!\ell^1_{p,q}S^{-1} \subset \mathcal{C}\!\ell^1_{p,q}\} = \{v_1 \cdots v_k \,|\, v_1, \ldots, v_k \in \mathcal{C}\!\ell^{1\times}_{p,q}\}$$

and its even subgroup

$$\Gamma^+_{p,q} := \{S \in \mathcal{C}\!\ell^{(0)\times}_{p,q} \,|\, S\mathcal{C}\!\ell^1_{p,q}S^{-1} \subset \mathcal{C}\!\ell^1_{p,q}\} = \{v_1 \cdots v_{2k} \,|\, v_1, \ldots, v_{2k} \in \mathcal{C}\!\ell^{1\times}_{p,q}\} \subset \Gamma^\pm_{p,q}.$$

The following groups are called *spin groups*:

$$\mathrm{Pin}(p,q) := \{S \in \Gamma^\pm_{p,q} \,|\, \widetilde{S}S = \pm e\} = \{S \in \Gamma^\pm_{p,q} \,|\, \widehat{\widetilde{S}}S = \pm e\},$$

$$
\begin{aligned}
\mathrm{Pin}_+(p,q) &:= \{S\in\Gamma^{\pm}_{p,q}\,|\,\widehat{\widetilde{S}}S=+e\}, \qquad \mathrm{Pin}_-(p,q) \;:=\; \{S\in\Gamma^{\pm}_{p,q}\,|\,\widetilde{S}S=+e\},\\
\mathrm{Spin}(p,q) &:= \{S\in\Gamma^{+}_{p,q}\,|\,\widetilde{S}S=\pm e\}=\{S\in\Gamma^{+}_{p,q}\,|\,\widehat{\widetilde{S}}S=\pm e\},\\
\mathrm{Spin}_+(p,q) &:= \{S\in\Gamma^{+}_{p,q}\,|\,\widetilde{S}S=+e\}=\{S\in\Gamma^{+}_{p,q}\,|\,\widehat{\widetilde{S}}S=+e\}.
\end{aligned}
\tag{60}
$$

Let us consider the twisted adjoint action

$$
\check{\mathrm{ad}}:\mathcal{C}\!\ell^{\times}_{p,q}\to \mathrm{End}\mathcal{C}\!\ell_{p,q},\qquad S\to \check{\mathrm{ad}}_S,\qquad \check{\mathrm{ad}}_S U=\widehat{S}US^{-1},\qquad U\in\mathcal{C}\!\ell_{p,q}.
$$

The following homomorphisms are surjective with the kernel $\{\pm 1\}$:

$$
\begin{gathered}
\check{\mathrm{ad}}:\mathrm{Pin}(p,q)\to \mathrm{O}(p,q),\quad \check{\mathrm{ad}}:\mathrm{Pin}_+(p,q)\to \mathrm{O}_+(p,q),\quad \check{\mathrm{ad}}:\mathrm{Pin}_-(p,q)\to \mathrm{O}_-(p,q),\\
\check{\mathrm{ad}}:\mathrm{Spin}(p,q)\to \mathrm{SO}(p,q),\quad \check{\mathrm{ad}}:\mathrm{Spin}_+(p,q)\to \mathrm{SO}_+(p,q).
\end{gathered}
$$

For each matrix $P=(p^a_b)\in \mathrm{O}(p,q)$, there are exactly two elements $\pm S\in \mathrm{Pin}(p,q)$ such that $\widehat{S}e_aS^{-1}=p^b_ae_b$; similarly for the other groups under consideration. The spin groups (60) are two-sheeted coverings of the corresponding orthogonal groups.

Theorem 10 *Let us consider the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$ with even $n=p+q$. Let $P\in \mathrm{SO}(p,q)$ be an orthogonal matrix such that*

$$
M:=\sum_{A,B} p^B_A e_B e^A \neq 0. \tag{61}
$$

Then we can compute the elements $\pm S\in \mathrm{Spin}(p,q)$ that correspond to the matrix $P=(p^b_a)\in \mathrm{SO}(p,q)$ under the two-sheeted covering $Se_aS^{-1}=p^b_ae_b$ in the following way:

$$
S=\pm\frac{M}{\sqrt{\alpha\,\widetilde{M}M}}, \tag{62}
$$

where $\widetilde{M}M\in \mathrm{Cen}(\mathcal{C}\!\ell_{p,q})=\mathcal{C}\!\ell^0_{p,q}\cong\mathbb{R}$ and the sign $\alpha:=\mathrm{sign}(p^{1\ldots p}_{1\ldots p})e=\mathrm{sign}(p^{p+1\ldots n}_{p+1\ldots n})e=\widetilde{S}S=\pm e$ depends on the component of the group $\mathrm{SO}(p,q)$.

Theorem 11 *Let us consider the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$ with odd $n=p+q$. Let $P\in \mathrm{O}(p,q)$ be an orthogonal matrix such that*

$$
M:=\sum_{A,B}(\det P)^{|A|} p^B_A e_B e^A \neq 0. \tag{63}
$$

Then we can compute the elements $\pm S\in \mathrm{Pin}(p,q)$ that correspond to the matrix $P=(p^b_a)\in \mathrm{O}(p,q)$ under the two-sheeted covering $\widehat{S}e_aS^{-1}=p^b_ae_b$ in the following way:

$$
S=\pm\frac{M}{\sqrt{\alpha\,\widetilde{M}M}}, \tag{64}
$$

where
$$\widetilde{M}M\in\mathcal{C}\!\ell^0_{p,q}\subset\mathrm{Cen}(\mathcal{C}\!\ell_{p,q})\cong\begin{cases}\mathbb{R}\oplus\mathbb{R}, & \text{if } p-q=1 \mod 4;\\ \mathbb{C}, & \text{if } p-q=3 \mod 4,\end{cases}$$
and the sign
$$\alpha:=\begin{cases}\mathrm{sign}(p^{p+1\ldots n}_{p+1\ldots n})e=\widetilde{S}S=\pm e, & \text{if } n=1 \mod 4;\\ \mathrm{sign}(p^{1\ldots p}_{1\ldots p})e=\widehat{\widetilde{S}}S=\pm e, & \text{if } n=3 \mod 4,\end{cases} \tag{65}$$
depends on the component of the group $\mathrm{O}(p,q)$.

Let us consider the particular case of Theorems 10 and 11 for elements of the group $\mathrm{Spin}_+(p,q)$ and the corresponding group $\mathrm{SO}_+(p,q)$. The elements of the group $\mathrm{Spin}_+(p,q)$ are often called rotors and are widely used in geometric algebra. Let
$$Se_a\widetilde{S}=\beta_a,\qquad \widetilde{S}=S^{-1},$$
where two frames e_a and β_a, $a=1,\ldots,n$ are related by rotation. If $M=\beta_Ae^A=e+\beta_ae^a+\cdots+\beta_{1\ldots n}e^{1\ldots n}\neq 0$, then
$$S=\pm\frac{M}{\sqrt{\widetilde{M}M}}.$$
The presented explicit formulas for computing the elements of spin groups generalize the previously well-known formulas that worked only in the case of small dimensions $n\leq 4$.

In **Section 2.5**, equations for spin connection of a general form are studied. A general solution of these equations is presented.

Let us consider the pseudo-Euclidean space $\mathbb{R}^{k,l}$ of dimension $\dim\mathbb{R}^{k,l}=k+l=m\geq 1$ with Cartesian coordinates x^μ, $\mu=1,\ldots,m$. The metric tensor of $\mathbb{R}^{k,l}$ is given by the diagonal matrix $g=(g_{\mu\nu})=(g^{\mu\nu})$ with k ones and l minus ones on the diagonal. Consider the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$, $p+q=n\geq 1$ with the generators e_a, $a=1,\ldots,n$, which satisfy the relations $e_ae_b+e_be_a=2\eta_{ab}e$ with the diagonal matrix $\eta=(\eta_{ab})=(\eta^{ab})$ with p ones and q minus ones on the diagonal.

We use the notation $\mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s$ for the set of tensor fields $U^\Phi_\Psi=U^{\phi_1\ldots\phi_r}_{\psi_1\ldots\psi_s}(x):\mathbb{R}^{k,l}\to\mathcal{C}\!\ell_{p,q}$ with values in the Clifford algebra:
$$U^\Phi_\Psi=u^{\phi_1\ldots\phi_r}_{\psi_1\ldots\psi_s}(x)e+u^{\phi_1\ldots\phi_r a}_{\psi_1\ldots\psi_s}(x)e_a+\cdots+u^{\phi_1\ldots\phi_r 1\ldots n}_{\psi_1\ldots\psi_s}(x)e_{1\ldots n}=u^{\Phi A}_\Psi(x)e_A\in\mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s,\quad u^{\Phi A}_\Psi:\mathbb{R}^{k,l}\to\mathbb{R}, \tag{66}$$
where we mean summation over an ordered multi-index A. We denote the multi-index $\phi_1\ldots\phi_r$ by Φ, the multi-index $\psi_1\ldots\psi_s$ by Ψ, and their lengths by $|\Phi|=r$, $|\Psi|=s$. We can raise and lower Greek indices using the matrix $g=(g^{\mu\nu})$ and raise and lower Latin indices using the matrix $\eta=(\eta^{ab})$. Let us consider the set of smooth functions $h_a:\mathbb{R}^{k,l}\to\mathcal{C}\!\ell_{p,q}$ with values in the Clifford algebra
$$h_a(x)=y_a(x)e+y^b_a(x)e_b+\cdots+y^{1\ldots n}_a(x)e_{1\ldots n}=y^A_a(x)e_A, \tag{67}$$

that satisfy the relations

$$h_a(x)h_b(x) + h_b(x)h_a(x) = 2\eta_{ab}e, \qquad a, b = 1, \ldots, n, \qquad \forall x \in \mathbb{R}^{k,l}. \tag{68}$$

In the case of odd $n = p + q$, we also require an additional condition $\langle h_1 \cdots h_n \rangle_0 = 0$ to get the independent elements h_A. The set $\{h_A(x)\} = \{e, h_a(x), \ldots, h_{1\ldots n}(x)\}$ is a basis of the algebra $\mathcal{C}\!\ell_{p,q}\mathrm{T}$ of smooth functions with values in the Clifford algebra. We denote the subspaces of fixed grades with respect to the new basis by

$$\mathcal{C}\!\ell[h]^k_{p,q}\mathrm{T} = \{\sum_{A:\,|A|=k} u^A(x)h_A(x)\}, \qquad k = 0, 1, \ldots, n. \tag{69}$$

The operation of projection onto the subspace $\mathcal{C}\!\ell[h]^k_{p,q}\mathrm{T}$ is denoted by $\pi[h]_k$.

Theorem 12 *The set*

$$\mathcal{C}\!\ell^{\circledS}_{p,q} := \mathcal{C}\!\ell_{p,q} \setminus \mathrm{Cen}(\mathcal{C}\!\ell_{p,q}), \qquad \textit{where} \quad \mathrm{Cen}(\mathcal{C}\!\ell_{p,q}) = \begin{cases} \mathcal{C}\!\ell^0_{p,q}, & \textit{if } n \textit{ is even;} \\ \mathcal{C}\!\ell^0_{p,q} \oplus \mathcal{C}\!\ell^n_{p,q}, & \textit{if } n \textit{ is odd,} \end{cases}$$

is a Lie algebra with respect to the commutator $[A, B] = AB - BA$.

For the elements $h_a \in \mathcal{C}\!\ell_{p,q}\mathrm{T}$ (67), (68), we have (for $n \geq 2$) $h_a \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}$, $a = 1, \ldots, n$.

Let us consider the following system of equations for unknown $C_\mu \in \mathcal{C}\!\ell_{p,q}\mathrm{T}_1$

$$\partial_\mu h_a - [C_\mu, h_a] = 0, \qquad \mu = 1, \ldots, m, \qquad a = 1, \ldots, n. \tag{70}$$

We call (70) *an equation for spin connection of a general form.* Note that if we have a solution $C_\mu = C_\mu(x) \in \mathcal{C}\!\ell_{p,q}\mathrm{T}_1$ of the system of equations (70) and $\alpha_\mu = \alpha_\mu(x)$ are arbitrary continuous components of the covector field with values at the center of the Clifford algebra $\mathrm{Cen}(\mathcal{C}\!\ell_{p,q})$, then the components $C_\mu + \alpha_\mu \in \mathcal{C}\!\ell_{p,q}\mathrm{T}_1$ also satisfy the equation (70). Therefore, assume $C_\mu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$.

Theorem 13 *Let $S : \mathbb{R}^{k,l} \to \mathcal{C}\!\ell^{\times}_{p,q}$ be a function with values in the group of all invertible elements of the Clifford algebra $\mathcal{C}\!\ell^{\times}_{p,q}$ such that $S^{-1}\partial_\mu S \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$. Then the following expressions*

$$\acute{h}_a = S^{-1}h_a S \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}, \qquad \acute{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$$

also satisfy the equation $\partial_\mu \acute{h}_a - [\acute{C}_\mu, \acute{h}_a] = 0, \quad \forall \mu = 1, \ldots, m, \quad a = 1, \ldots, n.$

Theorem 14 *The following zero curvature condition follows from (70):*

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = 0, \qquad \mu, \nu = 1, \ldots, m. \tag{71}$$

The conditions (71) are invariant under the gauge transformation $C_\mu \to \acute{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S$, where $S \in \mathcal{C}\!\ell^{\times}_{p,q}\mathrm{T}$ and $S^{-1}\partial_\mu S \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$.

Theorem 15 *Let $C_\mu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$. Then the following two systems of equations are equivalent:*

$$\partial_\mu h_a - [C_\mu, h_a] = 0 \quad \Leftrightarrow \quad C_\mu = \sum_{k=1}^{\acute{n}} \mu_k \pi[h]_k((\partial_\mu h^a)h_a), \tag{72}$$

where $\acute{n} = n$ for even n, $\acute{n} = n-1$ for odd n, and $\mu_k = \frac{1}{n-(-1)^k(n-2k)} = \frac{1}{n-\lambda_k}$. Here $\pi[h]_k$ are projection operators onto the subspaces (69).

We use the method of averaging in Clifford algebras to obtain another form of the unique solution (72) of the system (70).

Theorem 16 *From the system (70), it follows*

$$\partial_\mu h_A - [C_\mu, h_A] = 0, \qquad \mu = 1, \ldots, m \tag{73}$$

for all ordered multi-indices A of length from 0 to n.

Theorem 17 *The system (70) has a unique solution $C_\mu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$*

$$C_\mu = \frac{1}{2^n}(\partial_\mu h_A)h^A, \qquad \mu = 1, \ldots, m. \tag{74}$$

In the case of odd n, the expression (74) can be rewritten in the form

$$C_\mu = \frac{1}{2^{n-1}} \sum_{|A|=1}^{\frac{n-1}{2}} (\partial_\mu h_A)h^A, \qquad \mu = 1, \ldots, m. \tag{75}$$

In the particular case $h_a \in \mathcal{C}\!\ell^1_{p,q}\mathrm{T}$, the presented expressions (72), (74), (75) coincide with the well-known formula for the spin connection $C_\mu = \frac{1}{4}(\partial_\mu h_a)h^a \in \mathcal{C}\!\ell^2_{p,q}\mathrm{T}_1$.

In **Section 2.6**, we present a new class of covariantly constant solutions of the Yang–Mills equations. These solutions correspond to the solution of the equation for spin connection of a general form.

Let us consider an arbitrary tensor field (66) with values in the Clifford algebra. We can take the expressions $h_b(x) = y^A_b(x)e_A$ (67), (68) and obtain another basis $h_B(x) = y^A_B(x)e_A$ of the algebra $\mathcal{C}\!\ell_{p,q}\mathrm{T}$ for some $y^A_B = y^A_B(x) : \mathbb{R}^{k,l} \to \mathbb{R}$. We have

$$U^\Phi_\Psi(x) = u[h]^{\Phi B}_\Psi(x)h_B(x) \in \mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s, \qquad u[h]^{\Phi B}_\Psi(x) : \mathbb{R}^{k,l} \to \mathbb{R}, \qquad u^{\Phi A}_\Psi(x) = u[h]^{\Phi B}_\Psi(x)y^A_B(x). \tag{76}$$

Let us consider the following *operation of covariant differentiation*, which depends on the basis $\{h_A\}$ of the algebra $\mathcal{C}\!\ell_{p,q}\mathrm{T}$

$$D_\mu U^\Phi_\Psi := \partial_\mu U^\Phi_\Psi - [C_\mu, U^\Phi_\Psi], \qquad U^\Phi_\Psi \in \mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s, \tag{77}$$

where $C_\mu = C_\mu(x) \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$ is the unique solution of the system (70).

Theorem 18 *For an arbitrary tensor field (76) with values in $\mathcal{C}\!\ell_{p,q}$, we have*

$$D_\mu(U^\Phi_\Psi(x)) = \partial_\mu(u[h]^{\Phi B}_\Psi(x))h_B(x). \tag{78}$$

Consider the set of *covariantly constant tensor fields with values in the Clifford algebra* $U^\Phi_\Psi \in \mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s$

$$\mathrm{M}\mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s := \{U^\Phi_\Psi \in \mathcal{C}\!\ell_{p,q}\mathrm{T}^r_s, \quad D_\mu U^\Phi_\Psi = 0\}.$$

Let us consider the system of Yang–Mills equations (1), (2) in the Lie algebra $\mathfrak{g} = \mathcal{C}\!\ell^{\circledS}_{p,q}$, i.e. $A_\mu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$, $F_{\mu\nu} \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_2$, $J^\nu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}^1$. The case of pseudo-Euclidean space $\mathbb{R}^{k,l}$, $k+l=m$, with metric g is considered. For simplicity, we set the coupling constant equal to $\rho = 1$.

Theorem 19 *If a covariantly constant tensor field with values in the Clifford algebra $K_\mu \in \mathrm{M}\mathcal{C}\!\ell_{p,q}\mathrm{T}_1$ is a solution of the following system of algebraic equations*

$$[K_\mu, [K^\mu, K^\nu]] = J^\nu, \qquad \nu = 1, \ldots, m, \tag{79}$$

for some $J^\mu \in \mathrm{M}\mathcal{C}\!\ell_{p,q}\mathrm{T}^1$, then the tensor field

$$A_\mu(x) = C_\mu(x) + K_\mu(x), \qquad \mu = 1, \ldots, m, \tag{80}$$

is a solution of the Yang–Mills equations

$$\begin{aligned} &\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu}, \qquad \mu, \nu = 1, \ldots, m, \\ &\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = J^\nu, \qquad \nu = 1, \ldots, m, \end{aligned} \tag{81}$$

in the Lie algebra $\mathcal{C}\!\ell^{\circledS}_{p,q}$, where $C_\mu \in \mathcal{C}\!\ell^{\circledS}_{p,q}\mathrm{T}_1$ is the unique solution of the system

$$\partial_\mu h_a - [C_\mu, h_a] = 0, \qquad \mu = 1, \ldots, m, \qquad a = 1, \ldots, n.$$

Let us consider the particular case $k = p$, $l = q$. We have $m = n = p + q$, and the diagonal matrices coincide $\eta = g$. Let us consider a vector field $h^\mu \in \mathcal{C}\!\ell_{p,q}\mathrm{T}^1$ with values in the Clifford algebra $h^\mu = h^\mu(x) : \mathbb{R}^{p,q} \to \mathcal{C}\!\ell_{p,q}$

$$h^\mu(x) = y^\mu(x)e + y^{\mu a}(x)e_a + y^{\mu ab}(x)e_{ab} + \cdots + y^{\mu 1\ldots n}(x)e_{1\ldots n} = y^{\mu A}e_A, \tag{82}$$

that satisfies

$$h^\mu(x)h^\nu(x) + h^\nu(x)h^\mu(x) = 2\eta^{\mu\nu}e, \qquad \forall x \in \mathbb{R}^{p,q}. \tag{83}$$

In the case of odd n, we also require the additional condition $\langle h^1(x)\ldots h^n(x)\rangle_0 = 0$ to get the independent elements $h^{\mu_1\cdots\mu_k}$. The expression h^μ is called *a Clifford field vector*. The expression

$$U = ue + u_{\omega_1}h^{\omega_1} + u_{\omega_1\omega_2}h^{\omega_1\omega_2} + \cdots + u_{1\ldots n}h^{1\ldots n} = u_\Omega h^\Omega,$$

where $u_\Omega = u_{\omega_1\ldots\omega_j}$ are skew-symmetric tensor fields of rank j, is called *h-form*. The set of such h-forms is *the algebra of h-forms* $\mathcal{C}\!\ell[h]_{p,q}$. It is a generalization of the Atiyah–Kähler algebra, where instead of the expressions h^μ we have the differentials dx^μ. The set h^μ, $\mu = 1,\ldots,n = p+q$ generates a basis of the algebra $\mathcal{C}\!\ell[h]_{p,q}$: $\{h^\Omega, |\Omega| = 0,1,\ldots,n\} = \{e, h^{\omega_1}, h^{\omega_1\omega_2}, \ldots, h^{1\ldots n}\}$. Further in the dissertation, analogues of Theorems 13–19 are given not for the elements h_a, but for the vector quantities h^μ.

In **Section 2.7**, the generalized Pauli's theorem, proved in the author's previous papers for two sets of elements satisfying the anticommutative relations of the real or complexified Clifford algebra of dimension 2^n, is extended to the case when both sets of elements smoothly depend on the point of Euclidean space V of dimension r. Using the equation for spin connection of a general form, it is shown that the problem of the local Pauli theorem is equivalent to the problem of the existence of a solution of some special system of partial differential equations.

Theorem 20 *Let us consider smooth functions $h_a : V \to \mathcal{C}\!\ell$, $a = 1,\ldots,n$, that satisfy*

$$h_a(x)h_b(x) + h_b(x)h_a(x) = 2\eta_{ab}e, \qquad a,b = 1,\ldots,n, \qquad \forall x \in V$$

and the additional condition $\langle h_{1\ldots n}\rangle_0 = 0$ in the case of odd n.

Then there exists a function $S = S(x) : V \to \mathcal{C}\!\ell$ such that $\exists\, S^{-1}(x)\, \forall x \in V$, satisfying the system of equations

$$\partial_\mu S(x) = C_\mu(x)S(x), \qquad \mu = 1,\ldots,r, \qquad \forall x \in V, \tag{84}$$

where $C_\mu : V \to \mathcal{C}\!\ell \setminus \mathrm{Cen}(\mathcal{C}\!\ell)$ is the unique solution of the system of equations

$$\partial_\mu h_a - [C_\mu, h_a] = 0, \qquad a = 1,\ldots,n, \qquad \mu = 1,\ldots,r, \tag{85}$$

and there exists a function $T(x) = S(x)K$ for some independent on x invertible element $K \in \mathcal{C}\!\ell$, which is also a solution of the system (84) invertible in the entire Euclidean space, and connects two sets of elements

$$e_a = T^{-1}(x)h_a(x)T(x), \qquad a = 1,\ldots,n, \qquad \forall x \in V \tag{86}$$

in the case of even n and

$$e_a = h_{1\ldots n}e^{1\ldots n}T^{-1}(x)h_a(x)T(x), \qquad a = 1,\ldots,n, \qquad \forall x \in V \tag{87}$$

in the case of odd n, where $h_{1\ldots n}e^{1\ldots n} = \pm e$.

In the dissertation, the particular cases $n = 2$, $r \geq 1$ and $n \geq 2$, $r = 1$ are analyzed in details. In these cases, the solution of the problem takes a simpler form.

From the above theorem, we obtain an algorithm for computing the function $S = S(x)$. Using this algorithm and the algorithm for computing the element K provided by the algebraic Pauli theorem, we obtain an algorithm for computing the function $T(x) = S(x)K$, which connects two sets of elements $h_a(x)$, e_a, $a = 1, \ldots, n$.

In **Section 2.8**, some particular classes of constant solutions of the Yang–Mills–Proca and Yang–Mills equations in Clifford algebras are presented. Let us consider the system of Yang–Mills–Proca equations for constant solutions

$$[A_\mu, [A^\mu, A^\nu]] = -\lambda A^\nu, \qquad \lambda = \frac{m^2}{\rho^2} \geq 0, \tag{88}$$

where $A^\mu \in \mathfrak{g} = \mathcal{C}\!\ell^{\circledS}$ with $n \geq 2$. This system has the following class of solutions

$$(A^\mu)^2 = \frac{-\lambda \eta^{\mu\mu} e}{4(n-1)}, \quad \mu = 1, 2, \ldots, n; \qquad \{A^\mu, A^\nu\} = 0, \quad \mu \neq \nu. \tag{89}$$

After an appropriate normalization, the elements A^μ (89) will be generators of: 1) the Clifford algebra $\mathcal{C}\!\ell_{q,p}$, $p+q = n$ in the case of the real Clifford algebra; one can also take other signatures in the case of the complexified Clifford algebra; 2) Clifford algebras of smaller dimension 2^{n-1} for $q - p = 1 \mod 4$ in the case of the real Clifford algebra $\mathcal{C}\!\ell_{q,p}$ and for $p - q = 1, 3 \mod 4$ in the case of a complexified Clifford algebra; 3) Grassmann algebras for $\lambda = 0$.

In **Chapter 3**, we solve the problems related to Lie groups and Lie algebras of specific type in Clifford algebras.

In **Section 3.1**, we present a complete classification of Lie algebras of specific type in the complexified Clifford algebras $\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$. These 16 Lie algebras are direct sums of subspaces of quaternion types. We obtain isomorphisms between the considered Lie algebras and the classical matrix Lie algebras in the case of an arbitrary dimension and signature. We present 16 Lie groups: one Lie group for each Lie algebra associated with this Lie group. Relations between the considered Lie groups and spin groups (real $\mathrm{Spin}_+(p,q)$ and complex $\mathrm{Spin}(n, \mathbb{C})$) are studied.

Theorem 21 *The complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$ has the following Lie subalgebras*[97]

$$\begin{gathered} \overline{\mathbf{2}}, \quad \overline{\mathbf{02}}, \quad \overline{\mathbf{12}}, \quad \overline{\mathbf{23}}, \quad \overline{\mathbf{2}} \oplus i\overline{\mathbf{0}}, \quad \overline{\mathbf{2}} \oplus i\overline{\mathbf{1}}, \quad \overline{\mathbf{2}} \oplus i\overline{\mathbf{2}}, \quad \overline{\mathbf{2}} \oplus i\overline{\mathbf{3}}, \quad \overline{\mathbf{0123}}, \\ \overline{\mathbf{02}} \oplus i\overline{\mathbf{02}}, \quad \overline{\mathbf{12}} \oplus i\overline{\mathbf{12}}, \quad \overline{\mathbf{23}} \oplus i\overline{\mathbf{23}}, \quad \overline{\mathbf{02}} \oplus i\overline{\mathbf{13}}, \quad \overline{\mathbf{12}} \oplus i\overline{\mathbf{03}}, \quad \overline{\mathbf{23}} \oplus i\overline{\mathbf{01}}. \end{gathered} \tag{90}$$

Let us consider the following 16 Lie groups in $\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$ (for group definitions, see the second column of Table 2)[98]:

$$(\mathbb{C} \otimes \mathcal{C}\!\ell_{p,q})^\times, \quad \mathcal{C}\!\ell^\times_{p,q}, \quad \mathcal{C}\!\ell^{(0)\times}_{p,q}, \quad (\mathbb{C} \otimes \mathcal{C}\!\ell^{(0)}_{p,q})^\times, \quad (\mathcal{C}\!\ell^{(0)}_{p,q} \oplus i\mathcal{C}\!\ell^{(1)}_{p,q})^\times, \quad \mathrm{G}^{23i01}_{p,q},$$

[97] We omit the direct sum sign to simplify the notation: $\overline{\mathbf{0}} \oplus \overline{\mathbf{2}} = \overline{\mathbf{02}}$, $i\overline{\mathbf{1}} \oplus i\overline{\mathbf{3}} = i\overline{\mathbf{13}}$, $\overline{\mathbf{0}} \oplus \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}} = \overline{\mathbf{0123}}$, etc.

[98] We denote the complex conjugation of an element of the complexified Clifford algebra $U \in \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q}$ by $\bar{U}$, i.e. taking the complex conjugation of all coefficients $u_A \in \mathbb{C}$ of the expansion with respect to the basis $\{e_A\}$.

$$\mathrm{G}^{12i03}_{p,q},\quad \mathrm{G}^{2i0}_{p,q},\quad \mathrm{G}^{23i23}_{p,q},\quad \mathrm{G}^{12i12}_{p,q},\quad \mathrm{G}^{2i2}_{p,q},\quad \mathrm{G}^{2i1}_{p,q},\quad \mathrm{G}^{2i3}_{p,q},\quad \mathrm{G}^{12}_{p,q},\quad \mathrm{G}^{23}_{p,q},\quad \mathrm{G}^{2}_{p,q}.$$

Here $\mathrm{A}^{\times}$ means the set (group) of invertible elements of the set A.

Theorem 22 *The subsets of $\mathbb{C}\otimes\mathcal{C}\ell_{p,q}$ given in the second column of Table 2 are Lie groups. The subsets of $\mathbb{C}\otimes\mathcal{C}\ell_{p,q}$ given in the third column of Table 2 are the Lie algebras of the corresponding Lie groups from the second column of Table 2. The considered Lie groups and Lie algebras have the dimensions given in the fourth column of Table 2.*

Table 2: Lie groups and corresponding Lie algebras of specific type in Clifford algebras

	Lie group	Lie algebra	dimension
1	$(\mathbb{C}\otimes\mathcal{C}\ell_{p,q})^{\times}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{0123}}\oplus i\overline{\mathbf{0123}}$	2^{n+1}
2	$\mathcal{C}\ell^{\times}_{p,q}=\{U\in\mathcal{C}\ell_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{0123}}$	2^{n}
3	$\mathcal{C}\ell^{(0)\times}_{p,q}=\{U\in\mathcal{C}\ell^{(0)}_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}$	2^{n-1}
4	$(\mathbb{C}\otimes\mathcal{C}\ell^{(0)}_{p,q})^{\times}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell^{(0)}_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}\oplus i\overline{\mathbf{02}}$	2^{n}
5	$(\mathcal{C}\ell^{(0)}_{p,q}\oplus i\mathcal{C}\ell^{(1)}_{p,q})^{\times}=\{U\in\mathcal{C}\ell^{(0)}_{p,q}\oplus i\mathcal{C}\ell^{(1)}_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}\oplus i\overline{\mathbf{13}}$	2^{n}
6	$\mathrm{G}^{23i01}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\,\vert\,\bar{\tilde{U}}U=e\}$	$\overline{\mathbf{23}}\oplus i\overline{\mathbf{01}}$	2^{n}
7	$\mathrm{G}^{12i03}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\,\vert\,\bar{\hat{\tilde{U}}}U=e\}$	$\overline{\mathbf{12}}\oplus i\overline{\mathbf{03}}$	2^{n}
8	$\mathrm{G}^{2i0}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell^{(0)}_{p,q}\,\vert\,\bar{\tilde{U}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{0}}$	2^{n-1}
9	$\mathrm{G}^{23i23}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{23}}\oplus i\overline{\mathbf{23}}$	$2^{n}-2^{\frac{n+1}{2}}\sin\frac{\pi(n+1)}{4}$
10	$\mathrm{G}^{12i12}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\,\vert\,\hat{\tilde{U}}U=e\}$	$\overline{\mathbf{12}}\oplus i\overline{\mathbf{12}}$	$2^{n}-2^{\frac{n+1}{2}}\cos\frac{\pi(n+1)}{4}$
11	$\mathrm{G}^{2i2}_{p,q}=\{U\in\mathbb{C}\otimes\mathcal{C}\ell^{(0)}_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{2}}$	$2^{n-1}-2^{\frac{n}{2}}\cos\frac{\pi n}{4}$
12	$\mathrm{G}^{2i1}_{p,q}=\{U\in\mathcal{C}\ell^{(0)}_{p,q}\oplus i\mathcal{C}\ell^{(1)}_{p,q}\,\vert\,\bar{\tilde{U}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{1}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
13	$\mathrm{G}^{2i3}_{p,q}=\{U\in\mathcal{C}\ell^{(0)}_{p,q}\oplus i\mathcal{C}\ell^{(1)}_{p,q}\,\vert\,\bar{\hat{\tilde{U}}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{3}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
14	$\mathrm{G}^{23}_{p,q}=\{U\in\mathcal{C}\ell_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{23}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
15	$\mathrm{G}^{12}_{p,q}=\{U\in\mathcal{C}\ell_{p,q}\,\vert\,\hat{\tilde{U}}U=e\}$	$\overline{\mathbf{12}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
16	$\mathrm{G}^{2}_{p,q}=\{U\in\mathcal{C}\ell^{(0)}_{p,q},\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{2}}$	$2^{n-2}-2^{\frac{n-2}{2}}\cos\frac{\pi n}{4}$

In the dissertation, we prove isomorphisms of Lie groups with numbers 1–5 from Table 2 to linear classical matrix Lie groups, isomorphisms of Lie groups with numbers 6–8 to unitary, pseudo-unitary, and complex linear classical matrix Lie groups, isomorphisms of Lie groups with numbers 9–11 to complex orthogonal, symplectic, and linear classical matrix Lie groups, isomorphisms of Lie groups with numbers 12–16 to real, complex, and quaternion orthogonal, symplectic, linear, unitary, and pseudo-unitary classical matrix Lie groups (depending on p and q). As a consequence, we obtain isomorphisms for the corresponding Lie algebras.

In **Section 3.2**, we study inner automorphisms that leave invariant fixed subspaces of the real $\mathcal{C}\ell_{p,q}$ or complex $\mathcal{C}\ell(\mathbb{C}^n)$ Clifford algebra (we denote both cases by $\mathcal{C}\ell$) – subspaces of fixed grades and subspaces determined by the reversion and grade involution. We present group of elements that define such inner automorphisms and study their properties. Some of these Lie groups can be interpreted as generalizations of the Clifford group, Lipschitz group, and spin groups. The corresponding Lie algebras are studied.

Table 3: Lie groups that preserve fixed subspaces of $\mathcal{C}\!\ell$ under a similarity transformation and the corresponding Lie algebras

Lie group	n	Lie algebra	dimension
$\mathcal{C}\!\ell^{\times}$		$\mathcal{C}\!\ell$	2^n
$\Gamma = \bigcap_{k=0}^{n} \Gamma^k$	$1 \mod 2$	$\mathcal{C}\!\ell^{02n}$	$\frac{n(n-1)}{2}+2$
	$0 \mod 2$	$\mathcal{C}\!\ell^{02}$	$\frac{n(n-1)}{2}+1$
$\mathrm{P} = \Gamma^{(0)} = \Gamma^{(1)}$	$1 \mod 2$	$\mathcal{C}\!\ell^{(0)n}$	$2^{n-1}+1$
	$0 \mod 2$	$\mathcal{C}\!\ell^{(0)}$	2^{n-1}
$\mathrm{A} = \Gamma^{\overline{01}} = \Gamma^{\overline{23}}$	$1 \mod 4$	$\mathcal{C}\!\ell^{0\overline{23}n}$	$2^{n-1} - 2^{\frac{n-1}{2}} \sin(\frac{\pi(n+1)}{4}) + 2$
	$0,2,3 \mod 4$	$\mathcal{C}\!\ell^{0\overline{23}}$	$2^{n-1} - 2^{\frac{n-1}{2}} \sin(\frac{\pi(n+1)}{4}) + 1$
$\mathrm{B} = \Gamma^{\overline{03}} = \Gamma^{\overline{12}}$	$3 \mod 4$	$\mathcal{C}\!\ell^{0\overline{12}n}$	$2^{n-1} - 2^{\frac{n-1}{2}} \cos(\frac{\pi(n+1)}{4}) + 2$
	$0,1,2 \mod 4$	$\mathcal{C}\!\ell^{0\overline{12}}$	$2^{n-1} - 2^{\frac{n-1}{2}} \cos(\frac{\pi(n+1)}{4}) + 1$
$\mathrm{Q} = \mathrm{Q}' = \Gamma^{\overline{k}}$	$1,3 \mod 4$	$\mathcal{C}\!\ell^{0\overline{2}n}$	$2^{n-2} - 2^{\frac{n-2}{2}} \cos(\frac{\pi n}{4}) + 2$
$(k=0,1,2,3)$	$2 \mod 4$	$\mathcal{C}\!\ell^{0\overline{2}}$	$2^{n-2} - 2^{\frac{n-2}{2}} \cos(\frac{\pi n}{4}) + 1$
$\mathrm{Q} = \Gamma^{\overline{1}} = \Gamma^{\overline{3}}$	$0 \mod 4$	$\mathcal{C}\!\ell^{0\overline{2}}$	$2^{n-2} - 2^{\frac{n-2}{2}} \cos(\frac{\pi n}{4}) + 1$
$\mathrm{Q}' = \Gamma^{\overline{0}} = \Gamma^{\overline{2}}$	$0 \mod 4$	$\mathcal{C}\!\ell^{0\overline{2}n}$	$2^{n-2} - 2^{\frac{n-2}{2}} \cos(\frac{\pi n}{4}) + 2$

We use the following notation for groups of elements that preserve subspaces of fixed grades, fixed parity, fixed quaternion types, or their direct sums under a similarity transformation[99]:

$$\Gamma^k := \{T \in \mathcal{C}\!\ell^{\times} \mid T\mathcal{C}\!\ell^k T^{-1} \subseteq \mathcal{C}\!\ell^k\}, \quad k = 0,1,\ldots,n, \tag{91}$$
$$\Gamma^{(k)} := \{T \in \mathcal{C}\!\ell^{\times} \mid T\mathcal{C}\!\ell^{(k)} T^{-1} \subseteq \mathcal{C}\!\ell^{(k)}\}, \quad k = 0,1, \tag{92}$$
$$\Gamma^{\overline{k}} := \{T \in \mathcal{C}\!\ell^{\times} \mid T\mathcal{C}\!\ell^{\overline{k}} T^{-1} \subseteq \mathcal{C}\!\ell^{\overline{k}}\}, \quad k = 0,1,2,3, \tag{93}$$
$$\Gamma^{\overline{kl}} := \{T \in \mathcal{C}\!\ell^{\times} \mid T\mathcal{C}\!\ell^{\overline{kl}} T^{-1} \subseteq \mathcal{C}\!\ell^{\overline{kl}}\}, \quad k,l = 0,1,2,3. \tag{94}$$

In the particular case, we obtain the well-known Clifford group $\Gamma := \Gamma^1$. We denote by $\mathrm{Z}^{\times}$ the group of all invertible elements of the center Z of the Clifford algebra $\mathcal{C}\!\ell$

$$\mathrm{Z} := \begin{cases} \mathcal{C}\!\ell^0, & \text{if } n \text{ is even} \\ \mathcal{C}\!\ell^0 \oplus \mathcal{C}\!\ell^n, & \text{if } n \text{ is odd.} \end{cases} \tag{95}$$

Theorem 23 *We have*

$$\Gamma^{(0)} = \Gamma^{(1)} = \mathrm{P} := \mathrm{Z}^{\times}(\mathcal{C}\!\ell^{(0)\times} \cup \mathcal{C}\!\ell^{(1)\times}) = \begin{cases} \mathcal{C}\!\ell^{(0)\times} \cup \mathcal{C}\!\ell^{(1)\times}, & \textit{if } n \textit{ is even}, \\ \mathcal{C}\!\ell^{0n\times}\mathcal{C}\!\ell^{(0)\times}, & \textit{if } n \textit{ is odd.} \end{cases}$$

[99]We omit the direct sum sign to simplify the notation: $\mathcal{C}\!\ell^{\overline{kl}} = \mathcal{C}\!\ell^{\overline{k}} \oplus \mathcal{C}\!\ell^{\overline{l}}$, $\mathcal{C}\!\ell^{k\overline{l}} = \mathcal{C}\!\ell^{k} \oplus \mathcal{C}\!\ell^{\overline{l}}$, etc.

Theorem 24 *We have*

$$\Gamma^{\overline{01}}=\Gamma^{\overline{23}}=\mathrm{A}:=\{T\in\mathcal{C}\!\ell^\times\,|\,\widetilde{T}T\in\mathrm{Z}^\times\}.$$

Theorem 25 *We have*

$$\Gamma^{\overline{03}}=\Gamma^{\overline{12}}=\mathrm{B}:=\{T\in\mathcal{C}\!\ell^\times\,|\,\widehat{\widetilde{T}}T\in\mathrm{Z}^\times\}.$$

Let us consider the groups

$$\mathrm{Q} := \{T\in\mathrm{Z}^\times(\mathcal{C}\!\ell^{(0)\times}\cup\mathcal{C}\!\ell^{(1)\times})\,|\,\widetilde{T}T\in\mathrm{Z}^\times\}, \tag{96}$$

$$\mathrm{Q}' := \{T\in\mathrm{Z}^\times(\mathcal{C}\!\ell^{(0)\times}\cup\mathcal{C}\!\ell^{(1)\times})\,|\,\widetilde{T}T\in(\mathcal{C}\!\ell^0\oplus\mathcal{C}\!\ell^n)^\times\}. \tag{97}$$

We have $\mathrm{Q}=\mathrm{Q}'$ in the cases $n=1,2,3 \mod 4$.

Theorem 26 *In the cases $n\geq 4$, we have*

$$\mathrm{Q}=\Gamma^{\overline{1}}=\Gamma^{\overline{3}}\neq\mathrm{Q}'=\Gamma^{\overline{0}}=\Gamma^{\overline{2}},\qquad n=0\mod 4, \tag{98}$$

$$\mathrm{Q}=\Gamma^{\overline{0}}=\Gamma^{\overline{1}}=\Gamma^{\overline{2}}=\Gamma^{\overline{3}},\qquad n=1,2,3\mod 4. \tag{99}$$

In the exceptional cases, we have

$$\Gamma^{\overline{0}}=\mathcal{C}\!\ell^\times\neq\Gamma^{\overline{1}}=\Gamma^{\overline{2}}=\mathrm{Q}=\mathrm{P}=\mathcal{C}\!\ell^{(0)\times}\cup\mathcal{C}\!\ell^{(1)\times},\quad n=2,$$

$$\Gamma^{\overline{0}}=\Gamma^{\overline{3}}=\mathcal{C}\!\ell^\times\neq\Gamma^{\overline{1}}=\Gamma^{\overline{2}}=\mathrm{Q}=\mathrm{P}=\mathrm{Z}^\times\mathcal{C}\!\ell^{(0)\times},\quad n=3.$$

Theorem 27 *We have*

$$\Gamma=\mathrm{Q},\qquad n\leq 5;\qquad \Gamma\neq\mathrm{Q},\qquad n=6. \tag{100}$$

Theorem 28 *The Lie groups* P, A, B, Q, *and* Q′ *have the following Lie algebras* $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{q}$, *and* $\mathfrak{q}'$ *presented in Table 3 with the corresponding dimensions.*

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