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Zhuoke Yang

New approaches to Lie algebra weight system

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Contents

The results of this dissertation are reflected in three articles:

- Zhuoke Yang, *On the Lie superalgebra* $\mathfrak{gl}(m|n)$ *weight system*, Journal of Geometry and Physics. 2023. Vol. 187. Article 104808.
- Zhuoke Yang, *New approaches to* $\mathfrak{gl}_N$ *weight system*, `arXiv:2202.12225 [math.CO]`, Accepted by Izvestiya: Mathematics
- Zhuoke Yang, *On values of* $\mathfrak{sl}_3$ *weight system on chord diagrams whose intersection graph is complete bipartite*, `arXiv:2102.00888 [math.CO]`, Accepted by Moscow Mathematical Journal

Introduction

Finite order knot invariants, which were introduced in [29] by V. Vassiliev near 1990, may be expressed in terms of weight systems, that is, functions on chord diagrams satisfying the so-called Vassiliev 4-term relations. In paper [18], M. Kontsevich proved that over a field of characteristic zero every weight system corresponds to some finite order invariant. There are multiple approaches to constructing weight systems. In particular, D. Bar-Natan and M. Kontsevich suggested a construction of a weight system coming from a finite dimensional Lie algebra endowed with an invariant nondegenerate bilinear form. The $\mathfrak{sl}(2)$ Lie algebra weight system is the simplest case whose weight system is associated to the knot invariant known as the colored Jones polynomial. Its values lie in the center of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$, which, in turn, is isomorphic to the ring of polynomials in one variable (the Casimir element). The $\mathfrak{sl}(2)$ weight system was studied in many papers. Despite the fact that this weight system can be defined easily, it is difficult to compute its value on a chord diagram using the definition because it is necessary to work with elements of a non-commutative algebra in order to do this. The Chmutov–Varchenko recurrence relations [6] simplify these computations significantly and numerous computations have been done using it, see e.g. [12, 13, 30]. A theorem by S. Chmutov and S. Lando [7] states that the value of the $\mathfrak{sl}(2)$ weight system on a chord diagram depends only on the intersection graph of this chord diagram, i.e. if two chord diagrams have isomorphic intersection graphs, then the values of the weight system on these chord diagrams coincide.

On the other side, we don't have such good properties for the next reasonable case, namely, for the $\mathfrak{sl}(3)$ weight system. The $\mathfrak{sl}(3)$ Lie algebra weight system takes values in the center of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(3)$, which is isomorphic to the ring of polynomials in TWO variables (the Casimir elements of degrees 2 and 3). For the $\mathfrak{sl}(3)$ weight system, we do not have a result similar to the Chmutov—Varchenko recurrence relations for the $\mathfrak{sl}(2)$ weight system which could help us to compute its value. The Chmutov–Lando theorem also fails for the $\mathfrak{sl}(3)$ weight system, which means there are two different chord diagrams with different values of the $\mathfrak{sl}(3)$ weight system such that they have isomorphic intersection graphs.

The thesis is devoted to constructing efficient ways to computing the values of weight systems associated to various Lie algebras and Lie superalgebras, and to analyzing their properties. It has the following structure. In Sec. 7 we give the key definitions and state the main results.

Our first group of main results in Sec. 6 concerns explicit values of the $\mathfrak{sl}(3)$ weight system on chord diagrams whose intersection graph is complete bipartite, with the size of one part equal to 2. In our computations, we use certain results from [34]. Up to now, explicit values of the $\mathfrak{sl}(3)$ weight system were known only in few examples and simple series. Our results imply a nontrivial conclusion that for the chord diagrams whose intersection graph is the complete bipartite graph $K_{2,n}$, the value of the $\mathfrak{sl}(3)$ weight system depends on the second Casimir only.

A key role in our study is played by the Hopf algebra structure on the space of chord diagrams modulo 4-term relations introduced by Kontsevich. Chord diagrams whose intersection graph is complete bipartite generate a Hopf subalgebra in this Hopf algebra. By analyzing the structure of this Hopf subalgebra, P. Filippova managed in [12, 13] to deduce the values of the $\mathfrak{sl}(2)$ weight system on projections of the chord diagrams whose intersection graph is complete bipartite to the subspace of primitives. By combining our computations with her results, we obtain explicit expressions for the values of the $\mathfrak{sl}(3)$ weight system on primitives.

Much less is known about other Lie algebras; for them, explicit answers have been computed only for chord diagrams of very small order or for simple families of chord diagrams, see [31]. In particular, no recurrence similar to the Chmutov–Varchenko one exists (with the exception of the Lie superalgebra $\mathfrak{gl}(1|1)$, see [11, 6]). Sec. 7 is devoted to new ways to compute the values of the $\mathfrak{gl}(N)$ weight system.

One of these new ways is based on a suggestion due to M. Kazarian to define an invariant of permutations taking values in the center of the universal enveloping algebra of $\mathfrak{gl}(N)$. The restriction of this invariant to involutions without fixed points (such an involution determines a chord diagram) coincides with the value of the $\mathfrak{gl}(N)$-weight system on this chord diagram. We describe the recursion allowing one to compute the $\mathfrak{gl}(N)$-invariant of permutations and demonstrate how it works in a number of examples.

For $N' < N$, the center of the universal enveloping algebra of $\mathfrak{gl}(N')$ is naturally embedded into that of $\mathfrak{gl}(N)$, and the $\mathfrak{gl}(N)$-weight system is stable: its value on a permutation is a universal polynomial. The recursion we describe allows one to compute this polynomial simultaneously for all N.

Recall that calculations of the highest homogeneous part of the universal $\mathfrak{gl}(N)$ weight system in terms of Casimir elements for some special primitive elements given by open Jacobi diagrams form the central part in the proof of the lower estimate for the dimension of the Vassiliev knot invariants in [5, 9] (see also [5, §14.5.4]).

We also develop another efficient way for computing the $\mathfrak{gl}(N)$-weight system, which is based on the Harish–Chandra isomorphism.

In Sec. 9, we expand the results about the $\mathfrak{gl}(N)$ weight system to the weight system corresponding to the Lie Superalgebra $\mathfrak{gl}(m|n)$. We prove that it is a specialization of the $\mathfrak{gl}(N)$ weight system, for $N = m - n$.

The original references to the Lie superalgebras can be found in [15]. Weight systems arising from Lie superalgebras are defined in [28]. The straightforward approach to computing the values of a Lie superalgebra weight system on a general chord diagram amounts to elaborating calculations in the noncommutative universal enveloping algebra, in spite of the fact that the result belongs to the center of the latter. This approach is rather inefficient even for the simplest noncommutative Lie Superalgebra $\mathfrak{gl}(1|1)$. For this Lie Superalgebra, however, there is a recurrence relation due to Figueroa-O'Farrill, T. Kimura and A. Vaintrob [11]. Much less is known about other Lie superalgebras.

Our approach is based on defining an invariant of permutations taking values in the center of the universal enveloping algebra of $\mathfrak{gl}(m|n)$. The restriction of this invariant to involutions without fixed points (such an involution determines a chord diagram) coincides with the value of the $\mathfrak{gl}(m|n)$-weight system on this chord diagram. We prove the recursion for the $\mathfrak{gl}(m|n)$ weight system, which proves to be the same as the recursion for the $\mathfrak{gl}(N)$-one.

1 Chord diagrams and weight systems

Below, we use standard notions from the theory of finite order knot invariants; see, e.g. [5, 22].

In this section we define the Hopf algebra of chord diagrams modulo 4-term relations.

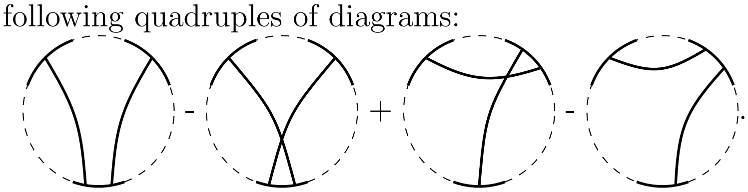
Definition 1.1 (chord diagram)**.** A *chord diagram* D of order n (or degree n) is an oriented circle (sometimes termed *Wilson loop*) with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. We denote the set of chords of a chord diagram D by $[D]$.

The vector space $\mathcal{A}$ spanned by chord diagrams over complex field $\mathbb{C}$ is graded,

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \oplus \dots .$$

Each component $\mathcal{A}_n$ is spanned by diagrams of the same order n.

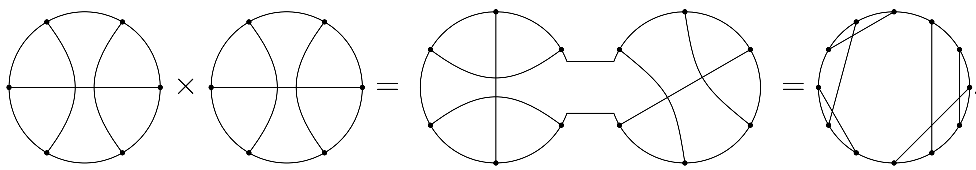
Definition 1.2 (4-term elements)**.** A 4*term* (or 4T) *element* is the alternating sum of the following quadruples of diagrams:



Here all the four chord diagrams contain, in addition to the two depicted chords, one and the same set of other chords whose ends belong to the dashed arcs. For any vector space V, a linear mapping $f \in \hom_{linear}(\mathcal{A}, V)$ that vanishes on all 4-term elements is called a *weight system*.

Now we define the Hopf algebra structure on $\mathcal{A}/\langle\text{4-term elements}\rangle := \mathcal{A}^{fr}$.

Definition 1.3. The *product* of two chord diagrams D_1 and D_2 is defined by cutting and gluing the two circles as shown



Modulo 4-term relationship, the product is well-defined, that is, it does not depend on the chosen points of cutting the Wilson loops.

Definition 1.4. The *coproduct* in the algebra $\mathcal{A}^{fr}$

$$\delta : \mathcal{A}_n^{fr} \to \bigoplus_{k+l=n} \mathcal{A}_k^{fr} \otimes \mathcal{A}_l^{fr}$$

is defined as follows. For a diagram $D \in \mathcal{A}_n^{fr}$ we put

$$\delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\bar{J}}.$$

The summation is taken over all subsets J of the set of chords of D. Here D_J is the chord diagram formed by those chords of D that belong to J and $\bar{J} = [D] \setminus J$ is the complementary subset of chords. To the entire space $\mathcal{A}^{fr}$, the operator δ is extended by linearity.

Claim 1.5. [18] *The vector space A^{fr} endowed with the above product and coproduct is a commutative cocommutative connected graded bialgebra.*

Definition 1.6. An element p of a bialgebra is said to be *primitive* if $\delta(p) = 1 \otimes p + p \otimes 1$.

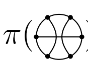
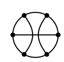
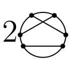
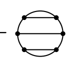
It is easy to show that primitive elements form a vector subspace $P(\mathcal{A}^{fr})$ in the bialgebra $\mathcal{A}^{fr}$. Since any homogeneous component of a primitive element is primitive, such a vector subspace of a graded bialgebra is also graded, $P_n = P(\mathcal{A}_n^{fr})$. An element of $\mathcal{A}_n$ is *decomposable* if it can be represented as a product of elements of order smaller than n.

Theorem 1.7 ([19, 27])**.** *The projection $\pi(D)$ of a chord diagram D to the subspace of primitive elements whose kernel is the subspace spanned by decomposable elements in the Hopf algebra $\mathcal{A}^{fr}$ is given by the formula*

$$\begin{aligned}\pi(D) &= D - 1! \sum_{[D_1]\sqcup[D_2]=[D]} D_1 \cdot D_2 + 2! \sum_{[D_1]\sqcup[D_2]\sqcup[D_3]=[D]} D_1 \cdot D_2 \cdot D_3 \ldots \\ &= D - \sum_{i=2}^{|[D]|} (-1)^i (i-1)! \sum_{\substack{\bigsqcup_{j=1}^{i} [D_j]=[D] \\ [D_j]\neq\emptyset}} \prod_{j=1}^{i} D_j\end{aligned}$$

For example

Example 1.8. The element

$\pi($  $) =$  $- 2$  $+$ 

is a primitive element, which is the projection of the argument in the left-hand side to the subspace of primitives.

2 Constructing weight systems from Lie algebras

Given a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant bilinear form, one can construct a weight system with values in the center of its universal enveloping algebra $U(\mathfrak{g})$. This is the form M. Kontsevich [18] gave to a construction due to D. Bar-Natan [3]. Kontsevich's construction proceeds as follows.

Definition 2.1 (Universal Lie algebra weight system for chord diagram)**.** Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle\cdot,\cdot\rangle$. Let d denote the dimension of $\mathfrak{g}$. Choose a basis $e_1, \ldots, e_d$ of $\mathfrak{g}$ and let $e_1^*, \ldots, e_d^*$ be the dual basis with respect to the form $\langle\cdot,\cdot\rangle$, $\langle e_i, e_j^*\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Given a chord diagram D with n chords, we first choose a base point on the circle, away from the ends of the chords of D. This gives a linear order on the endpoints of the chords, increasing in the positive direction of the Wilson loop. Assign to each chord a an index, that is, an integer-valued variable, i_a. The values of i_a will range from 1 to d, the dimension of the Lie algebra. Mark the first endpoint of the chord a with the symbol e_{i_a} and the second endpoint with $e_{i_a}^*$.

Now, write the product of all the e_{i_a} and all the $e^*_{i_a}$, in the order in which they appear on the Wilson loop of D, and take the sum of the d^n elements of the universal enveloping algebra $U(\mathfrak{g})$ obtained by substituting all possible values of the indices i_a into this product. Denote by $w_{\mathfrak{g}}(D)$ the resulting element of $U(\mathfrak{g})$.

Claim 2.2. [18] *The function $w_{\mathfrak{g}} : D \mapsto w_{\mathfrak{g}}(D)$ on chord diagrams has the following properties:*

1. *the element $w_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram;*
2. *it does not depend on the choice of the basis e_i of the Lie algebra $\mathfrak{g}$;*
3. *its image belongs to the ad-invariant subspace*
$$U(\mathfrak{g})^{\mathfrak{g}} = \{x \in U(\mathfrak{g}) | xy = yx \text{ for all } y \in \mathfrak{g}\} = ZU(\mathfrak{g});$$
4. *it is multiplicative, $w_{\mathfrak{g}}(D_1D_2) = w_{\mathfrak{g}}(D_1)w_{\mathfrak{g}}(D_2)$ for any pair of chord diagrams D_1, D_2;*
5. *this map from chord diagrams to $ZU(\mathfrak{g})$ satisfies the 4-term relations.*

Remark 2.3. If D is a chord diagram with n chords, then

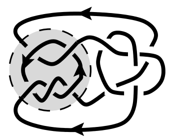
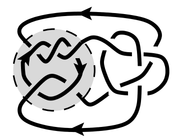
$$\phi_{\mathfrak{g}}(D) = c^n + \{\text{terms of degree less than } 2n \text{ in } U(g)\},$$

where $c = e_1 \otimes e_1^* + \cdots + e_m \otimes e_m^* \in U(\mathfrak{g})$ is the quadratic Casimir element. Indeed, we can permute the endpoints of chords on the circle without changing the highest term of $\phi_{\mathfrak{g}}(D)$ since all the additional summands arising as commutators have degrees smaller than $2n$. Therefore, the highest degree term of $\phi_{\mathfrak{g}}(D)$ does not depend on D with a given number n of chords. Finally, if D is a diagram with n isolated chords, that is, the nth power of the diagram with one chord, then $\phi_{\mathfrak{g}}(D) = c^n$.

3 Mutations of knots, chord diagrams and intersection graphs

The Figures of this section are borrowed from [7].

Two knots are said to be *mutant* if they differ by a rotation/reflection of a tangle with four endpoints; if necessary, the orientation inside the tangle may be replaced by the opposite one. Here is a famous example of mutant knots, the Conway ($11n34$) knot C of genus 3, and Kinoshita–Terasaka ($11n42$) knot KT of genus 2. (see [1]).

$C =$  $KT =$ 

Note that the change of the orientation of a knot can be achieved by a mutation in the complement to a trivial tangle.

Most known knot invariants cannot distinguish mutant knots. Neither the (colored) Jones polynomial, nor the HOMFLY polynomial, nor the Kauffman two variable polynomial distinguish mutants. All Vassiliev invariants up to order 10 do not distinguish mutants as well [21] (up to order 8 this fact was established by a direct computation [2, 5]). However, there is a Vassiliev invariant of order 11 distinguishing C and KT [20, 21]. It comes from the colored HOMFLY polynomial.

The main combinatorial objects of the Vassiliev theory of knot invariants are *chord diagrams*. To a chord diagram, its *intersection graph* (also called *circle graph*) is associated. The vertices of the graph correspond to chords of the diagram, and two vertices are connected by an edge if and only if the corresponding chords intersect.

The value of a Vassiliev invariant of order n on a singular knot with n double points depends only on the chord diagram of the singular knot. Hence any such invariant determines a function, a *weight system*, on chord diagrams with n chords. Conversely, any weight system induces, in composition with the Kontsevich integral, which is the universal finite order invariant, a finite order invariant of knots.

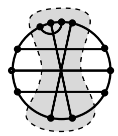
Definition 3.1. The *intersection graph* is associated to a chord diagram. The vertices of the graph correspond to chords of the diagram, and two vertices are connected by an edge if and only if the corresponding chords intersect.

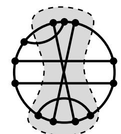
Not each abstract graph is isomorphic to the intersection graph of certain chord diagram. In the opposite direction, two different chord diagrams can have isomorphic intersection graphs.

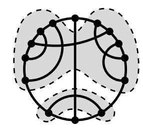
Knot mutation is defined as an operation of rotating a tangle in a knot. A combinatorial analog of the tangle in mutant knots is a *share* [2, 5]. Informally, a *share* of a chord diagram is a subset of chords whose endpoints are separated into at most two parts by the endpoints of the complementary chords. More formally,

Definition 3.2. A *share* is a part of a chord diagram consisting of two arcs of the outer circle possessing the following property: each chord one of whose ends belongs to these arcs has both ends on these arcs.

Here are some examples:


A share

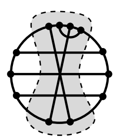
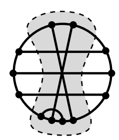
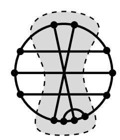

Not a share


Two shares

The complement of a share also is a share. The whole chord diagram is its own share whose complement contains no chords.

Definition 3.3. A *mutation of a chord diagram* is another chord diagram obtained by a rotation/reflection of a share.

For example, three mutations of the share in the first chord diagram above produce the following chord diagrams:

Obviously, mutations preserve the intersection graphs of chord diagrams.

Theorem 3.4. *[4, 8, 14] Two chord diagrams have the same intersection graph if and only if they are related by a sequence of mutations.*

Intersection graphs play a crucial role in studying mutant knots:

Theorem 3.5. *[7] A finite order knot invariant does not distinguish mutants if and only if the corresponding weight system does not distinguish mutant chord diagrams, that is, it depends on the intersection graph of a chord diagram rather than the diagram itself.*

In particular,

Theorem 3.6. *[7] The $\mathfrak{sl}(2)$ weight systems depend on the intersection graphs of chord diagrams rather than on the diagrams themselves.*

Therefore, we know that $\mathfrak{sl}(2)$ weight system does not distinguish mutant pairs. In contrast, the $\mathfrak{sl}(3)$-weight system distinguishes between the Conway and the Kinoshita–Terasaka mutant knots. The corresponding weight system distinguishes between two mutant chord diagrams of order 11.

4 Jacobi diagrams

When computing the values of the $\mathfrak{sl}(3)$ weight system, we will require the results in [34] about recurrence relations for the values of this weight system on Jacobi diagrams. To this end, we recall the notion of *closed Jacobi diagram*. These diagrams provide a better understanding of the primitive space $P\mathcal{A}$, see, e.g. [5].

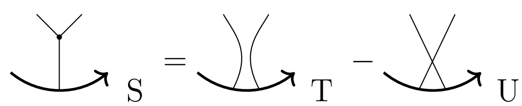
Definition 4.1. A *closed Jacobi diagram* (or, simply, a *closed diagram*) is a connected trivalent graph with a distinguished embedded oriented cycle, called Wilson loop, and a fixed cyclic order of half-edges at each vertex not on the Wilson loop.

Half the number of the vertices of a closed diagram is called the degree, or order, of the diagram. This number is always an integer. The vertices of a closed diagram belonging to the Wilson loop are called its *legs*.

In the pictures below, we shall always draw the diagram inside its Wilson loop, which will be assumed to be oriented counterclockwise unless explicitly specified otherwise. Inner vertices will also be assumed to be oriented counterclockwise.

Chord diagrams are exactly those closed Jacobi diagrams all of whose vertices lie on the Wilson loop.

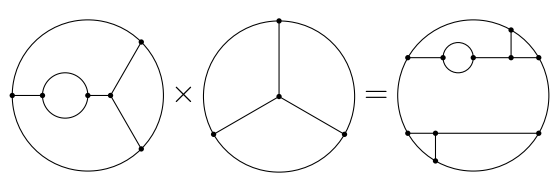
Definition 4.2. The vector space of closed diagrams $\mathcal{C}_n^{STU}$ is the space spanned by all closed diagrams $\mathcal{C}_n$ of degree n modulo the *STU relations*:



The three diagrams S, T and U must be identical outside the shown fragment. We write $\mathcal{C}^{STU}$ for the direct sum of the spaces $\mathcal{C}_n^{STU}$ for all $n \geq 0$.

Now we shall define a bialgebra structure in the space $\mathcal{C}^{STU}$.

Definition 4.3. The product of two closed diagrams is defined in the same way as for chord diagrams: the two Wilson loops are cut at arbitrary places and then glued together into one loop, in agreement with the orientations:



Definition 4.4. The *internal* graph of a closed diagram is the graph obtained by erasing the Wilson loop. A closed diagram is said to be *connected* if its internal graph is connected. The *connected components* of a closed diagram are defined as the connected components of its internal graph.

In the sense of this definition, any chord diagram of order n consists of n connected components — the maximal possible number of connected components in an order n Jacobi diagram.

Now, the construction of the coproduct proceeds in the same way as for chord diagrams, the chords being replaced by the more general connected components.

Definition 4.5. Let D be a closed diagram and $[D]$ the set of its connected components. For any subset $J \subseteq [D]$, denote by D_J the closed diagram with only those components that belong to J and by $D_{\bar{J}}$ the "complementary" diagram ($\bar{J} := [D] \backslash J$). We define the coproduct of D by

$$\delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\bar{J}}.$$

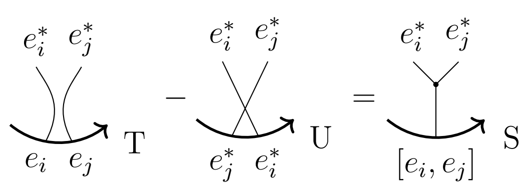
Now, for each $n = 0, 1, 2, \ldots$, we have a natural inclusion $\lambda : \mathcal{A}_n \to \mathcal{C}_n$.

Claim 4.6. [3] *The inclusion λ gives rise to an isomorphism of bialgebras $\lambda : \mathcal{A}^{fr} \to \mathcal{C}^{STU}$.*

By definition, connected closed diagrams are primitive with respect to the coproduct δ. It may sound surprising that the converse is also true:

Claim 4.7. [3] *The primitive space P of the bialgebra $\mathcal{C}^{STU}$ coincides with the linear span of connected closed diagrams.*

Since every closed diagram is a linear combination of chord diagrams, the weight system $\phi_{\mathfrak{g}}$ can be treated as a function on $\mathcal{C}^{STU}$ with values in $U(\mathfrak{g})$. The STU relation, the defining relation for the algebra $\mathcal{C}$, gives us a hint how to make this treatment explicit. Namely, if we assign elements e_i, e_j to the endpoints of chords of the T- and U- diagrams from the STU relations,



then it is natural to assign the commutator $[e_i, e_j]$ to the trivalent vertex on the Wilson loop of the S-diagram.

Generally, $[e_i, e_j]$ may not be a basis vector. A diagram with an endpoint marked by a linear combination of the basis vectors should be understood as the corresponding linear combination of diagrams marked by basis vectors. This understanding implies a useful

Lemma 4.8. *The degree of the value of a Lie algebra weight system on a closed diagram D is less or equal than the number of legs of D.*

5 The $\mathfrak{sl}(3)$ weight system

In this section, we concentrate on the weight system associated to the Lie algebra $\mathfrak{sl}(3)$.

Definition 5.1 (Weight systems associated with representations)**.** A linear representation $T : \mathfrak{g} \to \mathrm{End}(V)$ extends to a homomorphism of associative algebras $U(T) : U(\mathfrak{g}) \to \mathrm{End}(V)$. The composition of following three maps (with the last map being the trace)

$$\mathcal{A} \xrightarrow{\phi_{\mathfrak{g}}} U(\mathfrak{g}) \xrightarrow{U(T)} \mathrm{End}(V) \xrightarrow{Tr} \mathbb{C}$$

by definition gives the *weight system associated with the representation* T,

$$\phi_{\mathfrak{g}}^T = Tr \circ U(T) \circ \phi_{\mathfrak{g}}.$$

Consider the standard representation of the Lie algebra $\mathfrak{sl}(3)$ as the space of 3×3 matrices with zero trace. It is an eight-dimensional Lie algebra spanned by the matrices

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

whose commutators are

$$\begin{array}{llll} [E_i, F_j] = \delta_{ij} H_i, & [H_i, H_j] = 0, & [H_i, E_i] = 2E_i, & [H_i, F_i] = -2F_i, \\ [H_1, E_2] = -E_2, & [H_2, E_1] = -E_1, & [H_2, E_3] = E_3, & [H_1, E_3] = E_3, \\ [H_1, F_2] = F_2, & [H_2, F_1] = F_1, & [H_2, F_3] = -F_3, & [H_1, F_3] = -F_3. \end{array}$$

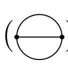
We shall use the symmetric bilinear form $\langle x, y\rangle = Tr(xy)$:

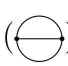
$$\langle E_i, E_j\rangle = 0, \qquad \langle F_i, F_j\rangle = 0, \qquad \langle H_i, E_j\rangle = 0, \qquad \langle H_i, F_j\rangle = 0,$$
$$\langle E_i, F_j\rangle = \delta_{ij}, \qquad \langle H_i, H_i\rangle = 2, \qquad \langle H_1, H_2\rangle = -1.$$

One can easily check that it is ad-invariant and nondegenerate. The corresponding dual basis is

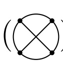
$$H_1^* = \frac{2}{3}H_1 + \frac{1}{3}H_2, H_2^* = \frac{1}{3}H_1 + \frac{2}{3}H_2, E_i^* = F_i, F_i^* = E_i,$$

and under the standard representation St of the Lie algebra $\mathfrak{sl}_3$ and the trace of the product of matrices as the preferred ad-invariant bilinear form, we have the $\mathfrak{sl}_3$ weight system associated with the standard representation $\phi^{St}_{\mathfrak{sl}_3} := Tr(\phi_{\mathfrak{sl}_3})$. This yields

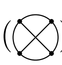
$\phi_{\mathfrak{sl}(3)}($$) = c_2 = \sum_i e_i e_i^* = \frac{2}{3}H_1^2 + \frac{2}{3}H_2^2 + \frac{1}{3}(H_1H_2 + H_2H_1) + \sum_{i=1}^{3}(E_iF_i + F_iE_i)$

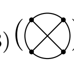
$\phi^{St}_{\mathfrak{sl}(3)}($$) = Tr\left(\frac{8}{3} \times id_3\right) = 8.$

In addition,

$\phi^{St}_{\mathfrak{sl}(3)}($$) = Tr\left(\sum_i e_i e_j e_i^* e_j^*\right) = Tr\left(-\frac{8}{9} \times id_3\right) = -\frac{8}{3}.$

Indeed,

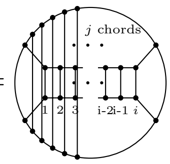
$\phi_{\mathfrak{sl}(3)}($$) = (c_2 - t)c_2,$

and we have $(\frac{8}{3} - t)\frac{8}{3} = -\frac{8}{9}$, hence $t = 3$. Therefore. $\phi_{\mathfrak{sl}(3)}($$) = (c_2 - 3)c_2$.

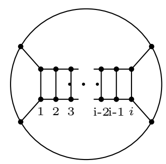
6 Values of the $\mathfrak{sl}(3)$ weight system on chord diagrams $K_{2,n}$

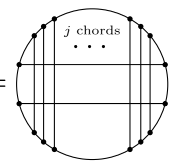
Given a weight system w, we write $\bar{w} := w \circ \pi$ for its composition with the projection to the subspace of primitives along the subspace of decomposable elements.

Denote by $J_{i,j}$ the order $i+j+2$ Jacobi diagram with $i-1$ cells and j chords crossing cells. $(i, j \geq 0)$

$J_{i,j} =$ 

Specifically,

$J_{i,0} =$ 

$J_{0,j} =$ 

and $J_{0,j}$ is the chord diagram whose intersection graph is the complete bipartite graph $K_{2,j}$.

Our first result is the following

Theorem 6.1. *For any simple Lie algebra $\mathfrak{g}$ endowed with the scalar product proportional to the Killing form with proportionality coefficient λ, one has*

$$\bar{w}_{\mathfrak{g}}(J_{i,j}) = \bar{w}_{\mathfrak{g}}(J_{i-1,j+1}) + \frac{1}{\lambda}\bar{w}_{\mathfrak{g}}(J_{i-1,j})$$

Theorem 6.1 implies the following

Corollary 6.2. *The following assertions are true:*

1. *the value $\bar{w}_{\mathfrak{g}}(J_{0,j})$ has degree at most 4;*
2. *we have $\bar{w}_{\mathfrak{g}}(J_{i,0}) = \sum_{k=0}^{i} \binom{i}{k} \lambda^{-k} \bar{w}_{\mathfrak{g}}(J_{0,i-k})$;*
3. *we have $\sum_{n=0} \bar{w}_{\mathfrak{g}}(J_{n,0}) \frac{x^n}{n!} = e^{\frac{x}{\lambda}} \sum_{n=0} \bar{w}_{\mathfrak{g}}(J_{0,n}) \frac{x^n}{n!}$.*

Then we get the values of $\mathfrak{sl}(3)$ weight system on chord diagrams whose intersection graph is complete bipartite $K_{2,n}$.

Main Theorem A. *We have*

$$\sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(K_{2,n}) \frac{x^n}{n!} = \sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(J_{0,n}) \frac{x^n}{n!} = e^{-6x} \sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(J_{n,0}) \frac{x^n}{n!},$$

$$\sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(K_{2,n}) \frac{x^n}{n!} = \frac{c_2}{40} \left((27c_2 - 72)e^{-8x} + (8c_2 + 72)e^{-3x} - 40c_2 e^{-6x} + 5c_2 \right),$$

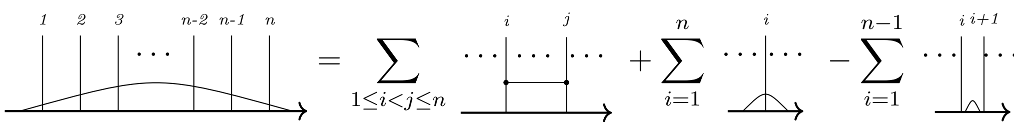
$$\sum_{n=0} w_{\mathfrak{sl}(3)}(K_{2,n}) \frac{x^n}{n!} = \frac{c_2}{40} \left((27c_2 - 72)e^{(c_2-8)x} + (8c_2 + 72)e^{(c_2-3)x} + 5c_2 e^{c_2 x} \right),$$

$$w_{\mathfrak{sl}(3)}(K_{2,n}) = \frac{c_2}{40} \left((27c_2 - 72)(c_2 - 8)^n + (8c_2 + 72)(c_2 - 3)^n + 5c_2^{n+1} \right).$$

Our computations show, in particular, that, for the chord diagrams whose intersection graph is $K_{2,n}$, their projection to the subspace of primitives can be represented as a linear combination of connected Jacobi diagrams with at most 4 legs. It has been conjectured earlier by S. Lando that the value of the weight system $\mathfrak{sl}(2)$ on a projection to the subspace of primitives of a chord diagram is a polynomial in the quadratic Casimir element c whose degree does not exceed half the length of the largest cycle in the intersection graph of the chord diagram. There is a lot of evidence supporting this conjecture, see, for example [12, 13]. The following more general conjecture may explain Lando's one, and is, probably, easier to prove.

Conjecture 6.3 (S. Lando, Z. Yang)**.** *Let D be a chord diagram, and let ℓ be the length of the longest cycle in it. Then the projection $\pi(D)$ of D to the subspace of primitives is a linear combination of connected Jacobi diagrams with at most ℓ legs. In particular, the value of a weight system associated to an arbitrary Lie algebra $\mathfrak{g}$ on this projection has degree at most ℓ.*

Theorem 6.4. *We have*



Corollary 6.5. *We have*

$$\deg(w_{\mathfrak{g}}(k_{m,n})) \le 2\min\{m,n\}.$$

Corollary 6.6. *We have*

$$\deg(w_{\mathfrak{g}}(V \times G)) \le 2|G|.$$

Corollary 6.7. *We have For $\mathfrak{sl}(2)$ weight system, we have*

$$w_{\mathfrak{sl}(2)}(D) = (c-2k)w_{\mathfrak{sl}(2)}(D_a) + 2\sum_{1\le i<j\le k}(w_{\mathfrak{sl}(2)}(D^{||}_{i,j}) - w_{\mathfrak{sl}(2)}(D^{\times}_{i,j}))$$

7 The $\mathfrak{gl}$ weight system for permutations

There is no recurrence relation for the weight system $w_{\mathfrak{gl}(N)}$ we know about. Instead, following the suggestion by M. Kazarian, we interpret an arc diagram as an involution without fixed points on the set of its ends and extend the function $w_{\mathfrak{gl}(N)}$ to arbitrary permutations of any number of permutated elements. For permutations, in contrast to chord diagrams, such a recurrence relation could be given.

For a permutation $\sigma \in S_m$, set

$$w_{\mathfrak{gl}(N)}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_{\sigma(1)}} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} \in U(\mathfrak{gl}(N)).$$

We claim that

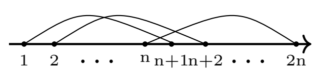
- $w_{\mathfrak{gl}(N)}$ lies in the center of $U(\mathfrak{gl}(N))$;
- this element is invariant under conjugation by a cyclic permutation: $w_{\mathfrak{gl}(N)}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} E_{i_1 i_{\sigma(1)}}$.

For example, the standard generator

$$C_m = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{m-1} i_m} E_{i_m i_1}$$

corresponds to the cyclic permutation $1 \mapsto 2 \mapsto \cdots \mapsto m \mapsto 1 \in S_m$.

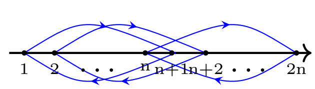
On the other hand, a chord diagram with n chords can be considered as an involution without fixed points on a set of $m = 2n$ elements. The value of $w_{\mathfrak{gl}(N)}$ on the corresponding involution is equal to the value of the $\mathfrak{gl}(N)$ weight system on the corresponding chord diagram.

Example 7.1. For the chord diagram $K_n =$ 

we have

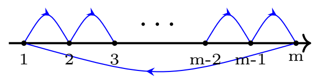
$$\begin{aligned} w_{\mathfrak{gl}(N)}(K_n) &= \sum_{i_1,\cdots,i_{2n}=1}^{N} E_{i_1 i_{n+1}} E_{i_2 i_{n+2}} \cdots E_{i_n i_{2n}} E_{i_{n+1} i_1} E_{i_{n+2} i_2} \cdots E_{i_{2n} i_n} \\ &= w_{\mathfrak{gl}(N)}((1\ n+1)(2\ n+2)\cdots(n\ 2n)). \end{aligned}$$

Definition 7.2 (digraph of the permutation)**.** Let us represent a permutation as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are ordered cyclically and are placed on a real line, subsequently connected with horizontal arrows looking right and numbered from left to right. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges, for example:

$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$ 

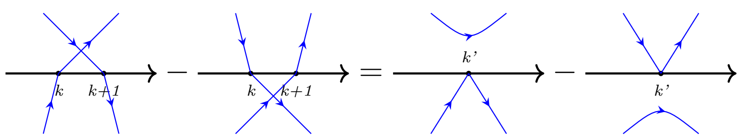
Example 7.3. The digraph of the Casimir element C_m, which corresponds to the cyclic permutation $1 \mapsto 2 \mapsto \cdots \mapsto m \mapsto 1 \in S_m$, is the following one:

$$G((1\ 2\ 3 \cdots m-1\ m)) =$$



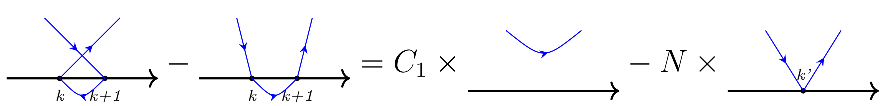
Main Theorem B. *The value of the $w_{\mathfrak{gl}(N)}$ invariant of permutations possesses the following properties:*

- *for an empty graph (with no vertices) the value of $w_{\mathfrak{gl}(N)}$ is equal to 1, $w_{\mathfrak{gl}(N)}(\bigcirc) = 1$;*
- *$w_{\mathfrak{gl}(N)}$ is multiplicative with respect to concatenation of permutations;*
- *for a cyclic permutation (with the cyclic order on the set of permuted elements compatible with the permutation), the value of $w_{\mathfrak{gl}(N)}$ is the standard generator, $w_{\mathfrak{gl}(N)}(1 \mapsto 2 \mapsto \cdots \mapsto k \mapsto 1) = C_k$.*
- (***Recurrence Rule***) *For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $k, k+1$, of the permuted set $\{1, 2, \ldots, m\}$, we have for the value of the $w_{\mathfrak{gl}(N)}$ weight system*



In the diagrams on the left, two horizontally neighboring vertices and the edges incident to them are depicted, while on the right these two vertices are replaced with a single one; the other vertices are placed somewhere on the line and their positions are the same on all diagrams participating in the relations, but the numbers of the vertices to the right of the latter are to be decreased by 1.

In particular, for the special case $\sigma(k+1) = k$, the recurrence looks like follows:



These relations are indeed a recursion, that is, they allow one to replace the computation of $w_{\mathfrak{gl}(N)}$ on a permutation with its computation on simpler permutations.

Remark 7.4. In the situation of permutations corresponding to chord diagrams, the difference at the right-hand side of the recurrence relation represents a Jacobi diagram with a triple vertex according to the STU relation, see Sec. 4. This gives a way to calculate the weight system $w_{\mathfrak{gl}(N)}$ on primitive elements given by Jacobi diagrams. For some special elements the calculation of this sort were given in [5, 9].

Corollary 7.5. *The value of $w_{\mathfrak{gl}(N)}$ on a permutation is well defined, can be represented as a polynomial in $N, C_1, C_2, \dots,$ and this polynomial is universal.*

Definition 7.6 (universal $\mathfrak{gl}$-weight system on permutations)**.** The *universal $\mathfrak{gl}$-weight system on permutation* $w_{\mathfrak{gl}}$ is the weight system taking values in the polynomial ring $\mathbb{C}[N, C_1, C_2, \cdots]$, which satisfies $w_{\mathfrak{gl}}(\sigma) = w_{\mathfrak{gl}(N)}(\sigma)$, for all permutations σ and is obtained by the above recurrence relations.

8 Symmetric functions and Harish–Chandra isomorphism

In this section, we show how to use the Harish-Chandra isomorphism for the Lie algebras $\mathfrak{gl}(N)$ to compute the corresponding weight systems.

Definition 8.1 (algebra of shifted symmetric polynomials)**.**
For a positive integer N, the algebra $\Lambda^*(N)$ of shifted symmetric polynomials in N variables $x_1, x_2, \cdots, x_N$ consists of polynomials that are invariant under changes of variables

$$(x_1, \cdots, x_i, x_{i+1}, \cdots, x_N) \mapsto (x_1, \cdots, x_{i+1} - 1, x_i + 1, \cdots, x_N),$$

for all $i = 1, \cdots, N-1$. Equivalently, this is the algebra of symmetric polynomials in the shifted variables $(x_1 - 1, x_2 - 2, \dots, x_N - N)$.

The universal enveloping algebra $U(\mathfrak{gl}(N))$ of the Lie algebra $\mathfrak{gl}(N)$ admits the direct sum decomposition

$$U(\mathfrak{gl}(N)) = (\mathfrak{n}_- U(\mathfrak{gl}(N)) + U(\mathfrak{gl}(N))\mathfrak{n}_+) \oplus U(\mathfrak{h}), \tag{1}$$

where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are the nilpotent subalgebras of, respectively, upper and lower triangular matrices in $\mathfrak{gl}(N)$, and $\mathfrak{h}$ is the subalgebra of diagonal matrices.

Definition 8.2 (Harish–Chandra projection in $U(\mathfrak{gl}(N))$)**.**
The *Harish–Chandra projection* for $U(\mathfrak{gl}(N))$ is the projection to the second summand in (1)

$$\phi : U(\mathfrak{gl}(N)) \to U(\mathfrak{h}) = \mathbb{C}[E_{11}, \cdots, E_{NN}],$$

where $E_{11}, \cdots, E_{NN}$ are the diagonal matrix units in $\mathfrak{gl}(N)$; they commute with one another.

Theorem 8.3 (Harish–Chandra isomorphism [35, 24])**.** *The Harish–Chandra projection, when restricted to the center $ZU(\mathfrak{gl}(N))$, is an algebra isomorphism to the algebra $\Lambda^*(N) \subset U(\mathfrak{h})$ of shifted symmetric polynomials in $E_{11}, \cdots, E_{NN}$.*

Thus, the computation of the value of the $\mathfrak{gl}(N)$ weight system on a chord diagram can be elaborated by applying the Harish-Chandra projection to each monomial of the polynomial. For such a monomial, the projection can be computed by moving variables E_{ij} with $i > j$ to the left, and/or variables E_{ij} with $i < j$ to the right by means of applying the commutator relations. If, in the process, we obtain monomials in $\mathfrak{n}_- U(\mathfrak{gl}(N))$ or $U(\mathfrak{gl}(N))\mathfrak{n}_+$, then we replace such a monomial with 0. A monomial in the (mutually commuting) variables E_{ii} cannot be simplified, and its projection to $U(\mathfrak{h})$ coincides with itself. The resulting polynomial in $E_{11}, \cdots, E_{NN}$ will be automatically shifted symmetric.

Example 8.4. Let's compute the projection of the quadratic Casimir element

$$C_2 = \sum_{i,j} E_{ij}E_{ji} \in ZU(\mathfrak{gl}(N))$$

to $U(\mathfrak{h})$. We have

$$\begin{aligned} C_2 &= \sum_i E_{ii}^2 + \sum_{i<j} E_{ij}E_{ji} + \sum_{i>j} E_{ij}E_{ji} \\ &= \sum_i E_{ii}^2 + 2\sum_{i>j} E_{ij}E_{ji} + \sum_{i<j}[E_{ij}, E_{ji}] \\ &= \sum_i E_{ii}^2 + 2\sum_{i>j} E_{ij}E_{ji} + \sum_{i<j}(E_{ii} - E_{jj}). \end{aligned}$$

In this expression, the first and the third summand depend on the diagonal unit elements E_{ii} only, while the second summand is in $\mathfrak{n}_- U(\mathfrak{gl}(N)) + U(\mathfrak{gl}(N))\mathfrak{n}_+$, whence the image under the projection is

$$\begin{aligned}\phi(C_2) &= \sum_i E_{ii}^2 + \sum_{i<j}(E_{ii} - E_{jj}) \\ &= \sum_i (E_{ii}^2 + (N+1-2i)E_{ii}).\end{aligned}$$

Similarly to the ring of ordinary symmetric functions, the ring $\Lambda^*(N)$ of shifted symmetric functions in N variables is isomorphic to a polynomial ring in N variables. There is a variety of convenient N-tuples of generators in $\Lambda^*(N)$. One of them is the tuple of shifted power sum polynomials

$$p_k = \sum_i \left(\left(E_{ii} + \frac{N+1}{2} - i \right)^k - \left(\frac{N+1}{2} - i \right)^k \right).$$

Representing $\phi(C_2)$ in the form

$$\phi(C_2) = \sum_i \left(\left(E_{ii} + \frac{N+1}{2} - i \right)^2 - \left(\frac{N+1}{2} - i \right)^2 \right),$$

we see that it is just p_2.

Remark 8.5. Since the Harish–Chandra isomorphism can be applied to arbitrary elements of $ZU(\mathfrak{gl}(N))$, we can also apply it to the values of $w_{\mathfrak{gl}}$ on permutations.

For $k > 2$, the expression for $\phi(C_k)$ is not reduced to just linear combinations of power sums. If fact, we have the following explicit formula, which follows from [26], [35, § 60] and [25, Remark 2.1.20],

$$\begin{aligned} 1 - Nu - \sum_{k=1}^{\infty} \phi(C_k) u^{k+1} &= \prod_{i=1}^{N} \frac{1 - (E_{ii} + N - i + 1)u}{1 - (E_{ii} + N - i)u} \\ &= (1 - Nu) e^{\sum_{k=1}^{\infty} \frac{(1 - \frac{N-1}{2}u)^{-k} - (1 - \frac{N+1}{2}u)^{-k}}{k} u^k p_k}. \end{aligned}$$

This provides an expression for the image $\phi(C_k)$ of C_k as a polynomial in $p_1, p_2, \dots,$ which is valid for all N. The projections of the Casimir elements $C_1, \cdots, C_N$ to $U(\mathfrak{h})$

can be expressed in shifted power sums $p_1, \cdots, p_N$ in the following way:

$$\begin{aligned}
\phi(C_1) &= p_1\\
\phi(C_2) &= p_2\\
\phi(C_3) &= -\frac{1}{4}N^2 p_1 + \frac{Np_2}{2} + \frac{p_1}{4} + p_3 - \frac{p_1^2}{2}\\
\phi(C_4) &= -\frac{1}{4}N^3 p_1 + N\left(-\frac{p_1^2}{2} + \frac{p_1}{4} + p_3\right) - p_1p_2 + \frac{p_2}{2} + p_4\\
&\dots
\end{aligned}$$

Computations using the Harish-Chandra isomorphism also are elaborative, and the results they produce are not universal, they depend on N. It is more efficient, therefore, to substitute the known values $\phi(C_k)$ into the answers obtained by the previous method.

9 Extension of the $\mathfrak{gl}(m|n)$-weight system to permutations

We define $w_{\mathfrak{gl}(m|n)}$ on permutations in the following way, which is similar to the definition for $w_{\mathfrak{gl}(N)}$.

For a permutation $\sigma \in S_k$, set

$$w_{\mathfrak{gl}(m|n)}(\sigma) = \sum_{i_1,\cdots,i_k=1}^{m+n} (-1)^{f_\sigma(\bar{i}_1,\dots,\bar{i}_k)} E_{i_1 i_{\sigma(1)}} E_{i_2 i_{\sigma(2)}} \cdots E_{i_k i_{\sigma(k)}},$$

where f_σ is the sign function which is a polynomial in $\bar{i}_1, \bar{i}_2, \dots, \bar{i}_k$ in the field $\mathbb{Z}_2$ defined below.

The sign function f_σ is a polynomial that has linear and quadratic terms only. For example, for the standard cyclic permutation $(12\dots k) : 1 \to 2 \to \dots \to k \to 1$, we have $f_{(12\dots k)}(\bar{i}_1, \dots, \bar{i}_k) = \bar{i}_2 + \dots + \bar{i}_k$.

We say that an index a, $1 \le a \le k$, is *distinguished* with respect to σ if $\sigma^{-1}(a) < a$. The set of distinguished indices is denoted by $P_1(\sigma) \subset \{1, \dots, k\}$. We say that a pair of indices (a, b), $1 \le a < b \le k$, is *distinguished* if the two pairs of distinct real numbers $(\sigma^{-1}(a) + \epsilon, a - \epsilon)$ and $(\sigma^{-1}(b) + \epsilon, b - \epsilon)$ alternate; here $\epsilon > 0$ is a small real number, say, $\epsilon = \frac{1}{3}$. The set of distinguished pairs of indices is denoted by $P_2(\sigma) \subset \{1, \dots, k\} \times \{1, \dots, k\}$.

Definition 9.1. The sign function f_σ of a permutation $\sigma \in S_k$ is defined by

$$f_\sigma(\bar{i}_1, \bar{i}_2, \dots) = \sum_{a \in P_1(\sigma)} \bar{i}_a + \sum_{(a,b) \in P_2(\sigma)} \bar{i}_a \bar{i}_b.$$

A more convenient treatment of the invariant $w_{\mathfrak{gl}(m|n)}(\sigma)$ and the sign function uses the language of digraphs from the previous section.

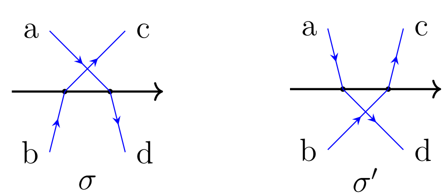
The set of indices participating in the summation will be labelled by the edges (rather than by vertices). For each vertex i, we denote by $in(i)$ and $out(i)$ the incoming edge and outcoming edge incident to the vertex i, respectively. With this notation, we have

$$w_{\mathfrak{gl}(m|n)}(\sigma) = \sum_{i_1,\cdots,i_k=1}^{m+n} (-1)^{f_\sigma(\bar{i}_1,\dots,\bar{i}_k)} E_{i_{in(1)} i_{out(1)}} \cdots E_{i_{in(k)} i_{out(k)}}.$$

The original formula corresponds to the numbering of the edges such that the edge $i \to j$ is numbered j. The result is obviously independent of the numbering.

With this notation, an edge is distinguished if it is directed from left to right. A pair of edges with pairwise distinct ends is distinguished if the corresponding pairs of vertices alternate. If the edges have common vertices, we first bring them to a general position by shifting slightly the beginning of each edge to the right and the endpoint of each edge to the left, and then check whether the pairs of ends of the shifted edges do alternate.

Assume that two permutations σ and σ' are conjugate by a transposition of two neighboring elements. Then these two elements are the endpoints of the four edges a, b, c, d as shown in the picture below (among the edges a, b, c, d there could be pairs of coincident ones).



Lemma 9.2. *The sign functions f_σ and $f_{\sigma'}$ are related by*

$$f_{\sigma'} = f_\sigma + (\bar{i}_a + \bar{i}_d)(\bar{i}_b + \bar{i}_c).$$

In other words, each of the four pairs of edges $(a, c), (a, d), (b, c), (b, d)$ changes the property of being distinguished when one passes from the permutation σ to σ'.

Since the sign function f_σ matches the sign in the Casimir elements and this lemma says the sign function f_σ matches the involution operation S, we have

Claim 9.3. *The $\mathfrak{gl}(m|n)$-weight system for chord diagrams in [28, 11] is a special case of the $\mathfrak{gl}(m|n)$-weight system for permutations, where we treat a chord diagram with k chords as an involution without fixed points on the set of $2k$ elements.*

Main Theorem C. *The weight system $w_{\mathfrak{gl}(m|n)}$ for permutations is the result of substituting $m-n$ for C_0, and the k th Casimir element in $\mathfrak{gl}(m|n)$ for C_k, $k>0$, in the weight system $w_{\mathfrak{gl}}$.*

10 Statement of the main results

The results of this dissertation are reflected in three papers:

- Zhuoke Yang, *On the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system*, Journal of Geometry and Physics. 2023. Vol. 187. Article 104808.
- Zhuoke Yang, *New approaches to $\mathfrak{gl}_N$ weight system*, `arXiv:2202.12225 [math.CO]`, accepted by Izvestiya Mathematics
- Zhuoke Yang, *On values of $\mathfrak{sl}_3$ weight system on chord diagrams whose intersection graph is complete bipartite*, `arXiv:2102.00888 [math.CO]`, Accepted by Moscow Mathematical Journal

In paper Zhuoke Yang, *On values of $\mathfrak{sl}_3$ weight system on chord diagrams whose intersection graph is complete bipartite* `arXiv:2102.00888 [math.CO]`, we get the values of $\mathfrak{sl}(3)$ weight system on chord diagrams whose intersection graph is complete bipartite $K_{2,n}$.

Main Theorem A. *We have*

$$\sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(K_{2,n})\frac{x^n}{n!} = \sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(J_{0,n})\frac{x^n}{n!} = e^{-6x}\sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(J_{n,0})\frac{x^n}{n!},$$

$$\sum_{n=0} \bar{w}_{\mathfrak{sl}(3)}(K_{2,n})\frac{x^n}{n!} = \frac{c_2}{40}\left((27c_2-72)e^{-8x}+(8c_2+72)e^{-3x}-40c_2e^{-6x}+5c_2\right),$$

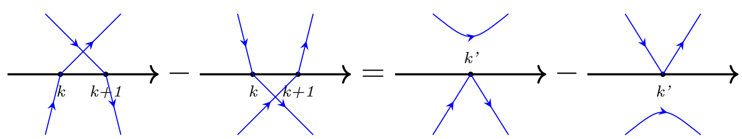
$$\sum_{n=0} w_{\mathfrak{sl}(3)}(K_{2,n})\frac{x^n}{n!} = \frac{c_2}{40}\left((27c_2-72)e^{(c_2-8)x}+(8c_2+72)e^{(c_2-3)x}+5c_2e^{c_2x}\right),$$

$$w_{\mathfrak{sl}(3)}(K_{2,n}) = \frac{c_2}{40}\left((27c_2-72)(c_2-8)^n+(8c_2+72)(c_2-3)^n+5c_2^{n+1}\right).$$

In paper Zhuoke Yang, *New approaches to* $\mathfrak{gl}_N$ *weight system*, `arXiv:2202.12225 [math.CO]`, accepted by Izvestiya Mathematics, we interpret an arc diagram as an involution without fixed points on the set of its ends and extend the function $w_{\mathfrak{gl}(N)}$ to arbitrary permutations of any number of permutated elements. For permutations, in contrast to chord diagrams, a recurrence relation could be given.

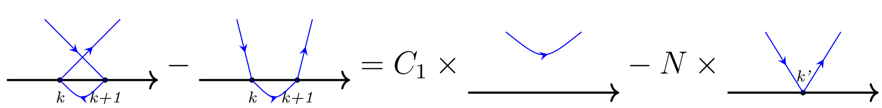
Main Theorem B. *The value of the* $w_{\mathfrak{gl}(N)}$ *invariant of permutations possesses the following properties:*

- *for an empty graph (with no vertices) the value of* $w_{\mathfrak{gl}(N)}$ *is equal to* 1, $w_{\mathfrak{gl}(N)}(\bigcirc) = 1$;
- $w_{\mathfrak{gl}(N)}$ *is multiplicative with respect to concatenation of permutations;*
- *for a cyclic permutation (with the cyclic order on the set of permuted elements compatible with the permutation), the value of* $w_{\mathfrak{gl}(N)}$ *is the standard generator,* $w_{\mathfrak{gl}(N)}(1 \mapsto 2 \mapsto \cdots \mapsto k \mapsto 1) = C_k$.
- (***Recurrence Rule***) *For the graph of an arbitrary permutation* σ *in* S_m, *and for any two neighboring elements* $k, k+1$, *of the permuted set* $\{1, 2, \ldots, m\}$, *we have for the value of the* $w_{\mathfrak{gl}(N)}$ *weight system*



In the diagrams on the left, two horizontally neighboring vertices and the edges incident to them are depicted, while on the right these two vertices are replaced with a single one; the other vertices are placed somewhere on the line and their positions are the same on all diagrams participating in the relations, but the numbers of the vertices to the right of the latter are to be decreased by 1.

In particular, for the special case $\sigma(k+1) = k$, *the recurrence looks like follows:*



These relations are indeed a recursion, that is, they allow one to replace the computation of $w_{\mathfrak{gl}(N)}$ *on a permutation with its computation on simpler permutations.*

In paper Zhuoke Yang, *On the Lie superalgebra* $\mathfrak{gl}(m|n)$ *weight system*, Journal of Geometry and Physics. 2023. Vol. 187. Article 104808., we extend this construction to the weight system associated to the Lie superalgebras $\mathfrak{gl}(m|n)$. Then we prove that the $\mathfrak{gl}(m|n)$-weight system is equivalent to the $\mathfrak{gl}$-one, under the substitution $C_0 = m - n$:

Main Theorem C. *The weight system $w_{\mathfrak{gl}(m|n)}$ for permutations is the result of substituting $m - n$ for C_0, and the k th Casimir element in $\mathfrak{gl}(m|n)$ for C_k, $k > 0$, in the weight system $w_{\mathfrak{gl}}$.*

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