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On the problems of boundary and distributed controllability for some systems described by differential and integro-differential equations

Dissertation summary

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1 Problem statement

Research in the field of control theory of systems with distributed parameters has a fairly long history. For the first time, these issues began to be considered, apparently, in the 60s of the 20th century. Pioneering works by such authors as, for example, J. L. Lyons ([1-4]) and A. G. Butkovsky ([5]) and D. L. Russell ([6]) were devoted to the problems of controlling the vibrations of a string, membrane and thin plate. The problems of both boundary controllability and global (or local) controllability were considered. The goals of the control were different. In some studies, the emphasis was given on driving the vibrations of the system to a given state, in others it was necessary to optimize various functionals having a physico-mechanical meaning. After that, similar tasks were already set for the heat equation.

In the problems listed above, additional restrictions can be imposed on the control function. These restrictions may have a different nature. For example, it is required to stop the vibrations of the membrane by means of the force action on the boundary (or part of the boundary) with the limitation of this force according to the norm of space L_2.

From the point of view of mechanics, it will be much more natural if the boundary control force is bounded in absolute value by an arbitrarily small, predetermined number. This additional condition significantly complicates the task. In this case, all previously used methods in their original form no longer work.

This dissertation sets and solves precisely this problem – to drive the oscillations described by classical systems of mechanics (the wave equation and the plate oscillation equation) to rest in a finite time by means of a force control action bounded in absolute value applied to the boundary (or to part of it) of the domain occupied by the system. In addition, the task is to drive to rest the vibrations of the system described by the «plate oscillation» equation on the torus in the case of a local (i.e. applied to part of the domain) force action. There are no restrictions imposed on the control function.

In addition to classical systems, in the last 15-20 years, research in the field of mechanical systems with so-called «memory» or integral aftereffect has become widespread. It all started with the study of the Gurtin-Pipkin equation ([7]), which can be used to describe, for example, the process of heat propagation with a finite velocity (there is a heat front). The kernel $K(t)$ can have a different form, reflecting the nature of the physical processes that this equation also describes. In the two simplest cases, this kernel can be identically equal to one or equal to a delta-function. In the first case, the equation is reduced to a wave equation by differentiating by a time variable, in the second case – turns into the heat equation. Below we will consider kernels of a more complex type. After the Gurtin-Pipkin equation, classical type equations were studied with the addition of integral terms (memory). These additional terms in some cases make it possible to describe certain processes of mechanics and physics more effectively.

For the Gurtin-Pipkin equation and similar systems, the paper also sets the task of driving the system to rest in a finite time. The control can be applied to the boundary, part or the whole domain. If the control function that drives the mechanical system to rest exists, this system is called *controllable*. In contrast, if there is no controllability, i.e. there are initial perturbations of the mechanical system that cannot be extinguished in a finite time, then such a system is called *uncontrollable*.

An additional difficulty in control problems for systems with memory is the lack of equivalence of the concepts «controllability to rest» and «null controllability». The reason is the presence of an integral term in the equation. Having reached the zero state at some point in time, the system can then leave it. As will be established later, most mechanical systems with memory are uncontrollable.

2 Relevance of the research topic

For classical systems of mechanics (membranes, thin plates), controllability issues are important in cases where the control action is applied either to the boundary or to a part of the domain. This is due to the fact that in practice it is difficult to control the whole system, but it is possible to influence only a part of it. In this regard, there is a problem of choosing this part, for example, for the wave equation such a choice is determined by a special condition (Geometric control condition, [8]), which consists in the fact that each optical ray of length T (control time) in the domain Ω enters the subdomain D (a force control action is applied to this subdomain). A significant complication in the formulation of control tasks is a restriction on the absolute value of the control function, such a restriction is due to the impossibility in real conditions to find an arbitrarily large force effect. In general, the relevance of the study of classical systems of mechanics is associated with the mathematical complexity of the statements in which there are additional restrictions on controls, but these restrictions are quite natural.

The study of controllability of mechanical systems with integrated memory is relevant due to the wide range of applications of these systems in practice. Integro-differential equations with non-local convolution-type terms often arise in applications such as heterogeneous media mechanics, viscoelasticity theory, thermophysics and kinetic theory of gases. For example, it is strictly proved that in the case of a two-phase heterogeneous medium consisting of a viscous liquid and elastic additives, the model described by the integro-differential equation will be effective and the corresponding convolution kernel consists of a finite or infinite sum of decreasing exponential functions.

If the viscosity of the liquid is small (high), then the effective equation contains (does not contain) third-order terms corresponding to the Kelvin-Voigt friction. This issue is described in [9] as well. In the theory of viscoelasticity relaxation kernels are approximated, as a rule, by sums of exponentials. In thermophysics, the laws of thermal conductivity with integral memory are the object of study in many research papers, in particular, see [7]. The presence of an integral memory in the law of thermal conductivity can lead to the appearance of a thermal front that moves at a finite speed. This creates a significant difference from the heat equation, whose solution sets the propagation of heat at an infinite speed. Also, the memory equations describe the process of diffusion of delayed neutrons in a nuclear reactor, acoustics in a medium that is a liquid with an admixture of solid particles.

3 The degree of the problem development

To date, the degree of the development of the controllability problem for classical systems of mechanics (membranes and plates) is very high. Let us note some of the most important results.

Previously, the controllability problem of the vibrations of a plane membrane using boundary forces was considered by many authors (see, for example, review articles [6, 10], as well as the references there). In [5], the problem of stopping the vibrations of a bounded string using boundary control is described and it is proved that it is possible to completely stop the vibrations of the string in a finite time with a restriction on the absolute value of the control action. Additionally, an estimation of the time required to completely stop the vibrations is given. In [1], problems of optimal control of systems with distributed parameters are considered and optimality conditions similar to the Pontryagin maximum principle are formulated for systems with a finite number of the degrees of freedom. At the same time, these conditions do not always lead to a efficient way of constructing optimal control. In [10], the problem of the complete stop of the membrane motion is presented, the existence of such boundary control is proved, and the time required for the complete stop of oscillations is estimated. Here, in many problem statements, the authors abandon the requirements of optimal control and consider only the problem of controllability, which greatly facilitates the study. The paper does not consider problems with a restriction on the absolute value of the control forces, and there are no explicit expressions for control actions, but only existence theorems are proved.

In addition to driving to rest there is a so-called solution stabilization problem for distributed oscillatory systems. This task consists in setting some feedback control at the boundary of the domain, which «stabilizes» the solution, i.e. the energy of the system tends to zero when the time t tends to infinity. For example, in [6], the problem of stabilizing the membrane energy by friction introduced at the boundary is considered. More precisely, the boundary of the domain occupied by the membrane consists of two parts: Γ_0 and Γ_1. A Dirichlet condition is introduced on Γ_0, i.e. this part of the boundary is rigidly fixed, and a boundary condition of the form is introduced on Γ_1

$$\frac{\partial w}{\partial \nu} = -k\frac{\partial w}{\partial t},$$

where ν is the external normal to Γ_1, $k > 0$. The friction set in this way leads to the dissipation of the energy of the system, and consequently to the stabilization of its oscillations. Since part of the boundary is fixed, the energy of the system coincides with the square of the norm of the direct product of spaces: $H^1 \times L_2$. Hence, with $t \to +\infty$ the solution of the problem and its first derivative in t (speed) tend to zero according to the norms of the spaces H^1 and L_2, respectively. Note that in this formulation, the initial data of the problem should be chosen sufficiently smooth and satisfy the conditions of coordination. A similar formulation was considered for the problem of boundary stabilization of transverse vibrations of a thin plate [11].

In general, the methods of boundary stabilization are quite effective, since they allow to drive the oscillations of the system in a finite time to an arbitrarily small neighborhood of zero, which in practice, as a rule, is equivalent to driving to rest. However, these methods have a drawback. The time spent on stabilization may be longer than in exact controllability problems. For example, for a plate, there are known methods that allow to drive the vibrations of the system to rest in an arbitrarily small time.

Controllability problems for systems with memory, as opposed to classical systems, are not investigated in such details. Let us give a brief overview of the results on this issue.

The presence of a non-local convolution type term in equations and systems leads to a large number of interesting qualitative effects that are not observed in classical differential equations and systems of differential equations. For example, systems of this type contain properties of both parabolic and hyperbolic equations. In spectral problems for such equations and systems, the spectrum consists of real and complex parts. The first part corresponds to the energy dissipation in the heat equation, the latter, in turn, corresponds to oscillations. Such equations can be solved using a method similar to the Fourier method. Moreover, systems of this type are usually uncontrollable to rest if we apply boundary control or control that is distributed over parts of the domain.

If the control is distributed over the entire domain, then the integral terms in some case «help» the control process. In this case, the control time can be significantly reduced. It should be noted that the spectral method proposed in [12] is sometimes successfully applied to the case of systems with non-local convolution type terms (for more details, see [13]).

The above-mentioned uncontrollability was discovered in [14] for a one-dimensional heat equation with memory.

In most cases, the controllability to rest is impossible. For example, in [14] it was proved that the solution of the heat equation with memory cannot be driven to rest in a finite time if some auxiliary analytical function has zeros. This result is true for boundary and local controllability. Moreover, the case of local controllability can be reduced to the case of the boundary control. We obtained similar results for problems with two-dimensional domains.

It should also be noted the paper [15], since uncontrollability is established in it for the heat equation with memory in some special case.

Positive controllability results for a multidimensional wave equation with memory were obtained in [13]. There it was shown that the system described by this equation can be driven to rest with the help of bounded distributed control. In this case, the kernel in the integral term is the sum of N decreasing exponential functions.

Problems for integro-differential equations close to the Gurtin-Pipkin equation

$$\dot{\theta}(t,x) = \int\limits_0^t K(t-s)\Delta\theta(s,x)ds, \tag{1}$$

widely studied. The equation (1) was first derived in [7]. The issues of solvability and asymptotic behavior of solutions of equations of this type were investigated in [16, 17]. In [18] it was proved that the energy of some dissipative system decreases polynomially, while the core decreases exponentially.

Problems related to the solvability of memory systems described by the equation (1) and the like were considered in [19]. It was proved that the solution belongs to some Sobolev space on the semiaxis (by the variable t) if the kernel $K(t)$ is the sum of exponential functions, each of which tends to zero at $t \to +\infty$.

Interesting exact formulas for the solution were obtained in [20] under the assumption that the kernel $K(t)$ is also the sum of decreasing exponential functions.

4 Goals and objectives of the study

1. Prove that the oscillations of a mechanical system described by a two-dimensional wave equation can be stopped in a finite time by a force control action bounded in absolute value (Neumann condition) applied to a part of the boundary. More precisely, a two-dimensional domain with a hole is considered and the control is applied to the outer contour of the border, the edges of the hole remain fixed. Note that the stopping of oscillations in this case means

reaching a state with zero displacement and zero velocity at the terminal moment of time.

2. Prove that the oscillations of a mechanical system described by a two-dimensional wave equation can be stopped in a finite time by a force control action bounded in absolute value (Neumann condition) applied to the entire boundary. In this case, a domain without holes is considered. Note that the stopping of oscillations in this case means reaching a state with a constant displacement and zero velocity at the terminal moment of time.

3. Prove that the vibrations of a mechanical system described by the equation of vibrations of a thin plate can be stopped in a finite time by a force control action bounded in absolute value (Neumann condition) applied to a part of the boundary. More precisely, it is considered, as in paragraph 1, a two-dimensional domain with a hole and the control is applied to the outer contour of the border, the edges of the hole remain rigidly fixed. At the initial moment, displacement and velocity are finite functions taken from some Sobolev classes. Note that the stopping of oscillations in this case means reaching at the terminal moment state time with zero displacement and zero velocity.

4. Prove local controllability for the «plate oscillation» equation on the torus. In this case, the subdomain to which the control is applied is arbitrary, and the order of smoothness of the control function increases with the smoothness of the initial data.

5. Prove the lack of local controllability for a mechanical system described by the Gurtin-Pipkin equation for a wide class of kernels. This class consists of functions continuous on the time semiaxis whose Laplace transform has at least one zero in the domain of holomorphism. The lack of local controllability in this case means that there are initial perturbations of the system that cannot be extinguished in a finite time by the control applied to the subdomain.

6. Prove the lack of global controllability for a mechanical system described by the Gurtin-Pipkin equation with a kernel represented by a series of decreasing exponential functions with «slowly» increasing exponents. The lack of global controllability in this case means that there are initial disturbances of the system that cannot be extinguished in a finite time by the control applied even to the entire domain.

7. Prove global bounded controllability for a special case of a wave equation with integral memory and a kernel consisting of the sum of a finite number of decreasing exponential functions. In this case, «bounded controllability» means that the control action function is

bounded in absolute value and at the terminal moment of time the oscillations of the system will be stopped.

8. Prove the lack of boundary controllability (in the one-dimensional case) for almost all models of «naive mechanics» (for details, see section 7, item 6).

5 Scientific novelty

The statement of the problem in this dissertation for the problems of controllability of classical systems differs significantly from the statements from [4, 6], since the value of the controlling boundary force must satisfy the condition:

$$|u(t,x)| \leqslant \varepsilon.$$

Note that the goal here is to find not optimal, but admissible control, satisfying this condition.

For example, in the first part of the dissertation, a membrane is considered in which one part of the border is fixed and control is applied to the other part. This control function is determined by the Neumann condition and is bounded in absolute value. Some important geometric conditions are imposed on both parts of the boundary (for exact formulations, see section 6, item 1). The purpose of the control process is to achieve a state of the system such that its displacement and velocity are zero. A similar statement is considered for the boundary controllability problem of a thin plate.

In the case of systems with memory, the whole spectrum of controllability problems also arises for various cases: boundary, local and global. For example, for the Gurtin-Pipkin equation with a kernel in the form of sums of an infinite number of exponents even global controllability does not take place. To prove the lack of controllability, the exponents of the series, which represents the convolution kernel, should slowly tend to minus infinity. A close result is proved in [21]. This result indicates the «unroughness» of the controllability property in this problem. Namely, the remainder of the series with which the convolution kernel is defined can be arbitrarily small and decrease arbitrarily quickly. If we discard this remainder, then the system becomes controllable, and its restoration leads to an uncontrollable system. This is the property of the «unroughness» of the task. It is in «dissonance» with the problem of controllability of linear finite-dimensional systems. According to the classical Kalman criterion of controllability of linear finite-dimensional systems full controllability is equivalent to the full rank of some rectangular matrix constructed for the problem in question, which in turn is equivalent to the difference from zero of several determinants from the elements of this matrix. It is clear that with a sufficiently small arbitrary perturbation of the problem data, this property of difference from zero determinants is preserved, which indicates the «roughness» of the controllability property.

In general, the properties of controllability problems for integro-differential systems are radically different from the properties of similar controllability problems for differential systems. So, if the problems of the boundary controllability for differential systems are usually solvable (while, of course, there are certain solvability conditions), then the solvability of similar problems for integro-differential systems are exceptional cases. To illustrate this fact, we can consider the boundary controllability problem for the one-dimensional Gurtin-Pipkin equation. It turns out that an obstacle to boundary controllability is, for example, that the density of the spectrum of the problem under consideration is equal to infinity, where the density is understood in the sense of some numerical characteristic ([22]). Often the spectra of integro-differential problems have an accumulation endpoint and therefore the spectrum is «dense», hence the lack of controllability takes place. So, in the one-dimensional case for the Gurtin-Pipkin equation with a kernel of the form (34) (see below), the boundary controllability problem is solvable only if the kernel consists of one decreasing exponential function; in cases where the kernel is the sum of two or more decreasing exponential functions, it is unsolvable due to the presence of accumulation endpoints in the spectra.

One feature should be pointed out in controllability problems for integro-differential equations. If the control is presented by force applied to the subdomain, then in most cases there is no controllability ([23]). At the same time, in this work it is proved that even if an arbitrarily small neighborhood inside the domain is not contained in the domain of the application of the control force, then the system is not controllable, i.e. there is an initial condition that we cannot drive to complete rest in a finite time, no matter what control action we apply that satisfies the conditions of the problem.

Recently, research has been quite actively conducted on the controllability problem for an integro-differential system when control force is applied to a moving sub-section (in the one-dimensional case). In this research controllability is established for some cases ([24]). Such a statement of the problem is, in some sense, intermediate between the problem of stopping vibrations using a force applied to a fixed part of the domain (interval) and the controllability problem in the case when the force is applied to the entire domain (interval). This formulation leads to interesting spectral problems about the existence of a biorthogonal system of functions for a system of exponents on a segment and about estimates for elements of this system, if it exists. At the same time, the study of the problem of full controllability is significantly more complicated compared to the case of a fixed subdomain, and positive results are obtained only for special cases. So, in [24], the equation of string vibrations with a convolution type term is considered, and the convolution kernel is equal to 1. The transition to a decreasing exponential function as the convolution kernel is not at all obvious. The paper also considers the boundary conditions of periodicity, and the transition to Dirichlet conditions is also not obvious.

6 Description of the research methodology

To prove the controllability of classical systems, the main methods known in the scientific literature are used. Namely, D. L. Russell's methods ([6]) on the construction of control through the extension of the solution to an unbounded domain or through the construction the friction (or its «counterparts») at the boundary of the domain. There is also a combination of these methods. To prove the local controllability of the system described by the «plate oscillation» equation on the torus, the cascade splitting method of the original problem and the HUM (Hilbert Uniqueness Method) are used.

To study the controllability of memory systems, the method of moments is used, and various methods of complex analysis that are used to prove the lack of controllability. We are talking about obstacles to the controllability of various mechanical systems. As such obstacles, one can, for example, indicate the existence of a limit point in the spectrum of the problem, the «slow» growth of the spectrum, the presence of the so-called «branching point», etc.

It should be noted that in many cases the direct application of the methods mentioned above (in their original form) was either impossible or difficult, therefore, in the presented work they often had to be adapted or significantly modified.

We will briefly describe the application of these methods to solve some problems of the thesis. In the first part of it, it is proved that a two-dimensional membrane can be driven to rest in a finite time using an absolute-bounded control force applied to the boundary of the membrane. In the formulation of the controllability problem, smoothness conditions and some boundary conditions are imposed on the initial data functions (displacement and velocity). The boundary force is determined by the inhomogeneous Neumann condition and in one case it is applied to the entire boundary of the domain, and in the other case – to its part. The solution of the problem is divided into two stages. At the first stage, the solution is stabilized into a sufficiently small neighborhood of the rest state by means of friction introduced at the boundary of the domain. At the same time, a sufficient smallness of the control value is achieved by choosing a value of the friction coefficient close to zero. In this case, the results from [6, 25, 26] are used. These works deal with the stabilization of membrane energy by means of boundary conditions of a special kind. At the second stage of control, the vibrations of the membrane are completely stopped. Here, a significant role is played by the method of extending the initial data to some bounded domain and considering some special initial-boundary value problem for a two-dimensional wave equation in this domain. Then the control is the derivative with respect to the normal to the boundary of the initial domain occupied by the membrane, taken from the solution of the specified initial boundary value problem. Note that the control method at this stage is actually determined by the way of the extension of the initial data to the mentioned bounded domain. The reversibility of the classical wave equation in time plays a decisive role in such a construction. Control of this kind was used in the works of many authors of the 70s-90s. In this case, the restriction on the absolute value of the control force action is carried out due to the fact that the solution of the original problem was brought into a sufficiently small neighborhood according to the norm of some Sobolev space at the first stage.

The possibility of stopping transverse vibrations of a thin plate is also investigated precisely in the case when the boundary control actions are bounded in absolute value. At the same time, significant restrictions are imposed on the geometry of the boundary of the domain occupied by the plate (see section 7, item 2). In addition, some conditions are also imposed on the initial data of the problem, namely the conditions of smoothness and agreement (see section 7, item 2). Controllability issues related to the weakening of these restrictions remain open. For example, in the presented study boundary of the domain occupied by the plate should consist of two parts. Namely, a plate with a hole is considered. It remains unclear whether vibrations can be driven to rest (by a bounded boundary control) if there is no hole and the region is simply connected. There is also a problem of reducing the degree of smoothness of the initial data. In this study, sufficiently strong smoothness conditions are imposed on the initial perturbation.

The second part of the theses deals with the problems of distributed (including local) and boundary controllability of oscillations of the systems described by the Gurtin-Pipkin equation and its counterparts. This equation contains a convolutional (by time variable) type term, this term is often called memory. The question is raised about the possibility of driving such systems to rest by means of various methods of control, namely, boundary, local and applied to the entire domain. Note that, generally speaking, the concept of «controllability to rest» for systems with memory is not equivalent to driving the system to a zero state. In addition, controllability to rest for such models is not always possible even if the control action is applied to the entire domain occupied by the mechanical system (see section 7, item 5). This is where the obstacles to controllability mentioned at the beginning of this section apply.

7 Main problems solved in the thesis

1. On the problem of boundary controllability for a system described by a two-dimensional wave equation. Let Ω be a bounded domain in R^2 with an infinitely smooth boundary, ν – external unit normal to the boundary of the domain Ω, Σ – lateral surface of the cylinder $Q_T = (0,T) \times \Omega$.

Here and further we will assume that Ω (or any other domain under consideration) is located locally on one side of its boundary.

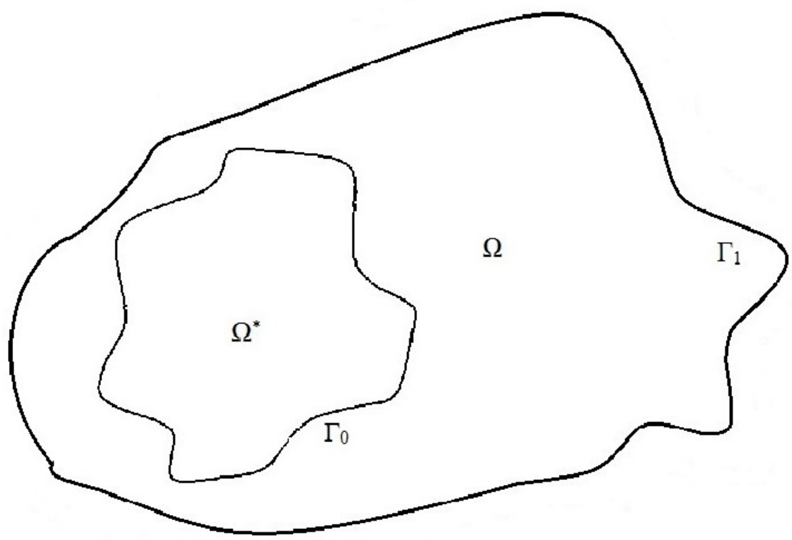
Let also the boundary of Ω consist of two connected parts Γ_0 and Γ_1, i.e.

$$\partial\Omega = \Gamma_0 \cup \Gamma_1.$$

We suppose additionally that

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$$

and Γ_0 is the boundary of some bounded domain Ω^*, such that $\Omega \cap \Omega^* = \emptyset$ (Pic. 1).



Pic. 1

We denote

$$\Sigma_i = (0,T) \times \Gamma_i, \quad i = 0, 1.$$

Let us consider the initial boundary value problem for the equation of membrane vibrations

$$w_{tt}(t,x) - \Delta w(t,x) = 0, \quad (t,x) \in Q_T, \tag{2}$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{3}$$

$$w(t,x) = 0, \quad (t,x) \in \Sigma_0, \tag{4}$$

$$\frac{\partial w}{\partial \nu} = u(t,x), \quad (t,x) \in \Sigma_1. \tag{5}$$

Let $\varepsilon > 0$ be an arbitrary number. The task is to construct such a control function u satisfying the inequality

$$|u(t,x)| \leqslant \varepsilon, \tag{6}$$

that the corresponding solution w and its first derivative with respect to t vanish at some point in time T, i.e. $w(T,x) = 0$, $w_t(T,x) = 0$ for all $x \in \Omega$. If this problem has a solution, then the system (2)–(5) will be called *controllable.*

Consider the space

$$\mathcal{H}_0^3(\Omega) = \{(w_1, w_2) \in H^3(\Omega) \times H^2(\Omega) : w_1(x) = w_2(x) = \Delta w_1 = 0,\ x \in \Gamma_0\}.$$

Theorem 1. *Let additionally the boundary Ω satisfies the condition: there exists a point $x_0 \in R^2$ such that*

1) $(x - x_0) \cdot \nu \leqslant 0,\ \ x \in \Gamma_0,$

2) $(x - x_0) \cdot \nu \geqslant \beta > 0,\ \ x \in \Gamma_1,$

besides $(\varphi(x), \psi(x)) \in \mathcal{H}_0^3(\Omega)$ and

$$\frac{\partial \varphi}{\partial \nu} = \psi = \frac{\partial \psi}{\partial \nu} = \Delta\varphi = 0 \ \text{ on } \Gamma_1. \tag{7}$$

Then the system (2)–(5) is controllable.

Consider the case when a part of the boundary of the domain is not fixed. Consider the initial boundary value problem for a two-dimensional wave equation

$$w_{tt}(t,x) - \Delta w(t,x) = 0, \quad (t,x) \in Q_T = (0,T) \times \Omega, \tag{8}$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{9}$$

$$\frac{\partial w}{\partial \nu} = u(t,x), \quad (t,x) \in \Sigma, \tag{10}$$

where $\Omega \subset R^2$ is a bounded, star-shaped relative to some closed disk domain with an infinitely smooth boundary. The initial data $\varphi(x)$ and $\psi(x)$ are given and will be selected from suitable Hilbert spaces, $u(t,x)$ is a control function defined on the boundary $\Gamma = \partial\Omega$.

Let $\varepsilon > 0$ be an arbitrary given number. We will impose a restriction on the control function

$$|u(t,x)| \leqslant \varepsilon. \tag{11}$$

The problem is to construct a control $u(t,x)$ satisfying the inequality (11) such that the corresponding solution $w(t,x)$ of the initial boundary value problem (8)–(10) and its the first derivative of t achieves the state $(C, 0)$ at some point in time T, i.e.

$$w(T,x) = C,\ \ w_t(T,x) = 0, \tag{12}$$

for all $x \in \Omega$. In this case, C is some constant. If we succeed to construct a control $u(t, x)$ such that the conditions (12) are reached, then the system (8)–(10) is called *controllable to rest*.

Note that the constant C in this case is not arbitrary, but depends on the choice of initial data and the parameter ε,

$$C = \frac{1}{|\Gamma|} \int\limits_{\Gamma} \varphi(x) d\Gamma + \frac{1}{k(\varepsilon)|\Gamma|} \int\limits_{\Omega} \psi(x) dx, \tag{13}$$

where $|\Gamma|$ – the length of Γ and $k(\varepsilon) \to 0$ if $\varepsilon \to 0$.

Theorem 2. *Let $\varphi(x) \in H^6(\Omega)$ and $\psi(x) \in H^5(\Omega)$ such that*

$$\frac{\partial \varphi(x)}{\partial \nu} = \Delta\varphi(x) = \frac{\partial \Delta\varphi(x)}{\partial \nu} = \Delta^2\varphi(x) = \frac{\partial \Delta^2\varphi(x)}{\partial \nu} = 0, \ \ x \in \Gamma,$$

$$\psi(x) = \frac{\partial \psi(x)}{\partial \nu} = \Delta\psi(x) = \frac{\partial \Delta\psi(x)}{\partial \nu} = \Delta^2\psi(x) = 0, \ \ x \in \Gamma. \tag{14}$$

Then the system (8)–(10) is controllable to rest.

Let us explain the meaning of the initial data smoothness conditions and conditions (14). The proof of the 2 theorem consists of two stages. At the first stage, the solution under consideration and its first derivative with respect to the variable t are stabilized in a small neighborhood of the equilibrium state $(C, 0)$ according to the norm of the space $C^4(\overline{\Omega}) \times C^3(\overline{\Omega})$, the second stage allows one to drive the system to rest from this small neighborhood. The first part of the proof (solution stabilization) is related to the introduction of friction at the boundary of the domain. This friction creates energy dissipation, which in turn leads to stabilization. This friction is the control. In this case, the restriction (11) will be fulfilled due to the sufficient «smallness» of this friction, and this «smallness» is achieved by varying a certain coefficient. The conditions (14) are imposed so that the problem statement remains correct for any selected coefficient of friction.

2. Suppression of Oscillations of a Thin Plate by Bounded Control Acting to the Boundary. Let Ω be a bounded domain on the plane R^2 with an infinitely smooth boundary Γ consisting of two connected parts: Γ_0 and Γ_1, i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\nu = (\nu_1, \nu_2)$ – the outer unit normal to the boundary of the domain Ω. Let the additional condition be met

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing.$$

Suppose that Γ_0 should also be the boundary of some bounded domain Ω^*, such that

$$\Omega \cap \Omega^* = \emptyset.$$

Consider the initial boundary value problem for the equation of transverse vibrations of thin plates

$$w_{tt}(t, x) + \Delta^2 w(t, x) = 0, \quad (t, x) \in Q_T = (0, T) \times \Omega, \tag{15}$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{16}$$

$$w = \frac{\partial w}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma_0, \tag{17}$$

$$\Delta w + (1-\mu)B_1 w = u_1(t,x), \quad \frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} = u_2(t,x), \quad (t,x) \in (0,T) \times \Gamma_1, \tag{18}$$

where μ is the Poisson constant ($0 < \mu < 1/2$), $\tau = (-\nu_2, \nu_1)$ is a tangent vector, and B_1, B_2 are boundary operators defined by formulas

$$B_1 w = 2\nu_1\nu_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} - \nu_1^2 \frac{\partial^2 w}{\partial x_2^2} - \nu_2^2 \frac{\partial^2 w}{\partial x_1^2},$$

$$B_2 w = (\nu_1^2 - \nu_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + \nu_1\nu_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2} \right).$$

Here and further we will assume that inequalities are met on the Γ boundary

$$x \cdot \nu = x_1\nu_1 + x_2\nu_2 \leqslant 0 \text{ на } \Gamma_0,$$

$$x \cdot \nu = x_1\nu_1 + x_2\nu_2 \geqslant 0 \text{ на } \Gamma_1.$$

Let $\varepsilon > 0$ be an arbitrary number. The problem is to build such control actions u_1 and u_2 satisfying inequalities

$$|u_i(t,x)| \leqslant \varepsilon, \ \ i = 1, 2, \tag{19}$$

that the corresponding solution w and its derivative with respect to t vanish at some point in time T, i.e. $w(T,x) = 0$, $w_t(T,x) = 0$ for all $x \in \Omega$. Zero displacement and zero velocity will be called *the state of rest* of the system under consideration.

Theorem 3. *Let the functions $\varphi(x) \in H^6(\Omega)$ and $\psi(x) \in H^4(\Omega)$, such that they are zero near the boundary of Γ (i.e. they are finite in the domain of Ω). Then there will be a moment T and control actions $u_1(t,x)$ and $u_2(t,x)$ satisfying the constraint (19), such that the system (15)–(18) is driven to the state of rest.*

3. Controllability to rest for the «plate oscillation» equation on the torus in the case of local force action. Consider the local distributed controllability problem for the «plate oscillation» equation on the torus

$$w_{tt}(t,x) + \Delta^2 w(t,x) = u(t,x), \ \ (t,x) \in Q_T = (0,t_*] \times T^2, \tag{20}$$

$$w|_{t=0} = w_0(x), \ \ w_t|_{t=0} = w_1(x), \ \ x \in T^2. \tag{21}$$

Here T^2 is a two-dimensional torus (a smooth, compact manifold without an edge), which is conveniently understood as a square $[-\pi,\pi]^2$ with identified opposite sides, u – control with a support by the variable x that does not coincide with T^2, $t_* > 0$ is a predetermined time.

The problem is to construct such a control function u, that the corresponding solution w and its first derivative with respect to t vanish at time t_*, i.e. $w(t_*,x) = 0$, $w_t(t_*,x) = 0$ for all $x \in T^2$. Zero displacement and zero velocity will be called *the state of rest* of the system under consideration.

For the initial data in the problem (20)–(21), we require that the following conditions are met

$$w_0 \in H^4(T^2),\ w_1 \in H^2(T^2),\ w_1 - i\Delta w_0 \in H^4(T^2). \tag{22}$$

Recall that the Sobolev space is $H^s(T^2)$ ($s \in R$) on T^2 can, for example, be understood as the domain of the operator $A^{s/2} = (1-\Delta)^{s/2}$, which is provided with the norm

$$\|w\|_s = \|A^{s/2}w\|_{L_2(T^2)}. \tag{23}$$

It is convenient to write the norm (23) in terms of Fourier coefficients when decomposing w into a series (formal series for $s < 0$) according to the exponent system $\{e^{i\alpha x}\}_{\alpha \in Z^2}$

$$\left(\sum (1+|\alpha|^2)^s |c_\alpha|^2\right)^{1/2}.$$

Theorem 4. *Let an arbitrary domain ω be set on the torus T^2 for which $\bar{\omega} \neq T^2$ and the conditions (22) are met. Then there will be a force control action u, identically equal to zero on the set $T^2 \setminus \bar{\omega}$, such that the system (20)–(21) is controllable to the state of rest during t_*.*

Note that the control construction method for the equation (20) implies the possibility of regularization of the control $u(t,x)$ for the original problem. That is, by increasing the Sobolev smoothness of the initial data, we can do $u(t,x)$ is arbitrarily smooth both in time and in spatial variables. For example, if we select the initial data satisfying the inclusions: $w_0 \in H^6(T^2)$, $w_1 \in H^4(T^2)$, $w_1 - i\Delta w_0 \in H^6(T^2)$, then one can construct a control for which the following will be true:

$$u \in C([0,t_*]; H^4(T^2)),\ \ u_t \in C([0,t_*]; H^2(T^2)),\ \ u_{tt} \in C([0,t_*]; L_2(T^2)).$$

4. Exact controllability of the system described by the wave equation with integral memory. Consider the initial boundary value problem

$$\theta_{tt}(t,x) - K(0)\Delta\theta(t,x) - \int_0^t K'(t-s)\Delta\theta(s,x)ds = u(t,x), \tag{24}$$

$$x \in \Omega, \quad t > 0.$$

$$\theta|_{t=0} = \varphi_0(x), \quad \theta_t|_{t=0} = \varphi_1(x), \tag{25}$$

$$\theta|_{\partial\Omega} = 0, \tag{26}$$

here

$$K(t) = \sum_{j=1}^{N} \frac{c_j}{\gamma_j} e^{-\gamma_j t},\ N \geqslant 2,$$

where c_j, γ_j given positive constants, $u(t,x)$ is a control function defined on some bounded domain (by x) $\Omega \subset R^n$ with an infinitely smooth boundary. and

$$|u(t,x)| \leqslant \varepsilon,$$

$\varepsilon > 0$ is the specified constant. It is required to drive the system to rest in a finite time.

Let $A := -\Delta$ – operator with the domain

$$Dom(A) := H^2(\Omega) \cap H_0^1(\Omega),$$

Following [19], we denote by $W_{2,\gamma}^2(R_+, A)$ the Sobolev space of functions $\theta : R_+ = (0, +\infty) \to Dom(A)$, equipped with the norm:

$$\|\theta\|_{W_{2,\gamma}^2(R_+,A)} = \left(\int\limits_0^{+\infty} e^{-2\gamma t} \left(\left\|\theta^{(2)}(t)\right\|_{L_2(\Omega)}^2 + \|A\theta(t)\|_{L_2(\Omega)}^2 \right) dt \right)^{\frac{1}{2}}, \quad \gamma \geqslant 0.$$

Definition 1. *A function $\theta(t,x)$ is called a strong solution of the problem (24)–(26) if for some $\gamma \geqslant 0$ such a function belongs to the space $W_{2,\gamma}^2(R_+, A)$, satisfies the equation (24) almost everywhere (in t) on the positive semiaxis R_+ and satisfies the initial conditions (25).*

Sufficient conditions for the solution of the problem (24)–(26) within the definition 1 are given in [19].

Theorem 5. *Let $\varphi_0 \in D(A^{\beta+\frac{1}{2}})$ and $\varphi_1 \in D(A^{\beta})$ where $\beta > \frac{n}{2}$. Then, depending on the value of ε, there are control $u(t,x) \in C([0,T] \times \Omega)$ and time $T > 0$ such that the following equalities are true for the solution of (24)–(26)*

$$\theta(T,x) = \theta_t'(T,x) = 0, \tag{27}$$

and the restriction

$$|u(t,x)| \leqslant \varepsilon,$$

for any $t \in (0,T]$, $x \in \Omega$ is fulfilled. If the constructed control $u(t,x)$ is extended by zero at $t > T$, then the system will remain in the zero state for $t > T$.

5. Lack of controllability to rest for the Gurtin-Pipkin equation. Consider the distributed controllability problem

$$\theta_t(t,x) = \int\limits_0^t K(t-s)\Delta\theta(s,x)ds + u(t,x), \quad x \in \Omega, \quad t > 0. \tag{28}$$

$$\theta|_{t=0} = \varphi(x), \tag{29}$$

$$\theta(t,x) = 0, \;\; x \in \partial\Omega. \tag{30}$$

The kernel $K(t)$ can, for example, have the form

$$K(t)=\sum_{j=1}^{+\infty}\frac{c_j}{\gamma_j}e^{-\gamma_j t},\ K(t)=\sum_{j=1}^{N}\frac{c_j}{\gamma_j}e^{-\gamma_j t},$$

where c_j, γ_j are given positive constants such that

$$0<\gamma_1<\gamma_2<...<\gamma_j<...,\ \gamma_j\to+\infty,\ j\to+\infty.$$

Let $\Omega\subset R^2$ be a bounded simply connected domain with an infinitely smooth boundary. For brevity, we will also write $\theta(t)$ and $u(t)$ instead of $\theta(t,x)$ and $u(t,x)$ respectively. This means that $\theta(t)$ and $u(t)$ are functions of the variable t with values in some appropriate spaces.

Consider the control function $u(t)\in L_2^{loc}([0,+\infty);L_2(\Omega))$ and the initial condition $\xi\in H_0^1(\Omega)$. Assume additionally that $K(t)$ is an arbitrary twice continuously differentiable function on $[0,+\infty)$ such that $K(0)=\mu>0$.

We say that for the problem (28)–(30) *there is no controllability to rest* if there is an initial condition φ such that for any control u (u is chosen from a suitable functional class), which is identically equal to zero (by the variable t) outside of some finite segment $[0,T]$, the corresponding solution is not identically equal to zero outside of any finite segment (by t).

Definition 2. *A function*

$$\theta(t)\in H^1_{loc}([0,+\infty);L_2(\Omega))\cap L_2^{loc}([0,+\infty);DomA)$$

is called a solution of the problem (28)–(30) if $\theta(t)$ satisfies (28):

$$\theta_t(t)-\int_0^t K(t-s)\Delta\theta(s)ds=u(t)$$

and also satisfies the initial condition (29): $\theta(0)=\xi$.

Note that the solution of (28)–(30) in terms of the definition 2 may not exist in general, but due to the assumed smoothness of $K(t)$, problem (28)–(30) will be solvable if we impose additional smoothness conditions on ξ and $u(t)$ (see [27]).

Let D be an arbitrary bounded domain such that $\overline{D}\subset\Omega$. Let us define $\widetilde{L}_2(D)$ as the space of all elements from $L_2(D)$, extended by zero on the set $\Omega\setminus D$.

Theorem 6. *If the control function $u(t,x)$ is an element of the space*

$$L_2^{loc}([0,+\infty);\widetilde{L}_2(D))$$

and $\hat{K}(\lambda)$ has at least one zero λ_0 in the domain of holomorphism, then in the problem (28)–(30) there is no controllability to rest.

Here $\hat{K}(\lambda)$ is the Laplace transform of the function $K(t)$.

There are examples of kernels for which there is no global controllability to rest, i.e. in the case when the control action is applied to the entire domain occupied by the system. This is the Abel kernel and a series of exponents with slowly growing indicators. Consider the kernel

$$K(t) = \sum_{j=1}^{+\infty} \frac{c_j}{\gamma_j} e^{-\gamma_j t}, \tag{31}$$

such that

$$\sum_{j=1}^{+\infty} \frac{c_j}{\gamma_j} < +\infty.$$

Now let Ω be a bounded domain in R^n with an infinitely smooth boundary. Existence and uniqueness of a solution to the problem (28)–(30) under additional conditions imposed on the kernel (31), ξ and the right part u are proved in [19].

Definition 3. *A number* $\{z_k\}$

$$\tau = \inf \left\{ \alpha > 0 : \sum_{k=1}^{+\infty} \frac{1}{|z_k|^\alpha} < +\infty \right\}.$$

is called the convergence index of a sequence of complex numbers.

Theorem 7. *Suppose that* $\tau > 1$ *for a sequence of indicators* $\{\gamma_k\}$ *of the kernel (31). Then there is no controllability to rest for the problem (28)–(30) if the control is applied even to the entire domain.*

6. Three exceptional cases in boundary controllability problems for models of «naive mechanics». A wide class of models with integral memory can be obtained within the framework of the so-called «naive mechanics». The problem is to define the determining relation between stress and deformation (we consider here a one-dimensional case in terms of a spatial variable). This relation, in turn, is written based on various ways of connecting «springs» and «plungers» (terminology of A. A. Ilyushin and B. E. Pobedrya, [28]). As a result of this connection, an element is formed, which is the simplest cell of a continuous medium and the determining relation for this element is then in a number of cases (with a certain type of connection between these elements) transferred to the entire continuous medium. It is proved ([28]) that in the framework of «naive mechanics» the determining ratio has the form

$$P\sigma = Q\varepsilon, \tag{32}$$

where

$$P = \sum_{i=0}^{n} a_i \frac{d^i}{dt^i}, \quad Q = \sum_{i=0}^{n+1} b_i \frac{d^i}{dt^i},$$

$a_i > 0$, $b_i > 0$, $i = 1, 2, ..., n$, $b_{n+1} \geqslant 0$. Consider the polynomial

$$P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i.$$

Let the roots be $\lambda_1, \lambda_2, ..., \lambda_n$ $P(\lambda)$ are real, negative and pairwise distinct, and there is no zero among them. Then we formally express $\sigma(t, x)$ from the determining relation (32)

$$\sigma = C_0 \varepsilon + C_1 \dot{\varepsilon} + \int_0^t K(t-s) \varepsilon(s, x) ds. \tag{33}$$

Here the kernel $K(t)$ has the form

$$K(t) = \sum_{i=1}^{n} K_i e^{\lambda_i t}. \tag{34}$$

Suppose that $C_0 \geqslant 0$, $C_1 \geqslant 0$ and all constants K_i, $i = 1, 2, ..., n$, are less than or equal to zero. This condition is due to the fact that in all such viscoelastic models the signs are taken exactly like this.

Let $\theta(t, x)$ be the state of the system, then its relation to stress and deformation has the form

$$\sigma_x = \ddot{\theta}, \;\; \varepsilon = \theta_x. \tag{35}$$

Using (35), we obtain from (33) the integro-differential equation

$$\ddot{\theta} = C_0 \theta_{xx} + C_1 \dot{\theta}_{xx} + \int_0^t K(t-s) \theta_{xx}(s, x) ds. \tag{36}$$

The equation (36) describes a fairly wide class of models in mechanics. Moreover, this class directly or indirectly includes the basic classical equations (string, heat and telegraphic equations).

For (36), one can set a controllability problem. For example, put the null condition of the solution θ at the right end of the segment $[0, \pi]$, and the function $v(t) \in L_2^{loc}(0, +\infty)$ (control) at the left end. At the zero moment of time there are two initial conditions ξ_1, ξ_2 (displacement, velocity, respectively). We say that in this problem *there is no boundary controllability to rest*, if there are initial conditions ξ_1, ξ_2 such that for any control $v(t)$ that is identically equal to zero outside of some finite segment $[0, T]$, the corresponding solution is not identically equal to zero outside any finite segment (by t).

Theorem 8. *There is no boundary controllability to rest in all but three cases*

a) $C_0 > 0$, $C_1 = K_1 = K_2 = ... = K_n = 0$,

b) $C_1 > 0$, $C_0 = K_1 = K_2 = ... = K_n = 0$,

c) $K_1 > 0$, $C_0 = \frac{K_1}{\lambda_1}$, $C_1 = K_2 = ... = K_n = 0$.

The Case (a) defines the string equation. It is a well known fact that boundary controllability takes place for this equation. For (b) and (c), the situation is more complicated. Formally there is no boundary controllability for the equations described by these cases if the definition of controllability is understood as it was introduced above. Meanwhile, if we consider a narrower class of initial data: we set the second initial condition ξ_2 equal to zero, then by means of the integration by the variable t the equation in the case of (b) reduces to the heat equation, and in the case of (c) to the Gurtin-Pipkin equation, which can be further investigated for controllability. Note that the equation in the case (b) for $\xi_2 = 0$ can also be reduced to a telegraphic equation.

8 Personal contribution of the author to the study of the problem

Most of the published scientific papers, which reflect the scientific results of the thesis, were written in collaboration with professor A. S. Shamaev. He mainly worked on the problem statements and general considerations related to the methods and techniques of constructing control and lack of controllability. The author has made precise formulations of the theorems, carried out proofs and indicated the main consequences of the results obtained.

9 General conclusions of the study

In the presented dissertation, the issues of controllability for two broad classes of systems with distributed parameters are comprehensively investigated: classical (membranes, plates) and systems with memory (the Gurtin-Pipkin equation and its counterparts).

For membranes and plates the issues of boundary controllability are investigated in the case when a restriction is imposed on the absolute value of the force control action, which significantly complicates the task. In all cases, additional conditions must be met for the initial data and for the geometry of the domains (boundaries). In addition, the issue of local controllability for the system described by the «plate oscillation» equation on the torus is investigated.

For systems with memory, the absence of boundary controllability (in the one-dimensional case) is proved for all viscoelastic models of «naive mechanics», except the three cases. The absence of local and boundary controllability is also proved for the Gurtin-Pipkin equation and a wide class of kernels in the integral term of the equation, examples of kernels for which there is no global controllability are found.

10 List of published articles reflecting the main results of the dissertation

14 scientific papers have been published on the topic of the dissertation. Two articles from the journal Q1 according to the WoS and Scopus, the same journal is included in List A, 5 articles from Q2 Scopus journals.

1. *Romanov I., Shamaev A.* Controllability to Rest for the «Plate Oscillation» Equation on the Torus in the Case of Local Force Action. *Mathematical Notes.* 2023, V. 113, №4, P. 598-600. (Scopus **Q2**)

2. *Romanov I.* About the Lack of Controllability in Naive Mechanics Models: Three exceptional cases. *Mechanics of Solids.* 2022, V. 57, №8, P. 2123-2127.

3. *Romanov I.* Investigation of Controllability for some Dynamic System with Distributed Parameters Described by Integrodifferential Equations. *Journal of Computer and Systems Sciences International* 2022, V. 61, №2, P. 191-194. (Scopus **Q2**)

4. *Romanov I., Shamaev A.* Exact Control of a Distributed System Described by the Wave Equation with Integral Memory. *Journal of Mathematical Sciences.* 2022, V. 262, P. 358-373.

5. *Romanov I., Shamaev A.* Exact Bounded Boundary Controllability to Rest for the Two-Dimensional Wave Equation. *Journal of Optimization Theory and Applications.* 2021, V. 188, №3, P. 925-938. (WoS, Scopus **Q1**)

6. *Romanov I., Shamaev A.* Suppression of Oscillations of Thin Plate by Bounded Control Acting to the Boundary. *Journal of Computer and Systems Sciences International.* 2020, V. 59, №3, P. 371-380.

7. *Romanov I., Shamaev A.* On a Boundary Controllability Problem for a System Governed by the Two-Dimensional Wave Equation. *Journal of Computer and Systems Sciences International.* 2019, V. 58, №1, P. 105-112.

8. *Romanov I., Romanova A.* Some Problems of Controllability of Distributed Systems Governed by Integrodifferential Equations. *IFAC-PapersOnLine.* 2018, V. 51, №2, P. 132-137.

9. *Romanov I., Shamaev A.* Some Problems of Distributed and Boundary Control for Systems with Integral Aftereffect. *Journal of Mathematical Sciences.* 2018, V. 234, №4, P. 470-484.

10. *Romanov I., Shamaev A.* Noncontrollability to Rest of the Two-Dimensional Distributed System Governed by the Integrodifferential Equation. *Journal of Optimization Theory and Applications.* 2016, V. 170, №3, P. 772-782. (WoS, Scopus **Q1**)

11. *Romanov I., Shamaev A.* Exact Bounded Boundary Controllability of Vibrations of

a Two-Dimensional Membrane. *Doklady Mathematics.* 2016, V. 94, №2, P. 607-610. (Scopus **Q2**)

12. *Romanov I., Shamaev A.* Exact Controllability of the Distributed System, Governed by String Equation with Memory. *Journal of Dynamical and Control Systems.* 2013, V. 19, №4, P. 611-623. (Scopus **Q2**)

13. *Romanov I., Shamaev A.* On the Problem of Precise Control of the System Obeying the Delay String Equation. *Automation and Remote Control.* 2013, V. 74, №11, P. 1810-1819. (Scopus **Q2**)

14. *Romanov I.* On the Impossibility of Bringing a Flat Membrane to Rest with Boundary Forces. *Automation and Remote Control.* 2012, V. 73, №12, P. 1994-2000.

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