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Finite groups acting on algebraic and complex manifolds

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Contents

Introduction

This thesis studies several problems related to actions of groups (in particular, finite groups) in algebraic and complex analytic geometry.

The first major topic of this thesis is the study of finite subgroups in the groups of birational automorphisms of projective varieties, as well as in the groups of bimeromorphic automorphisms of compact complex spaces. We are interested in the following "qualitative" boundedness properties for finite subgroups in these groups, which generalize the classical theorems of C. Jordan [11] for general linear groups over $\mathbb{C}$ and of H. Minkowski [23] for general linear groups over $\mathbb{Q}$, respectively.

Definition 0.1. Let G be a group. We say that G is *Jordan* (or has the *Jordan property*) if there is a constant $J(G) \in \mathbb{N}$ such that for any finite subgroup $H \subset G$ there is a normal abelian subgroup $A \subset H$ of index at most $J(G)$.

We say that a group G has *bounded finite subgroups* if there is a natural number $B(G)$ such that for any finite subgroup $H \subset G$ we have $|H| \leqslant B(G)$.

The study of these properties for (birational) automorphism group was initiated in the works of J.-P. Serre [24] and V. Popov [17]. A number of important results were obtained in the works of S. Meng and D.-Q. Zhang [14] and of Yu. Prokhorov and C. Shramov [18, 19]. An important direction for future research is to generalize these results to non-projective compact complex (for instance, Kaehler) spaces.

In Chapter 1 of this thesis we generalize the methods from [18] and prove that bimeromorphic automorphism groups of non-uniruled compact Kaehler threefolds are Jordan.

Note that the Jordan property does not provide any information on finite abelian subgroups of the group in question. A closely related problem is to find constraints on finite abelian subgroups in $\mathrm{Bir}(X)$ of large orders. A natural invariant of a finite abelian group G is its rank $r(G)$, that is, the minimal number of generators. In Chapter 2 we develop methods to find upper bounds for ranks of "large" finite abelian subgroups in birational automorphism groups of projective varieties and, under some additional assumptions, in bimeromorphic automorphism groups of compact Kaehler spaces. Generalizing the results of I. Mundet i Riera [16] and J. Xu [29], we obtain the upper bound

$$r(G) \leqslant 2\dim(X)$$

and describe the birational isomorphism classes of varieties for the cases $r(G) = 2\dim(X)$ and $r(G) = 2\dim(X) - 1$. In particular, the equality $r(G) = 2\dim(X)$ is attained if and only if X is birational to an abelian variety.

It is also desirable to find methods to study the Jordan property for (biholomorphic and bimeromorphic) automorphism groups of non-Kaehler compact complex manifolds. In Chapter 3 we focus on automorphism groups of compact complex parallelizable manifolds. These manifolds are isomorphic to quotients G/Γ of connected complex Lie groups by discrete cocompact subgroups [27] and are not Kaehler (except compact complex tori). Using a description of the automorphism groups of these manifolds [28], we prove that the Jordan property holds for (biholomorphic and bimeromorphic) automorphism groups of compact parallelizable manifolds.

Another important problem both for algebraic and analytic geometry is the existence problem for Kaehler–Einstein metric on Fano varieties. Recall that a Kaehler metric ω on a smooth Fano variety X is Kaehler–Einstein if it satisfies the equation

$$\mathrm{Ric}(\omega) = \omega,$$

where Ricci(ω) is the Ricci curvature of ω. By [4, 25] an algebraic condition that ensures existence of a Kaehler–Einstein metric on a Fano variety is that of K-stability [7]. Moreover, if there is an action of a group G on X by automorphisms, then there is a notion of G-equivariant K-stability, which is sufficient to check the Kaehler–Einstein property [5]. In [1, 9] the notion of K-stability was reformulated in terms of valuative invariants of X. A natural expectation is that an analogous valuative criterion exists in the G-equivariant setting.

In Chapter 4 we define a G-equivariant version of the δ-invariant and prove a criterion of G-equivariant K-semistability (and of uniform K-stability) in terms of δ_G. The other sections of Chapter 4 is devoted to various applications and ramifications of this result. These include a formula relating the δ_G-invariant and the greatest Ricci lower bound of X and an explicit formula for the δ_G-invariant of a spherical Fano variety in terms of the corresponding combinatorial data. Also, for the case of a finite group G we suggest an alternative definition of δ_G and prove a ramification formula, with a view towards a general version of the above valuative criterion.

1 Non-uniruled Kaehler threefolds

Chapter 1 of the thesis is devoted to the study of finite subgroups in the groups of bimeromorphic automorphisms of non-uniruled compact Kaehler threefolds. The goal of this chapter is to apply the ideas of the Minimal Model Program, developed in [18], to show the Jordan property in the setting of compact Kaehler spaces.

We recall basic definitions concerning groups with the Jordan property in section 1. Then in section 2 we present, following [10], some definitions and technical results related to compact Kaehler spaces with singularities and notions of positivity for $(1,1)$-classes.

In section 3 we prove the following proposition, which is a stronger version of a well-known result from algebraic geometry [13, Lemma 4.3].

Proposition 1.1. *Let $f\colon X \dashrightarrow X'$ be a bimeromorphic map between compact Kähler spaces with terminal singularities. Suppose that K_X and $K_{X'}$ are modified nef. Then f is an isomorphism in codimension one.*

In section 4 we prove, after some technical preparations, we generalize to the singular setting a result of A. Fujiki [8, Corollary 3.3].

Proposition 1.2. *Let $f\colon X \dashrightarrow X'$ be a bimeromorphic map of normal compact Kähler spaces with rational singularities. Suppose that f is an isomorphism in codimension one. If there exists a Kähler class $\alpha \in H^{1,1}(X,\mathbb{R})$ such that the class $\alpha' = f_*\alpha$ is also Kähler then f is biholomorphic.*

Finally, section 5 of the first chapter is devoted to the proofs of the main results. First, we prove that pseudoautomorphism groups of compact Kaehler spaces are Jordan, generalizing a result of J. Kim [12, Theorem 1].

Theorem 1.3. *Let X be a normal compact Kähler space with rational singularities. Then the group* $\mathrm{Psaut}(X)$ *is Jordan.*

We use the above theorem and existence of minimal models for non-uniruled compact Kaehler threefolds [10, Theorem 1.1] to deduce the main result of Chapter 1.

Theorem 1.4. *The group* $\mathrm{Bim}(X)$ *is Jordan for any non-uniruled compact Kähler space X of dimension 3.*

Combined with [20, Theorem 1.3] this result gives a complete characterization of compact Kaehler threefolds X such that the group $\mathrm{Bim}(X)$ is Jordan.

2 Finite abelian subgroups

In Chapter 2 we consider finite abelian subgroups of large orders in the group $\mathrm{Bir}(X)$ for a complex projective variety X. The starting point for us a recent theorem by I. Mundet i Riera [16, Theorem 1.15].

Theorem 2.1 (I. Mundet i Riera)**.** *Let X be a compact Kähler manifold. Suppose that there exists $r \in \mathbb{N}$ such that for infinitely many positive integers N_i the group $\mathrm{Aut}(X)$ contains a subgroup isomorphic to $(\mathbb{Z}/N_i\mathbb{Z})^r$. Then $\mathrm{Aut}(X)$ contains a subgroup isomorphic to a compact real torus of dimension r. Moreover, one has $r \leqslant 2\dim(X)$, and if $r = 2\dim(X)$ then X is biholomorphic to a compact complex torus.*

I. Mundet i Riera asked if the same upper bound is valid also for birational automorphism groups of projective varieties (or for bimeromorphic automorphisms of compact Kaehler manifolds). The goal of the second chapter of the thesis is to provide an affirmative answer to this question for projective varieties.

Section 1 contains technical preliminaries concerning sequences of finite abelian groups. We introduce the notion of the unbounded rank of a sequence of finite groups. We also recall in section 1 the construction of the maximal rationally connected (MRC) fibration for compact Kaehler manifolds from [3].

In section 2 we consider groups of birational automorphisms of projective varieties from two separate classes: the non-uniruled ones and the rationally connected ones. For the non-uniruled case we obtain a generalization of [29, Theorem 2.9].

Theorem 2.2. *Let X be a non-uniruled projective variety. Suppose that there exists $r \in \mathbb{N}$ such that the group $\mathrm{Bir}(X)$ contains a subgroup isomorphic to $(\mathbb{Z}/N_i\mathbb{Z})^r$ where N_i tend to infinity. Then the inequality*

$$r \leqslant 2\dim(X)$$

holds and moreover the group $\mathrm{Bir}(X)$ contains a subgroup isomorphic to an abelian variety of dimension $\lceil r/2 \rceil$. In the case $r = 2\dim(X)$ the variety X is birational to an abelian variety.

As for the rationally connected varieties, the results of [19] and [2] imply the following.

Theorem 2.3. *Let X be a rationally connected variety of dimension n over a field k of zero characteristic. Suppose that there exists $r \in \mathbb{N}$ such that for arbitrarily large $N_i \in \mathbb{N}$ the group $\mathrm{Bir}(X)$ contains a subgroup $G_i \simeq (\mathbb{Z}/N_i\mathbb{Z})^r$. Then $r \leqslant n$.*

Also, in section 2 we consider the MRC fibration of a projective variety X and combine the above results to prove the main result of the second chapter.

Theorem 2.4. *Let X be a projective variety over an algebraically closed field of zero characteristic. Suppose that the group $\mathrm{Bir}(X)$ contains finite abelian subgroups isomorphic to $(\mathbb{Z}/N_i\mathbb{Z})^r$ for some fixed r and arbitrarily large N_i. Then the unbounded rank r of this sequence of subgroups does not exceed $2\dim(X)$ and in case of equality X is birational to an abelian variety.*

Combining these ideas with some technical results from Chapter 1, we were able to prove the analogous result for groups of bimeromorphic selfmaps of compact Kähler spaces under an additional assumption.

Theorem 2.5. *Let X be a compact Kähler space. Suppose that the group* $\mathrm{Bim}(X)$ *contains subgroups isomorphic to* $(\mathbb{Z}/N_i\mathbb{Z})^r$ *for some fixed r and arbitrarily large N_i. Suppose also that the base of the MRC-fibration of X has dimension at most 3. Then the upper bound* $r \leqslant 2\dim(X)$ *holds and in case of equality X is bimeromorphic to a compact complex torus.*

3 Compact parallelizable manifolds

In Chapter 3 of the thesis we continue to study the Jordan property, this time for automorphism groups of compact complex parallelizable manifolds.

Definition 3.1. A compact complex manifold X is called parallelizable if its holomorphic tangent bundle is trivial.

A compact manifold in this class is isomorphic to a quotient of a complex Lie group G by a discrete cocompact subgroup Γ ([27, Theorem 1]). Compact parallelizable manifolds (except compact complex tori) do not admit Kaehler metrics. Therefore, new methods are needed to prove the Jordan property for (biholomorphic and bimeromorphic) automorphism groups of these manifolds.

The automorphism groups of compact complex parallelizable manifolds have been explicitely described by J. Winkelmann [28] in terms of the Lie group G and the lattice Γ. After studying various group-theoretical properties of lattices in complex Lie groups, following [21, 26], we recall this description in section 3. Then in sections 4 and 5 we state and prove a few results related to outer automorphisms of lattices and their deformation spaces, including A. Weil's local rigidity and G. Mostow's strong rigidity [15] for lattices in semisimple Lie groups.

The main result of Chapter 3 is the following theorem, proved in section 6.

Theorem 3.2. *Let X be a compact complex parallelizable manifold. Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

In the course of the proof of Theorem 3.2 we establish a boundedness result for the orders of finite subgroups in the group of outer automorphisms $\mathrm{Out}(\Gamma)$ for a cocompact lattice Γ in a complex Lie group.

Theorem 3.3. *Let Γ be a cocompact lattice in a connected complex Lie group G. Then the group* $\mathrm{Out}(\Gamma)$ *has bounded finite subgroups.*

4 Delta-invariants with group actions

The final Chapter 4 of the thesis studies equivariant K-stability of Fano varieties with group actions. We adopt the valuative approach to K-stability, developed in [1, 9]. Section 1 is devoted to preliminary information on valuations on the fields of rational functions of varieties, algebraic group actions and singularities of linear systems. In section 2 we study G-equivariant K-stability of Fano varieties. After formulating the definition of K-stability both in terms of test configurations and in terms of valuations, we define the δ_G-invariant as the infimum of a certain function on the space of G-invariant divisorial valuations.

Definition 4.1. Let (X, L) be a polarized variety and $G \subset \mathrm{Aut}(X, L)$ be a connected subgroup. We define

$$\delta_G(X, L) = \inf_{v \in \mathrm{DivVal}_X^G} \frac{A_X(v)}{S_L(v)}.$$

Here $A_X(v) = 1 + \mathrm{ord}_E(K_{Y/X})$ is the log discrepancy and

$$S_L(v) = \frac{1}{\mathrm{Vol}(L)} \int_0^\infty \mathrm{Vol}(\varphi^* L - tE) dt$$

is the expected vanishing order of the divisorial valuation $v = \mathrm{ord}_E$.

Then we prove, following the ideas of [1, 9], the following theorem, which is the main result of Chapter 4.

Theorem 4.2. *Let $(X, -K_X)$ be a klt $\mathbb{Q}$-Fano variety with the anticanonical polarization. Let $G \subset \mathrm{Aut}(X)$ be a closed connected subgroup. Then $(X, -K_X)$ is uniformly K-stable (resp. K-semistable) with respect to G-equivariant degenerations if and only if the δ_G-invariant of $(X, -K_X)$ is greater than one (resp. greater or equal to one).*

The rest of Chapter 4 is devoted to various applications of Theorem 4.2. In section 3 we compare the δ_G-invariant to the greatest Ricci lower bound [22] of a smooth Fano variety. More precisely, we show that the following formula holds.

Proposition 4.3. *Let X be a smooth Fano variety and let $G \subset \mathrm{Aut}(X)$ be a connected reductive subgroup. Then there is an equality*

$$\beta_G(X) = \min\{1, \delta_G(X)\}, \tag{4.1}$$

where $\beta_G(X)$ is the greatest Ricci lower bound of X. We also have the following relation between δ-invariants

$$\delta(X) = \min\{1, \delta_G(X)\}. \tag{4.2}$$

In section 4 we study equivariant K-stability of Fano varieties with actions of tori $T = (\mathbb{G}_m)^k$ of any dimension k. We show that the delta-invariants with and without T-action are equal, so the corresponding notions of K-stability are equivalent.

Section 5 is devoted to equivariant K-stability of spherical Fano varieties. We provide a purely algebraic and more straightforward proof of the combinatorial criterion for K-stability of spherical Fano varieties, first proved by T. Delcroix [6] by analytic methods.

Proposition 4.4. *Let X be a Fano variety which is spherical under the action of a connected reductive group G. Then δ_G-invariant of X can be expressed as follows:*

$$\delta_G(X) = \min_{\substack{\mathrm{ord}_{D_i} \in \mathcal{V} \cap \mathcal{C} \\ \mathcal{C} \in \mathbb{F}_X}} \frac{a_{D_i}}{a_{D_i} - \langle 2\rho_Q - \mathrm{bar}_{DH}(\Delta^+), \pi^{-1}(\mathrm{ord}_{D_i}) \rangle}.$$

Here $\mathrm{bar}_{DH}(\Delta^+)$ is the barycenter of Δ^+ with respect to the Duistermaat–Heckman measure. The minimum is taken over a finite set $\mathrm{ord}_{D_1}, \ldots, \mathrm{ord}_{D_N}$ of valuations corresponding to primitive generators of edges in $\mathcal{C} \cap \mathcal{V}, \mathcal{C} \in \mathbb{F}_X$.

Finally, in section 6 we consider the case of a variety with an action of a finite group G. In this case we suggest an alternative definition of δ_G using G-invariant divisors and prove the ramification formula.

Proposition 4.5. *Let X be a variety with klt singularities and $-K_X$ big. Let $G \subset \mathrm{Aut}(X)$ be a finite group. Denote by $Y = X/G$ the quotient variety and let $B = \sum_i (1 - 1/m_i) B_i$ be the branch divisor on Y. Then we have*

$$\delta_G(X) = \delta(Y, B)$$

where $\delta(Y, B)$ is the δ-invariant of the klt pair (Y, B).

Publications containing the main results of the thesis

The main results of the thesis are published in the following research papers.

- Golota, Aleksei. Delta-invariants for Fano varieties with large automorphism groups. // International Journal of Mathematics, Vol. 31, No. 10 (2020) 2050077 (31 pages).
- Golota, Aleksei. Jordan property for groups of bimeromorphic automorphisms of compact Kähler threefolds (Russian) // Sb. Math. 214:1 (2023), 31–42.

References

[1] Blum, Harold; Jonsson, Mattias. Thresholds, valuations, and K-stability. *Adv. Math.* 365 (2020), 107062, 57 pp.

[2] Birkar, Caucher. Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)* 193 (2021), no. 2, 347–405.

[3] Campana, Frédéric. Orbifolds, special varieties and classification theory: an appendix. *Ann. Inst. Fourier (Grenoble)* 54 (2004), no. 3, 631–665.

[4] Chen, Xiuxiong; Donaldson, Simon; Sun, Song. Kähler–Einstein metrics on Fano manifolds, I-III. J. Amer. Math. Soc. 28 (2015), 183–197, 199–234, 235–278.

[5] Datar, Ved; Székelyhidi, Gábor. Kähler-Einstein metrics along the smooth continuity method. *Geom. Funct. Anal.* 26 (2016), no. 4, 975–1010.

[6] Delcroix, Thibaut. K-stability of Fano spherical varieties. *Ann. Sci. Éc. Norm. Supér.* (4) 53 (2020), no. 3, 615–662.

[7] Donaldson, Simon K. Scalar curvature and stability of toric varieties. *J. Differential Geom.* 62 (2002), no. 2, 289–349.

[8] Fujiki, Akira. A theorem on bimeromorphic maps of Kähler manifolds and its applications. *Publ. Res. Inst. Math. Sci.* 17 (1981), no. 2, 735–754.

[9] Fujita, Kento. A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. *J. Reine Angew. Math.* 751 (2019), 309–338.

[10] Höring, Andreas; Peternell, Thomas. Minimal models for Kähler threefolds. *Invent. Math.* 203 (2016), no. 1, 217–264.

[11] Jordan, M. Camille. Mémoire sur les équations différentielles linéaires à intégrale algébrique. *J. Reine Angew. Math.* 84 (1878), 89–215.

[12] Kim, Jin Hong. Jordan property and automorphism groups of normal compact Kähler varieties. *Commun. Contemp. Math.* 20 (2018), no. 3, 1750024, 9 pp.

[13] Kollár, János. Flops. Nagoya Math. J. 113 (1989), 15–36.

[14] Meng, Sheng; Zhang, De-Qi. Jordan property for non-linear algebraic groups and projective varieties. *Amer. J. Math.* 140 (2018), no. 4, 1133–1145.

[15] Mostow, G. D. Strong rigidity of locally symmetric spaces. Annals of Mathematics Studies, No. 78. *Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo,* 1973. v+195 pp.

[16] Mundet i Riera, Ignasi. Topological rigidity of tori and finite group actions on hypertoral manifolds. arXiv:2112.05599.

[17] Popov, Vladimir L. On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. *Affine algebraic geometry,* 289–311, CRM Proc. Lecture Notes, 54, *Amer. Math. Soc., Providence, RI,* 2011.

[18] Prokhorov, Yuri; Shramov, Constantin. Jordan property for groups of birational selfmaps. *Compos. Math.* 150 (2014), no. 12, 2054–2072.

[19] Prokhorov, Yuri; Shramov, Constantin. Jordan property for Cremona groups. *Amer. J. Math.* 138 (2016), no. 2, 403–418.

[20] Prokhorov, Yu. G.; Shramov, K. A. Finite groups of bimeromorphic selfmaps of uniruled Kähler threefolds. (Russian). *Izv. Ross. Akad. Nauk Ser. Mat.* 84 (2020), no. 5, 169–196.

[21] Raghunathan, M. S. Discrete subgroups of Lie groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. *Springer-Verlag, New York-Heidelberg,* 1972. ix+227 pp.

[22] Rubinstein, Yanir A. Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics. *Adv. Math.* 218 (2008), no. 5, 1526–1565.

[23] Serre, Jean-Pierre. Bounds for the orders of the finite subgroups of $G(k)$. *Group representation theory,* 405–450, *EPFL Press, Lausanne,* 2007.

[24] Serre, Jean-Pierre. A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.* 9 (2009), no. 1, 193–208, back matter.

[25] Tian, Gang. K-stability and Kähler–Einstein metrics. *Comm. Pure Appl. Math.* 68 (2015), no. 7, 1085–1156.

[26] Vinberg, È. B.; Gorbatsevich, V. V.; Shvartsman, O. V. Discrete subgroups of Lie groups. (Russian) *Current problems in mathematics. Fundamental directions, Vol. 21 (Russian),* 5–120, 215, Itogi Nauki i Tekhniki, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow,* 1988.

[27] Wang, Hsien-Chung. Complex parallisable manifolds. *Proc. Amer. Math. Soc.* 5 (1954), 771–776.

[28] Winkelmann, Jörg. Complex analytic geometry of complex parallelizable manifolds. *Mém. Soc. Math. Fr. (N.S.)* No. 72-73 (1998), x+219 pp.

[29] Xu, Jinsong. Finite p-groups of birational automorphisms and characterizations of rational varieties. arXiv:1809.09506.