Faculty of mathematics

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# Functions of a complex and quaternionic variable in lattice models and theory of circular surfaces 

Summary of the thesis<br>for the purpose of obtaining academic degree Doctor of Science in Mathematics

## Introduction

This thesis is devoted to the study of the functions of a complex and quaternion variable and their applications, mainly in geometry. We consider functions both on the plane and the Riemann surfaces, and on the lattices.

In theory of functions of a complex variable, especially on Riemann surfaces, the evidence for the existence of basic objects is often non-constructive, which makes it difficult to calculate them in practice. One of the promising approaches to computation in this theory is its discretization and the subsequent limit transition.

Various constructions of complex analysis on planar graphs were introduced by R.Isaacs, R.Duffin, C.Mercat [19, 17, 31, 32], I.Dynnikov-S.Novikov [18], A.Bobenko-C.MercatYu.Suris [4], and A.Bobenko-U.Pinkall-B.Springborn [5]. The first of these theories, the so-called linear discretization of complex analysis on a quadrilateral lattice, is currently being developed especially actively. This is due to its applications in statistical physics (S.Smirnov-D.Chelkak [46]; M.Khristoforov, S.Smirnov, and the applicant [25]), numerical analysis [22], and combinatorial geometry [24]. See the surveys by S.Smirnov and L.Lovasz [29, 46].

Complex analysis on graphs, like classical complex analysis, is related to potential theory. Therefore, physical terminology is often used: a weighted graph is called a directcurrent (respectively, alternating current) circuit, if the weight of each edge is positive (respectively, it has a positive real part).


Figure 1: Physical interpretation of tilings: of a rectangle by squares, of a polygon by squares, of a square by similar rectangles (from the left to the right).

Historically, among the first applications of discrete complex analysis (in the appearance of electric networks) was the problem of tiling a given polygon by rectangles. A celebrated physical interpretation of such tilings by R.Brooks, K.Smith, K.Stone, and U.Tutte uses direct-current networks [6]; see the left and the middle parts of Figure 1. This interpretation allowed to completely solve a number of problems on cutting polygons into squares. Recently this idea is experiencing a new peak of popularity, thanks to the application to models of quantum gravity, due to S.Sheffield and coauthors.

A more general problem of cutting a polygon into polygons of a given shape is also related to the discrete functions of a complex variable. To solve it, further development of the theory of such functions is required.

Further, we consider functions of a quaternion variable, mainly polynomials and rational functions. We study their applications in the classification of surfaces containing


Figure 2: Circular arc structures
several circles through each point (those circles completely lie on the surface). This is motivated, in particular, by potential applications in architecture.

In contemporary architecture, there is a trend towards freeform structures which is very clearly seen in the works by star architects such as F.Gehry or Z.Hadid. While digital models of architectural freeform surfaces are easily created using standard modeling tools, the actual fabrication and construction of architectural freeform structures remains a challenge. In order to make a freeform design realizable, an optimization process known as rationalization has to be applied [35, 3]. This means replacing a smooth surface by a lattice composed of separate panels with special properties.

As a contribution towards rationalization of architectural freeform structures, P.Bo and coauthors [3] have recently suggested so-called circular arc structures. A circular arc structure is a mesh whose edges are realized as circular arcs instead of straight line segments and which possesses congruent nodes with well-defined tangent planes. Figure 2 shows two examples of circular arc structures with quad-mesh and triangle-mesh combinatorics, respectively. In the first case, the construction consists of two discrete sets of arc splines which intersect each other under the right angle. In the second case, we see three sets of arc splines which intersect each other under the 60 degree angle. In the latter case we have an example of a so-called hexagonal web, a classical object of geometry; see Figure 3 and [1, 50, 51, 40, 38, 39, 41].

The following fundamental problem is closely related to the applied one we are discussing. Keeping aside the requirement of equality of angles, let us require that the structure is formed by whole arcs of circles, not just splines from them. The natural question is what constructions can be achieved from two families of circles. It has remained open for a long time despite partial advances tracing back to the works by G.Darboux from the 19th century. Although the solution of this problem has no direct applications


Figure 3: Webs from circles on Darboux cyclides: hyperboloid and its image under inversion, Dupin cyclide, canal cyclide, general cyclide with different webs (from the left to the right).
in architecture, there are reasons to believe that the developed methods could be useful for subsequent applied research.

The resulting question naturally leads to the question when and to what extent the factorization of quaternion polynomials in two variables into irreducible factors is unique. This question was studied earlier for the case of one variable, and the case of two variables was not amenable to the previously known methods.

## Discrete complex analysis

Let us give a brief overview of the results known earlier and obtained in the thesis.
Classical methods allowed to construct a discretization of complex analysis and prove its convergence to the continuum theory for square and rhombic lattices. For example, the convergence of the solution of the Dirichlet problem on a square lattice approximating a domain to the solution of the Dirichlet problem in this domain was proved by R.Courant-K.Friedrichs-H.Lewy [13], and for the case of rhombic lattices - by D.Chelkak-S.Smirnov [8] and implicitly by P.Ciarlet-P.Raviart [10]. For more general quadrilateral lattices, the classical methods do not work (the reason for this is that the property of the discrete analyticity of the integral of a discrete analytic function disappears). They allow to obtain convergence results only in a very weak sense, not related to solutions of boundary value problems. We propose new methods that allow to prove convergence for the socalled orthogonal lattices (article 5 in the list of publications), as well as on triangulated Riemannian surfaces (article 6 in the list of publications). This solves a problem posed by S.Smirnov [46, Question 1].

A related question on the convergence of the Galerkin finite element method has been studied thoroughly; see, for example, textbook [9]. However, the question of the convergence of the finite element method for the Delaunay triangulations of Riemann surfaces remained open until our joint work with A.Bobenko (article 6 in the list of publications).

Historically, among the first applications of discrete complex analysis was the problem of tiling a given polygon by rectangles of given shapes. A celebrated physical interpretation of such tilings by R.Brooks, K.Smith, K.Stone, and U.Tutte [6] is exposed in our elementary introductions [42, 43]. In a joint work with M.Prasolov (article 1 in the list of
publications) we develop a new approach to this tiling problem using alternating-current networks and electrical impedance tomography [7, 11, 12, 14, 15]; see Figure 1 to the right. In particular, we describe all polygons which can be tiled by squares. This solves a problem of C.Freiling et al. [21]. This was known in the particular cases when the polygon is an L-shaped hexagon (R.Kenyon [24]).

Close ideas are used by us to analyze various lattice models (articles 10 and 9 in the list of publications); see also reviews [23, 27, 48]. Note that article 9 from the list of publications uses functions on the lattice with both complex and quaternion values.

## Functions of a quaternion variable and circular surfaces

Let us give a brief overview of the results known earlier and obtained in the thesis.
Examples of surfaces (in 3-dimensional space) containing several circles through each point have been known for a long time. For brevity, such surfaces are called circular. G.Darboux introduced a special class of 4th degree surfaces, which are called cyclides. Darboux cyclides include quadrics and Dupin cyclides as special cases. Other examples are shown in Figure 3 to the right. G.Darboux showed that up to 10 circles pass through each point of a cyclide, but he did not distinguish between real and complex circles. R.Blum [2] constructed examples of cyclides with 6 real circles through each point.


Figure 4: Laguerre minimal surfaces: helicoid, cycloid, the Plücker conoid, general ruled surface, general surfaces enveloped by families of cones (from the left to the right).

There are very few known results on the classification of circular surfaces. N.Takeuchi showed that a surface of genus 0 or 1, different from the sphere, cannot contain more than 6 circles through each point. It was known that a surface containing 2 cospheric or 2 orthogonal circles through each point is necessarily a cyclide. However, there are circular surfaces that are different from cyclides: for example, a surface obtained by the translation of one circle along another circle. Surfaces containing two conics (respectively, a conic and a straight line) through each point were classified by J.Schicho [37] (respectively, H.Brauner).

In a joint work with F.Nilov (article 4 in the list of publications), all surfaces containing a conic and a line through each point were described.

In a joint work with H.Pottmann and L.Shi (article 3 in the list of publications), all so-called hexagonal webs from circles were found on all cyclides other than a sphere and a plane; see Figure 3. By means of related methods, together with M. Barton, P.Bo, Ph.Grohs, H.Pottmann, we have also described all surfaces containing a family of isotropic
circles and all ruled surfaces that are minimal in the sense of E.Laguerre (articles 8 and 2 in the list of publications, cf. [26, 28, 33, 34, 36, 49]); see Figure 4. Functions of a complex variable played the main role in the proof of the latter result.

Finally, in a joint work with R.Krasauskas (article 7 in the list of publications), the problem of classification of circular surfaces was completely solved using functions of a quaternion variable.

More detailed expositions of the proofs (than in published articles 7 and 2 from the list of publications) are given on the preprint server https://arxiv.org/abs/1011.0272v2 and https://arxiv.org/abs/1512.09062v3.

## Main results

Let us give precise statements of the main results.
We start with the results on the convergence of discrete analytic functions.
A quadrilateral lattice is a graph $Q \subset \mathbb{C}$ with rectilinear edges such that each bounded face is a quadrilateral (not necessarily convex). Depending on the shape of the faces, one speaks of square, rhombic, orthogonal lattices (the latter means that the diagonals of each face are orthogonal).

A function $f$ defined on the vertices of the lattice and assuming complex values is discrete analytic, if for each quadrilateral the difference quotients of this function along both diagonals are equal. In other words, for each quadrilateral $A B C D$ we have

$$
\frac{f(A)-f(C)}{A-C}=\frac{f(B)-f(D)}{B-D}
$$

where points $A, B, C, D$ of the plane are identified with complex numbers. The real part of a discrete analytic function is called a discrete harmonic function. The boundary of the lattice $Q$ is the boundary of its outer face. We will always assume that the graph $Q$ is finite, and its boundary consists of one closed curve without self-intersections.

Let $g$ be an arbitrary smooth function defined on the complex plane and assuming real values. The Dirichlet problem on $Q$ is to find a discrete harmonic function $u=u_{Q, g}$ equal to the given function $g$ on the boundary of the lattice $Q$.

Theorem 1. The Dirichlet boundary value problem on any finite quadrilateral lattice has a unique solution.

A sequence of lattices $\left\{Q_{n}\right\}$ approximates a domain $\Omega$, if for $n \rightarrow \infty$ :

- the maximal distance from a point of the boundary of $Q_{n}$ to the boundary of $\Omega$ tends to zero;
- the maximal distance from a point of the boundary of $\Omega$ to the boundary of $Q_{n}$ tends to zero;
- the maximal edge length of $Q_{n}$ tends to zero.

A sequence of lattices $\left\{Q_{n}\right\}$ is nondegenerate uniform, if there is a constant const (not depending on $n$ ) such that for each member of the sequence:

- the ratio of the diagonals of each quadrilateral is less than const and the angle between them is greater than $1 /$ const;
- the number of vertices in an arbitrary disk of radius equal to the maximal edge length is less than Const.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be a bounded simply-connected domain. Let $g: \mathbb{C} \rightarrow \mathbb{R}$ be a smooth function. Let $\left\{Q_{n}\right\}$ be a nondegenerate uniform sequence of finite orthogonal lattices approximating the domain $\Omega$. Then the solution $u_{Q_{n}, g}$ of the Dirichlet problem on $Q_{n}$ uniformly converges to the solution $u_{\Omega, g}$ of the Dirichlet problem on $\Omega$ (with the same boundary values).

Let us give an example of an application of discrete complex analysis to tiling problems.
Theorem 3. For a number $c>0$ the following 2 conditions are equivalent:

- a rectangle with side ratio c can be tiled by rectangles similar to it but not all homothetic to it;
- the number $c^{2}$ is a root of an integral polynomial such that all the other complex roots are negative real numbers.

Now we proceed to the classification of circular surfaces.
Theorem 4. If through each point of an analytic surface in $\mathbb{R}^{3}$ one can draw two transversal circular arcs fully contained in the surface (and analytically depending on the point) then some composition of inversions takes the surface to a subset of one of the following sets:
(E) the set $\{p+q: p \in \alpha, q \in \beta\}$, where $\alpha, \beta$ are two circles in $\mathbb{R}^{3}$;
(C) the set $\left\{2 \frac{p \times q}{|p+q|^{2}}: p \in \alpha, q \in \beta, p+q \neq 0\right\}$, where $\alpha, \beta$ are two circles in the unit sphere $S^{2}$;
(D) the set $\left\{(x, y, z): Q\left(x, y, z, x^{2}+y^{2}+z^{2}\right)=0\right\}$, where $Q \in \mathbb{R}[x, y, z, t]$ has degree 2 or 1 .

Let us outline the proof of this theorem. We solve the problem in $S^{4}$ instead of $\mathbb{R}^{3}$. Using the Schicho parametrization of surfaces containing two arcs of conics through each point, we reduce the problem to solving the equation

$$
X_{1}^{2}+X_{2}^{2}+X_{3}^{2}+X_{4}^{2}+X_{5}^{2}=X_{6}^{2}
$$

in polynomials $X_{1}, X_{2}, \ldots, X_{6}$ in $\mathbb{R}[u, v]$ of degree at most 2 in each of the variables. Such "Pythagorean 6-tuples" of polynomials define the surface $X_{1}(u, v): \ldots: X_{6}(u, v)$ in $S^{4}$, containing two (possibly degenerate) circles $u=$ const and $v=$ const through each point.

We solve this equation (that is, we find a simple parametrization of the set of solutions) using the decomposition of quaternionic polynomials into irreducible factors up to "Möbius transformations".

Let us discuss more particular results.
Theorem 5. If through each point of a smooth surface in $\mathbb{R}^{3}$ one can draw both a straight line segment and a circular arc fully contained in the surface (and continuously depending on the point) then the surface is a subset of either a one-sheeted hyperboloid, or a quadratic cone, or an elliptic cylinder, or a plane.

A cyclide is a surface in 3-dimensional space given by an equation of the form

$$
a\left(x^{2}+y^{2}+z^{2}\right)^{2}+(b x+c y+d z)\left(x^{2}+y^{2}+z^{2}\right)+Q(x, y, z)=0
$$

where $a, b, c, d$ are constants and $Q(x, y, z)$ is a polynomial of degree at most 2 that do not vanish simultaneously.

Theorem 6. Let a smooth closed surface be homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least 4 distinct circles fully contained in the surface (and continuously depending on the point) then the surface is a cyclide.

## Approbation of the work

The main results of the thesis were presented at the following conferences and seminars:

1. Talk «Feynman checkers: quantum field theory on a checkered paper», Satellite mini-workshop of the conference "Diophantine analysis, dynamics and related topics", Haifa, Israel, February 2023.
2. Talk "Surfaces containing two circles through each point", seminar by J.Schicho, March 2017, University of Linz, Linz, Austria.
3. Talk "Surfaces containing two circles through each point", conference «Real Geometry», Lumini, France, September 2017, https://conferences.cirm-math.fr/ 1782.html
4. Talk «Conservation of energy in lattice field theories», conference «Contemporary mathematics, in honor of the 80th birthday of Vladimir Arnold (1937-2010)», Moscow, Russia, December 2017, http://me.hse.ru/lando/.
5. Talk «Conservation of energy in lattice field theories», conference «DYNAMICAL SYSTEMS AND PERTURBATIONS, The conference celebrating S.Yu. Pilyugin's 70th birthday», St. Petersburg, Russia, October 2017, http://pilyugin70.spb. ru/.
6. Talk «Surfaces containing two circles through each point», Conference «Mathematics, Theoretical Physics and Data Science 2016, dedicated to anniversaries of Yakov Sinai and Grigory Margulis», July 2016, Moscow, Russia, http://mpd2016.iitp. ru/schedule.
7. Talk «Tiling by rectangles and alternating current», Transversal aspects of tilings, Oleron, France, June 2016, https://oleron.sciencesconf.org/.
8. Talk «Surfaces on which two circles can be drawn through each point», Weekly seminar of the Laboratory of Algebraic Geometry and Its Applications, HSE University, Moscow, Russia, November 2016.
9. Talk «Surfaces containing two circles through each point», Meeting of the Moscow Mathematical Society, Moscow, Russia, December 2016
10. Talk «Discrete complex analysis: convergence results», International Congress of mathematicians, Seoul, Korea, August 2014.
11. Talk «Discrete complex analysis: convergence results», Embedded graphs, St. Petersburg, Russia, October 2014.
12. Talk «Discrete Riemann surfaces: convergence results», International conference "Discrete curvature", Marseille, France, November 2012.
13. Talk «Tiling of a rectangle, alternating current, and continued fractions», International conference "Multidimensional continued fractions", Graz, Austria, July 2012.
14. Talk «Discrete complex analysis: convergence results», I.M. Gelfand Centennial Conference, Moscow, Russia, July 2012.
15. Talk «Discrete analytic functions: convergence results», Christmas mathematical meetings of the "Dynasty" Foundation, Moscow, Russia, January 2012.
16. Talk «Triangulations of surfaces by circular arcs», Algebra and number theory, Saratov, Russia, September 2012.
17. Talk «Surfaces containing several circles through each point», International Conference dedicated to the 65 th anniversary of Askold G. Khovanskii, Moscow, Russia, June 2012.
18. Talk «Discrete analytic functions: convergence results», Oberwolfach workshop "Discrete differential geometry", Oberwolfach, Germany, July 2011.
19. Talk «Discrete Riemann surfaces: convergence results», seminar by A.I. Bobenko, Berlin, Germany, March 2011.

## Publications

The main results of the thesis are published in the following articles:

1. M. Prasolov, M. Skopenkov, Tiling by rectangles and alternating current, J. Combin. Theory A 118:3 (2011), 920-937.
2. M. Skopenkov, H. Pottmann, P. Grohs, Ruled Laguerre minimal surfaces, Math. Z. 272 (2012), 645-674.
3. H. Pottmann, L. Shi, M. Skopenkov, Darboux cyclides and webs from circles, Computer Aided Geom. Design 29:1 (2012), 77-97.
4. F. Nilov, M. Skopenkov, A surface containing a line and a circle through each point is a quadric, Geom. Dedicata 163:1 (2013), 301-310.
5. M. Skopenkov, The boundary value problem for discrete analytic functions, Adv. Math. 240 (2013) 61-87.
6. A. Bobenko, M. Skopenkov, Discrete Riemann surfaces: linear discretization and its convergence, J. Reine Angew. Math. 2016:720 (2016) 217-250.
7. M. Skopenkov, R. Krasauskas, Surfaces containing two circles through each point, Math. Ann. 373 (2019) 1299-1327.
8. M. Skopenkov, P. Bo, M. Barton, H. Pottmann, Characterizing envelopes of moving rotational cones and applications in CNC machining, Computer Aided Geom. Design 83 (2020), 101944.
9. M. Skopenkov, Lattice gauge theory and a random-medium Ising model, Math. Phys. Anal. Geom. 25:18 (2022).
10. M. Skopenkov, A. Ustinov, Feynman checkers: towards algorithmic quantum theory, Russian Math. Surveys 77:3(465) (2022), 73-160.

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