STEKLOV MATHEMATICAL INSTITUTE
OF RUSSIAN ACADEMY OF SCIENCES

Topology of plane real and complex algebraic curves

Stepan Yurievich Orevkov

Summary of the thesis for the purpose of obtaining
academic degree Doctor of Science in Mathematic

Moscow - 2023

In this thesis we study topological properties of plane algebraic curves. Most of results concern real curves. The references [P0], [P1], ..., [P9] refer to the papers which are included into this thesis as its part.

1. A brief historical survey

Topological study of plane real projective algebraic curves can be traced back at least to the works of A. Harnack [18] and F. Klein [20]. For a given degree d there are only finitely many homeomorphism types of pairs $(\mathbb{RP}^2, \mathbb{R}A)$ where $\mathbb{R}A$ is a smooth real algebraic curve of degree d. These homeomorphism types are known as topological arrangements of real algebraic curves (real schemes of a curve according to the terminology of V.A. Rokhlin's school), and a natural question is to list them all. This problem was popularized by D. Hilbert, and the corresponding problem became known as the 16th Hilbert problem (more precisely, its first part). This problem remains one of the few open problems in Hilbert's famous problem list. As it was shown in [18], a planar real algebraic curve of degree d may have up to $\frac{1}{2}(d-1)(d-2)+1$ components. The curves with exactly $\frac{1}{2}(d-1)(d-2)+1$ components are known as M-curves. (Note that $g = \frac{1}{2}(d-1)(d-2)$ is the genus of a smooth complex curve of degree d in the plane by the adjunction formula.)

A complete topological classification of smooth plane real algebraic curves of degree 5 is an elementary consequence of Harnack's bound combined with an observation that any line cannot cut a curve of degree d at more than d points. The case of degree 6 is the first non-trivial case which was especially emphasized by Hilbert in his 16th problem. The first result on degree 6 was obtained by I. G. Petrovsky [31]. He proved that 11 ovals of such curve cannot be one outside another. This was a particular case of the Petrovsky inequality for curves of any even degree d:

$$-\tfrac{3}{8}d^2 + \tfrac{3}{4}d \le p - n \le \tfrac{3}{8}d^2 - \tfrac{3}{4}d + 1$$

where p (resp. n) is the number of even (resp. odd) ovals, i.e. ovals encircled by an even (resp. odd) number of another ovals.

Later on, D. A. Gudkov [17] completed the classification for degree 6. Any such classification naturally splits in two parts: constructions and restrictions (often called prohibitions in English translations from the Russian). For restrictions, Gudkov used the approach proposed by Hilbert and then developed by Rohn and by himself (Hilbert-Rohn-Gudkov method). Roughly speaking, the idea is the following. Suppose by contradiction that a certain arrangement of ovals is realized by a curve of degree 6 given by $F = 0$. We consider a continuous family of curves $F_t = 0$ where $F_t = F + tG^2$, $\deg G = 3$. Then the domain $F_t \ge 0$ grows, hence a singularity appears for some t. Then we replace G by another cubic polynomial vanishing at the singular point and continue. In this way we arrive to a curve with 10 double points. After this, we continue the deformation already in non-linear one-dimensional families. At each step there are several a priory possible positions for newly appearing singularities and their nature. So, we obtain a rather big tree of possibilities. At each branch of this tree we finally obtain a contradiction either with Bezout theorem for auxiliary lines and conics or (and this is the main Gudkov's contribution) using the fact that the positions of singular points can be chosen generically which excludes many cases of splitting the curve into irreducible components. For constructions, Gudkov perturbed singular curves obtained by quadratic Cremona transformations starting with curves of smaller degree.

Analyzing the available information, Gudkov formulated the conjecture that

$$p - n \equiv k^2 \mod 8$$

for any M-curve of degree $2k$. This conjecture stimulated important progress in the domain. It was proved by Arnold [2] mod 4 and then by Rokhlin [32] mod 8. They introduced a completely new technique based on the 4-dimensional topology for the study of the double covering of $\mathbb{CP}^2$ branched along the complexification of the curve.

The Gudkov-Rokhlin congruence and a similar congruence for $(M-1)$-curves yields the restriction part of degree 6 classification as an immediate consequence. Then Viro considerably simplified the construction part by introducing a very powerful method which later was called patchworking and which was one of the starting points of the tropical geometry. This method allowed him to complete the construction part of classification in degree 7 (see [40]). The restriction part in degree 7 was also done by Viro [38] using a combination of Rokhlin's formulas for the complex orientations [33] and Fiedler's theorem about the alternation of the complex orientations in pencils of lines. By these (and some other) methods Viro also essentially advanced in the classification of curves of degree 8 which was further continued by E.I. Shustin, A.B. Korchagin, B. Chevallier using more or less the same methods. When I started to work on this subject there were 9 unknown cases for M-curves of degree 8.

I finish my historical sketch here because this is not a survey of all the real algebraic geometry. This is just an introduction to my thesis, thus I do not discuss many interesting and important results obtained in other branches of the domain by A.I. Degtyarev, I.V. Itenberg, V.V. Nikulin, V.M. Kharlamov, G.B. Mikhalkin, G.M. Polotovskii, J.-Y. Welschinger, V.I. Zvonilov, and many other mathematicians.

2. Real algebraic and real pseudo-holomorphic curves

2.1. Quasipositive braids and pseudo-holomorphic curves.

In [P1] I proposed (and further developed in [10, P2, P3, P6]) a new approach to the study of the topology of plane real algebraic curves. This approach is based on the obvious observation that the boundary braid of a multivalued algebraic function in a disk without poles is a *quasipositive braid*, i.e., a braid which is a product of conjugates of the standard generators of the braid group. Any algebraic curve can be viewed as a graph of a multivalued function as soon as we fix a pencil of lines which plays the role of the pencil of vertical lines when speaking of graphs. On the other hand, if the algebraic function is real and the disk is contained in the upper half-plane and its boundary is sufficiently close to the real axis (in fact, if the disk contains all branching points which are in the upper half-plane), then sometimes the braid can be recovered from the embedded type of the real curve, maybe up to some unknown parameters. More precisely, if the d-valued function has at least $d-2$ real values (counted with multiplicities) at each real point, then the braid is uniquely determined by the fiberwise arrangement of the curve with respect to the pencil. This is so, for example, when the center of the pencil is chosen inside $(d-2)/2$ nested ovals. If the center of the pencil is inside $(d-4)/2$ nested ovals, then the braid is determined up to k unknown integer parameters where k is the number of segments of the pencil with 4 non-real intersections with the curve.

Thus in such cases (when the braid can be recovered from the topology of the curve) the problem of realizability of an arrangement of a real curve with respect to some pencil of lines is partially reduced to the quasipositivity of certain collection of braids. "Partially" because the quasipositivity is a necessary but not sufficient condition of algebraic realizability. However this condition is necessary and sufficient for pseudo-holomorphic realizability which is the subject of the rest of this section.

Let X be a compact 4-manifold endowed with a symplectic form ω and an almost complex structure $\mathcal{J}$ tamed by ω, i.e. $\omega(v, \mathcal{J}v) > 0$ for any nonzero tangent vector v. A smooth embedded surface is called a *$\mathcal{J}$-holomorphic curve* (or *pseudo-holomorphic curve* when $\mathcal{J}$ is not specified) if all its tangent planes are $\mathcal{J}$-invariant. The famous theory of pseudoholomorphic curves created by M. L. Gromov in [15] shows that such curves share many important properties with algebraic curves.

Assume now that $X = \mathbb{CP}^2$, ω is the Fubini-Studi symplectic form, and $\mathcal{J}$ is anti-invariant under the complex conjugation: $\mathrm{conj}^* \circ \mathcal{J} = \mathcal{J}^{-1} \circ \mathrm{conj}$. We say that a $\mathcal{J}$-holomorphic curve A is real if $\mathrm{conj}(A) = A$. In this case we set $\mathbb{R}A = A \cap \mathbb{RP}^2$. Then $\mathbb{R}A$ is a disjoint union of embedded circles. Due to Gromov's theory real pseudo-holomorphic curves are very similar to real algebraic curves in many aspects. In particular, they are flexible curves in the sense of Viro (see [39]) which implies that most of general restrictions of topological nature are valid for them. In fact Viro in [39] gave a formal definition of topological restrictions as those which are valid for his flexible curves. A long list of such restrictions can be found in [40]. In particular, it includes Gudkov-Rokhlin congruence (and its analog for $(M-1)$-curves), Petrovsky inequality, Arnold inequalities, Rokhlin and Rokhlin-Mishachev formulas for complex orientations, all restrictions based on construction of many 2-cycles on the double coverings.

Almost all non-topological restrictions discussed in the survey [40] were based either on Bezout theorem for auxiliary lines or conics, or on consideration of auxiliary pencils of lines, maybe, after Cremona transformation. Anyway, only these restrictions were applied for the classification in degree up to 9 (the only cases where it was done or at least started). It was observed in [P2] that all these restrictions extend to the pseudoholomorphic case as well.

Thus one of the reasons why the study of real pseudoholomorphic curves is important in the context of the 16th Hilbert Problem, is that is shows the limits of applicability of the standard methods commonly being used in the domain. Namely, if a certain configuration of ovals is realized pseudo-holomorphically, then nobody will waste time and efforts trying to prohibit it by standard methods.

However pseudo-holomorphic curves also allow one to advance in classification problem in both directions: restrictions and constructions. The reason in both cases is that sometimes a hypothetically existing smooth algebraic curve can be degenerated to a singular pseudo-holomorphic curve. This idea was used in [P2] for restrictions and in [27], [28] for constructions. However, the usage of pseudo-holomorphic curves for constructions needs some comments. Formally speaking, they have not been used in [27] and [28]. Even the word "pseudo-holomorphic" never occurs in these papers. However, it was hardly possible to find the needed singular curve to be perturbed without knowledge that this is one of a few deep degenerations which are pseudo-holomorphically realizable.

2.2. Classification of M-curves of degree 8. When I started my research on plane real algebraic curves, 9 cases remained open for configuration of ovals of a real algebraic M-curve of degree 8 (see Figures 1–3 where each number n means that there are n unnested ovals in the corresponding region). I realized one of them in [28][1] and excluded two in [P2]. Moreover, all the remaining 6 cases I realized in [P1], [P2] pseudo-holomorphically thus completing the classification of real pseudoholomorphic M-curves of degree 8 up to isotopy.

Theorem 2.1. (Theorem 1.2 in [P2].) *The isotopy types in Figure 2 are not realizable by real pseudoholomorphic curves of degree* 8*. The isotopy types in Figure 3 are realizable by real pseudoholomorphic curves of degree* 8*.*

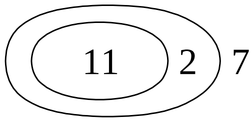

FIGURE 1. Algebraic curve constructed in [28]

FIGURE 2. Pseudoholomorphically unrealizable configurations [P2]

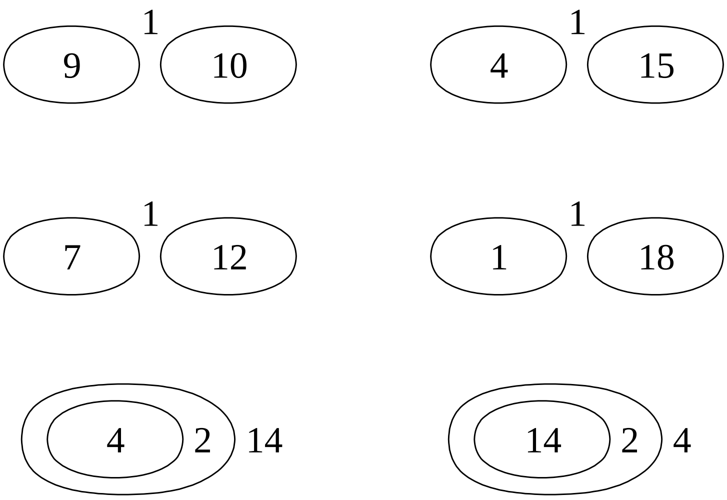

FIGURE 3. Pseudoholomorphic curves constructed in [P1], [P2]

A complete list of the isotopy types realizable by real pseudoholomorphic M-curves of degree 8 is given in Table 1. The encoding of the isotopy types is described in [39], [40]. The isotopy types whose algebraic realizability remains unknown are

[1]This result is not included in the thesis by formal requirements.

Table 1. Isotopy types of pseudo-holomorphic M-curves of degree 8

p=19, n=3	p=15, n=7	p=11, n=11	p=7, n=15	p=3, n=19
$\langle 18\sqcup 1\langle 3\rangle\rangle^{\mathrm{Ha}}$	$\langle 14\sqcup 1\langle 7\rangle\rangle^{\mathrm{G}}$	$\langle 10\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$	$\langle 6\sqcup 1\langle 15\rangle\rangle^{\mathrm{V}}$	$\langle 2\sqcup 1\langle 19\rangle\rangle^{\mathrm{V}}$
$\langle 17\sqcup 1\langle 1\rangle\sqcup 1\langle 2\rangle\rangle^{\mathrm{Ha}}$	$\langle 13\sqcup 1\langle 1\rangle\sqcup 1\langle 6\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 1\rangle\sqcup 1\langle 10\rangle\rangle^{\mathrm{K}}$	$\langle 5\sqcup 1\langle 1\rangle\sqcup 1\langle 14\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 1\rangle\sqcup 1\langle 18\rangle\rangle^{\mathrm{O}\,*}$
	$\langle 13\sqcup 1\langle 2\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{G}}$	$\langle 9\sqcup 1\langle 2\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 2\rangle\sqcup 1\langle 13\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 2\rangle\sqcup 1\langle 17\rangle\rangle^{\mathrm{V}}$
	$\langle 13\sqcup 1\langle 3\rangle\sqcup 1\langle 4\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 3\rangle\sqcup 1\langle 8\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 3\rangle\sqcup 1\langle 12\rangle\rangle^{\mathrm{V}}$	
		$\langle 9\sqcup 1\langle 4\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 4\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 4\rangle\sqcup 1\langle 15\rangle\rangle^{\mathrm{O}\,*}$
		$\langle 9\sqcup 1\langle 5\rangle\sqcup 1\langle 6\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 5\rangle\sqcup 1\langle 10\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 5\rangle\sqcup 1\langle 14\rangle\rangle^{\mathrm{V}}$
			$\langle 5\sqcup 1\langle 6\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	
			$\langle 5\sqcup 1\langle 7\rangle\sqcup 1\langle 8\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 7\rangle\sqcup 1\langle 12\rangle\rangle^{\mathrm{O}\,*}$
				$\langle 1\sqcup 1\langle 8\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$
				$\langle 1\sqcup 1\langle 9\rangle\sqcup 1\langle 10\rangle\rangle^{\mathrm{O}\,*}$
$\langle 17\sqcup 3\langle 1\rangle\rangle^{\mathrm{W}}$	$\langle 12\sqcup 2\langle 1\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 13\rangle\rangle^{\mathrm{S}}$	$\langle 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 17\rangle\rangle^{\mathrm{S}}$
	$\langle 12\sqcup 1\langle 1\rangle\sqcup 2\langle 3\rangle\rangle^{\mathrm{V}}$	$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{S}}$	$\langle 1\langle 1\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{S}}$
		$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 1\langle 5\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{S}}$
		$\langle 8\sqcup 1\langle 3\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{S}}$	
			$\langle 4\sqcup 1\langle 3\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	
			$\langle 4\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{S}}$	
$\langle 1\sqcup 1\langle 2\sqcup 1\langle 17\rangle\rangle\rangle^{\mathrm{Hi}}$	$\langle 1\sqcup 1\langle 6\sqcup 1\langle 13\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 10\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 14\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{Hi}}$	$\langle 1\sqcup 1\langle 18\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$
$\langle 2\sqcup 1\langle 2\sqcup 1\langle 16\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 2\sqcup 1\langle 6\sqcup 1\langle 12\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 2\sqcup 1\langle 10\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 2\sqcup 1\langle 14\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$	
$\langle 3\sqcup 1\langle 2\sqcup 1\langle 15\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 3\sqcup 1\langle 6\sqcup 1\langle 11\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 3\sqcup 1\langle 10\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 3\sqcup 1\langle 14\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$	
$\langle 4\sqcup 1\langle 2\sqcup 1\langle 14\rangle\rangle\rangle^{\mathrm{O}\,*}$	$\langle 4\sqcup 1\langle 6\sqcup 1\langle 10\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 4\sqcup 1\langle 10\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 4\sqcup 1\langle 14\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$	
$\langle 5\sqcup 1\langle 2\sqcup 1\langle 13\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 5\sqcup 1\langle 6\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 10\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 14\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{Hi}}$	
$\langle 6\sqcup 1\langle 2\sqcup 1\langle 12\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 6\sqcup 1\langle 6\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 6\sqcup 1\langle 10\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$		
$\langle 7\sqcup 1\langle 2\sqcup 1\langle 11\rangle\rangle\rangle^{\mathrm{O}}$	$\langle 7\sqcup 1\langle 6\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 7\sqcup 1\langle 10\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$		
$\langle 8\sqcup 1\langle 2\sqcup 1\langle 10\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 8\sqcup 1\langle 6\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 8\sqcup 1\langle 10\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$		
$\langle 9\sqcup 1\langle 2\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 6\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 10\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$		
$\langle 10\sqcup 1\langle 2\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 10\sqcup 1\langle 6\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$			
$\langle 11\sqcup 1\langle 2\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 11\sqcup 1\langle 6\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$			
$\langle 12\sqcup 1\langle 2\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 12\sqcup 1\langle 6\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$			
$\langle 13\sqcup 1\langle 2\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 13\sqcup 1\langle 6\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$			
$\langle 14\sqcup 1\langle 2\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{O}\,*}$				
$\langle 15\sqcup 1\langle 2\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{K}}$				
$\langle 16\sqcup 1\langle 2\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{C}}$				
$\langle 17\sqcup 1\langle 2\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$				

Ha=Harnack [18]
Hi=Hilbert [19]
W=Wiman [43]
G=Gudkov [16]
K=Korchagin [21, 22]
V=Viro [37, 40]
S=Shustin [34, 35]
C=Chevallier [5]
O=Orevkov [P1,P2,28]

marked by an asterisk. Near each real scheme, we indicate the author of its first realization.

2.3. New formulas of complex orientations.

A real pseudoholomorphic (in particular, real algebraic) curve A is called *separating* or *Type I* if $A \setminus \mathbb{R}A$ is disconnected. In this case $A \setminus \mathbb{R}A$ has two connected components A_+ and A_- exchanged by the complex conjugation, and a *complex orientation* of $\mathbb{R}A$ is the boundary orientation induced from A_+ or from A_-. These orientations are obtained from each other by simultaneous reversion of the orientation of each connected component of $\mathbb{R}A$.

Rokhlin [33] proved the following formula for complex orientations for separating curves of even degree $2k$ in $\mathbb{RP}^2$:

$$2(\Pi_+ - \Pi_-) = k^2 - l,$$

where l is the number of ovals and Π_+ (resp. Π_-) is the number of *positive* (resp. *negative*) *injective pairs* of ovals, i.e. pairs of ovals such that one of them is in the interior of the other one and the complex orientations are as shown in Figure 4.

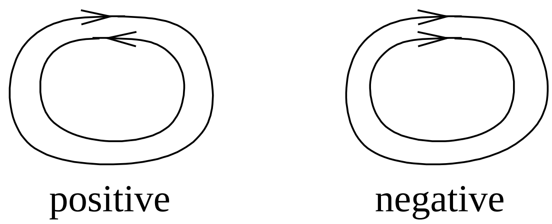

FIGURE 4. Positive and negative injective pairs

This formula was generalized by Mishachev [26] to curves of odd degree $2k+1$:

$$2(\Pi_+ - \Pi_-) + \Lambda_+ - \Lambda_- = k^2 + k - l,$$

where Λ_+ (resp. Λ_-) is the number of positive (resp. negative) ovals; an oval O is called *positive* (resp. *negative*) if the class of $[O] = -2[J]$ (resp. $[O] = 2[J]$) in $H_1(M)$ where M is the non-oriented component of $\mathbb{RP}^2 \setminus O$ and J is the non-contractible component of $\mathbb{R}A$ (also called *pseudoline*); see Figure 5.

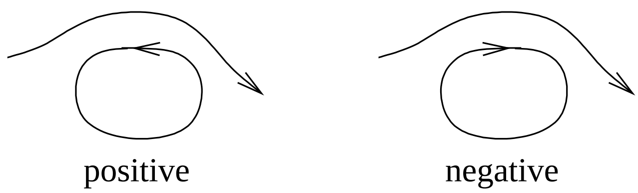

FIGURE 5. Positive and negative ovals

The complex orientation formulas played a crucial role in the restriction part of the classification of real algebraic curves of degree 7 and 8 (see [38], [P2]).

In [P1] I found new complex orientation formulas for curves with a deep nest, i.e., curves with a nest of depth $\lfloor m/2 \rfloor$ where m is the degree. Let us formulate them.

When two ovals O and O' form an injective pair, we set $[O : O'] = 1$ if this pair is positive and $[O : O'] = -1$ if it is negative. Let A be a pseudoholomorphic curve of degree m. In the case when m is even and O is not outer, we say that O is *positive* if $[O : O'] = 1$ where O' is the outer oval surrounding O. Otherwise O is called *negative.* If m is even, we assume also that *any outer non-empty oval is negative by definition.*

Suppose $\mathbb{R}A$ has a nest $(O_1, \dots, O_{k-1})$ of depth $k-1$ where $k = [m/2]$. This means that the oval O_j is surrounded by O_k for $j > k$. It follows from Bézout theorem that all the other ovals are empty.

Theorem 2.2. (Theorem 1.5A in [P1].) *Let k^+ (resp. k^-) be the number of positive (resp. negative) non-empty ovals, λ_+ (resp. λ_-) the number of positive (resp. negative) empty ovals, and let π_s^S, $S, s \in \{+,-\}$ be the number of pairs (O, o) where o is an empty oval surrounded by O and (S, s) are the signs of (O, o). Then*

$$\pi_-^+ - \pi_+^+ = (k^+)^2, \qquad \pi_+^- - \pi_-^- = (k^-)^2 \qquad \textit{(m is even)};$$
$$\pi_-^+ - \pi_+^+ = (k^+)^2, \qquad \pi_+^- - \pi_-^- + (\lambda_+ - \lambda_-)/2 = (k^-)^2 + k^- \qquad \textit{(m is odd).}$$

These formulas, as well as their direct generalizations in [42] and in [30], found numerous applications, see, e.g., [8], [9], [29], [P1], [P2].

2.4. Trigonal curves.

The question of realizability of a given fiberwise isotopy type (not necessarily smooth) by a real algebraic trigonal curve (i.e., a curve given by an equation $F(x, y) = 0$ with $\deg_y F = 3$) is completely answered in [P5]. A similar question for real pseudoholomorphic curves is answered in [P7, §6] (see §4 below). By a *fiberwise isotopy type* we mean an equivalence class of smooth plane curves with finitely many singular points (the curves are supposed to be analytic near them) where two curves are equivalent if they are related by an isotopy of $\mathbb{R}^2$ which maps any vertical line to another vertical line.

With each real trigonal plane curve A given by $F(x, y) = 0$ we associate the planar graph $\Gamma = j^{-1}(\mathbb{RP}^1) \subset \mathbb{CP}^1$ where $j(x)$ is the Weierstrass j-invariant of the elliptic curve

$$\{(y, z) \mid F(x, y) = z^2\}.$$

Then the fiberwise type of $\mathbb{R}A$ determines the combinatorial type of Γ near $\mathbb{RP}^1$, and the problem of realizability of a given fiberwise type reduces to the problem of existence of extension of the graph from a neighborhood of $\mathbb{RP}^1$ to the whole $\mathbb{CP}^1$. This is a combinatorial problem which is algorithmically solvable.

The results and ideas from [P5] found numerous applications (the paper [P5] has 34 citation according to the database MathSciNet).

2.5. Affine M-sextics.

An affine smooth irreducible real algebraic curve A in $\mathbb{R}^2$ of degree d is an *affine M-curve* if it has maximal possible number of connected components, which is equal to $g + d$ where $g = (d-1)(d-2)/2$ is the genus of the complexification of A. This condition is equivalent to the fact that the projective closure of A is an M-curve (i.e. has $g+1$ connected components) and all intersections with the infinite line are real and transverse and sit on the same connected component of the closure of A.

A classification of algebraic affine M-sextics up to isotopy was started in [23] and completed in [P1], [P3], [P6], [P9]. In [P1] a classification of real pseudoholomorphic M-sextics is obtained. The two classifications do not coincide. They are as follows.

The notation for isotopy types of affine M-sextics, which we represent as pairs of a projective sextic and a real line in $\mathbb{R}P^2$ (the line at infinity) is shown in Figure 6, where a, b, c denote the number of non-nested ovals in the corresponding domains.

Theorem 2.3. (a). (Theorem 1 and Section 7.2 in [P1].) *The following is a complete list of algebraic affine M-sextics up to isotopy.*

$$\begin{cases}
A_1(a,b), & (a,b) = (1,8), (5,4), \\
A_2(a,b,c), & (a,b,c) = (1,8,1), (8,1,1), (0,5,5), (1,4,5), \\
 & \qquad\qquad (4,1,5), (5,0,5), (0,1,9), (1,0,9), \\
A_3(a,b,c), & (a,b,c) = (4,5,1), (7,2,1), (2,3,5), (4,1,5), (0,1,9), \\
 & \qquad\qquad (0,5,5), \\
A_4(a,b,c), & (a,b,c) = (1,8,1), (5,4,1), \\
B_1(a,b), & (a,b) = (1,8), (5,4), \\
B_2(a,b,c), & (a,b,c) = (1,8,1), (0,5,5), (5,0,5), (0,1,9), (1,0,9), \\
B_3(a,b,c), & (a,b,c) = (3,6,1), (1,4,5), (2,3,5), \\
C_1(a,b,c), & (a,b,c) = (0,9,1), (7,2,1), (0,5,5), (3,2,5), (0,1,9), \\
C_2(a,b,c), & (a,b,c) = (1,7,2), (5,3,2)
\end{cases}$$

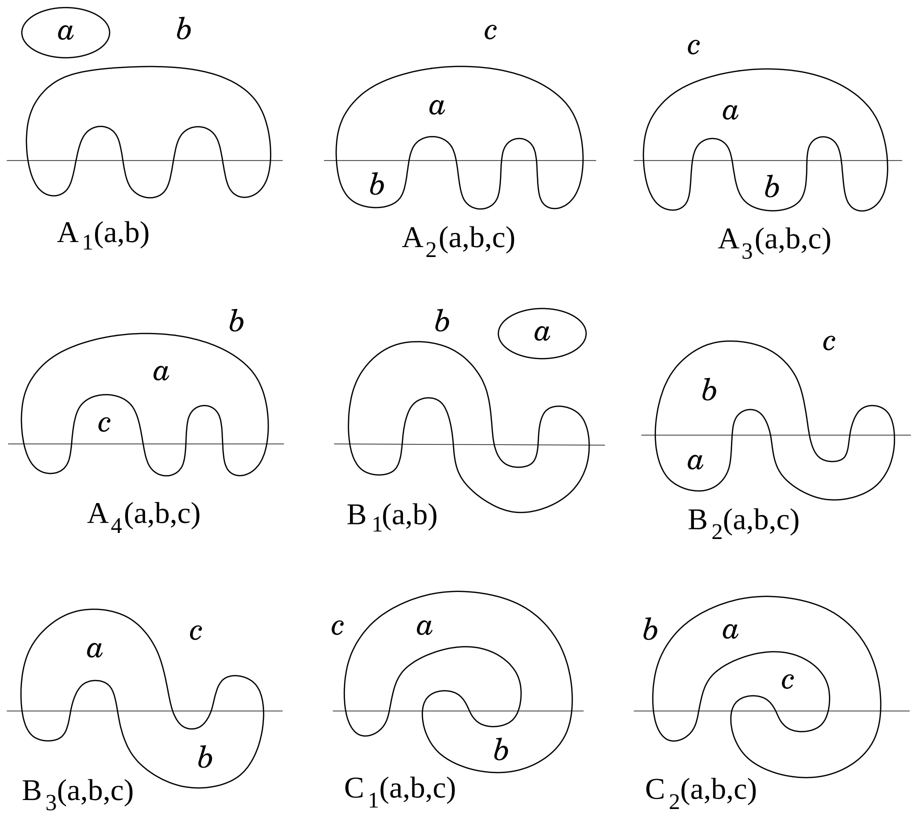

FIGURE 6. Encoding the isotopy types of affine M-sextics

(b) (Theorem 5 in [P3], Theorem 1 in [P9], and Theorem 1.1 in [P6] respectively.) *The following isotopy types are realizable pseudoholomorphically but not algebraically.*

$$A_4(1,4,5), \quad B_2(1,4,5), \quad C_2(1,3,6)\ . \tag{1}$$

3. On algebraic unrealizability of isotopy types realizable by real pseudoholomorphic curves

As we mentioned above, real pseudoholomorphic curves share many topological properties of real algebraic curves. Therefore, in the cases when an isotopy type is pseudoholomorphically realizable, essentially new methods should be involved to prove that it is unrealizable algebraically. In this section we present the methods developed and/or applied by the author to this end.

3.1. Hilbert – Rohn – Gudkov method (joint work with E. I. Shustin). We already gave a brief outline of this method in Section 1. This method was developed and used by Gudkov to obtain a classification of real algebraic curves of degree 6. Later on, a topological proof was found, and Gudkov's result about sextic curves became a partial case of much more general facts concerning curves of any even degree. So, there was an impression that the Hilbert–Rohn–Gudkov method is no longer needed. However, the discovery of algebraically unrealizable pseudoholomorphic curves gave a new life to it.

In the series of joint papers with E. I. Shustin started with [P3], [P6] we applied this method to exclude the three affine sextics mentioned in Section 2.5 as well as to prove the algebraic unrealizability of certain curves on the quadratic cone. To apply the Hilbert–Rohn–Gudkov method in our setting, we found new sufficient conditions for existence of a one-dimensional equisingular deformation of a curve with A_n-singularities passing through a given set of fixed points, such that a certain quantity monotonically decreases. The latter condition guarantees that the obtained family of curves converges to some more degenerate curve.

3.2. Trigonal curves.

The results of [P5] (see Section 2.4) provide an algorithm to decide if a given fiberwise arrangement is realizable by an algebraic trigonal curve of a given bidegree. For example, the fiberwise arrangement in Figure 7 is unrealizable by a curve of bidegree $(3, 15)$ on the Hirzebruch surface Σ_5. The real locus $\mathbb{R}\Sigma_5$ is a Klein bottle. It is represented in Figure 7 by a rectangle with identified opposite sides. The horizontal sides represent the exceptional section (whose self-intersection number is -5) and the vertical sides represent a fiber of the fibration $\Sigma_5 \to \mathbb{P}^1$.

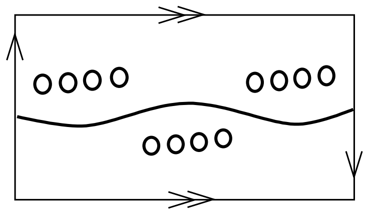

FIGURE 7. A fiberwize arrangement which is unrealizable by a smooth real algebraic trigonal curve of bidegree $(3, 15)$ in Hirzebruch surface Σ_5 but realizable pseudoholomorphically in the same homology class

The algebraic unrealizability of Figure 7 is used in the final step of the proof in [P9] of the algebraic unrealizability of the pseudoholomorphic affine sextic $B_2(1, 4, 5)$ (see (1) in Section 2.5 above). Notice that this fiberwise arrangement is realizable pseudoholomorphically. In order to prove this fact, it is enough to show that the braid

$$b := \sigma_2^{-4}\,\sigma_1\sigma_2^{-1}\,\sigma_1^{-4}\,\sigma_2\sigma_1^{-1}\,\sigma_2^{-4}\,\sigma_1\sigma_2^{-1}\,\Delta^3$$

is quasipositive (Δ is the Garside half-twist $\sigma_1\sigma_2\sigma_1$). Indeed, one easily checks that

$$b = (\sigma_2^{-3}\sigma_1\sigma_2^3)(\sigma_1^{-2}\sigma_2\sigma_1^2)(\sigma_2^{-1}\sigma_1^{-3}\sigma_2\sigma_1^3\sigma_2).$$

3.3. Cubic resolvents of quadrigonal curves.

Let A be a real algebraic quadrigonal curve, i.e., a curve given by $F(x, y) = 0$, $\deg_y F = 4$. By a birational change of variables, the question of realizability of a given fiberwise isotopy type by such a curve can be reduced to realizability of some other fiberwise isotopy type by a curve of the form

$$y^4 + a_2(x)y^2 + a_1(x)y + a_0(x) = 0.$$

The fiberwise isotopy type of this curve determines that of its cubic resolvent $R(x, y) = 0$ with respect to y and its relative position with respect to the line $y = 0$. Recall that the *cubic resolvent* (called also *resolvent cubic*) of a polynomial

$$y^4 + a_2y^2 + a_1y + a_0 = (y - y_1)(y - y_2)(y - y_3)(y - y_4)$$

is the polynomial $(y - z_1)(y - z_2)(y - z_3)$ where

$$z_1 = (y_1 + y_2)(y_3 + y_4), \quad z_2 = (y_1 + y_3)(y_2 + y_4), \quad z_3 = (y_1 + y_4)(y_2 + y_3).$$

Since $R = 0$ is a trigonal curve, the question of its algebraic realizability can be solved (see Section 2.4). Moreover, even when the required fiberwise type is algebraically realizable by $R(x, y) = 0$, sometimes it is possible to prove that the required fiberwise type of $yR(x, y)$ is unrealizable even pseudoholomorphically, which implies the algebraic unrealizability of the initial quadrigonal fiberwise arrangement.

This method was successfully applied in [P6], [P9] in order to prove the algebraic unrealizability of the affine sextic curves (1) (see Section 2.5).

3.4. Separating morphisms and Abel Theorem.

Recall that by a non-singular real algebraic curve in $\mathbb{RP}^2$ we mean a non-singular algebraic curve in $\mathbb{CP}^2$ invariant under the complex conjugation $(x : y : z) \mapsto (\bar{x} : \bar{y} : \bar{z})$. If such a curve is denoted by A, then we denote the set of its real points by $\mathbb{R}A$. A curve A is called *separating* (or *type I*) if $A \setminus \mathbb{R}A$ is not connected. In this case $A \setminus \mathbb{R}A$ has two connected components exchanged by the complex conjugation, and the boundary orientation induced by the complex orientation of any of these components is called a *complex orientation* of $\mathbb{R}A$. It is defined up to simultaneous reversing of the orientation of each connected component of $\mathbb{R}A$.

The main result of the paper [P10] is an inequality for the isotopy type of a plane nonsingular real algebraic curve endowed with a complex orientation (i.e., for the *complex scheme* of such curve according to Rokhlin's terminology [33]) which implies in particular that the oriented isotopy type shown in Figure 8, that is the complex scheme (in the notation of Viro [39])

$$J \sqcup 9_- \sqcup 1_-\langle 1_+\langle 1_-\rangle\rangle \tag{2}$$

is unrealizable by a real algebraic curve of degree 9 in $\mathbb{RP}^2$. Since this complex scheme is easily realizable by a real pseudoholomorphic curve it provides the first example of a complex scheme of a non-singular plane real projective curve which is algebraically unrealizable but pseudoholomorphically realizable. Similar examples for any degree congruent to 9 modulo 12 are also constructed in [P10].

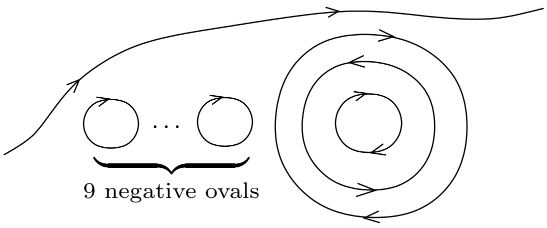

FIGURE 8. The complex scheme (2)

Let A be a non-singular separating real algebraic curve in $\mathbb{RP}^2$ of an odd degree $m = 2k+1$. We fix a complex orientation on $\mathbb{R}A$. Let r be the number of connected components of $\mathbb{R}A$. Then $l = r-1$ is the number of *ovals* (components of $\mathbb{R}A$ whose complement in $\mathbb{RP}^2$ is not connected). The component which is not an oval is called *pseudo-line* and we denote it by J. Recall that an oval is even (resp. odd) if it is encircled by an even (resp. odd) number of other ovals. An oval O is called *positive* if $[O] = -2[J]$ in $H_1(M)$ where M is the closure of the non-orientable component of $\mathbb{RP}^2 \setminus O$. Otherwise O is called *negative*. Traditionally, the number of even (resp. odd) ovals is denoted by p (resp. by n), and the number of positive (resp. negative) ovals is denoted by Λ_+ (resp. Λ_-). Let

$$\begin{aligned}\Lambda^{\mathrm{p}}_+ &= \text{the number of positive even ovals,}\\ \Lambda^{\mathrm{p}}_- &= \text{the number of negative even ovals,}\\ \Lambda^{\mathrm{n}}_+ &= \text{the number of positive odd ovals,}\\ \Lambda^{\mathrm{n}}_- &= \text{the number of negative odd ovals.}\end{aligned}$$

Theorem 3.1. (Theorem 1.1 in [P10].) *If $k > 0$, then*

$$\Lambda^{\mathrm{p}}_+ + \Lambda^{\mathrm{n}}_- + 1 \ge \frac{l - k^2 + 2k}{2} \qquad \textit{and} \qquad \Lambda^{\mathrm{n}}_+ + \Lambda^{\mathrm{p}}_- \ge \frac{l - k^2 + 2k}{2}. \tag{3}$$

Setting $l = g - 2s$ one can equivalently rewrite (3) in the form

$$\Lambda^{\mathrm{p}}_+ + \Lambda^{\mathrm{n}}_- + 1 \ge \frac{k^2 + k}{2} - s \qquad \textit{and} \qquad \Lambda^{\mathrm{n}}_+ + \Lambda^{\mathrm{p}}_- \ge \frac{k^2 + k}{2} - s. \tag{4}$$

For the complex scheme (2) we have $l = 12$ and $\Lambda^{\mathrm{p}}_+ = \Lambda^{\mathrm{n}}_- = 0$, thus the left inequality in (3) is not satisfied for $k = 4$. So we obtain:

Corollary 3.2. (Corollary 1.2 in [P10].) *The complex scheme (2) is unrealizable by a real algebraic curve of degree* 9.

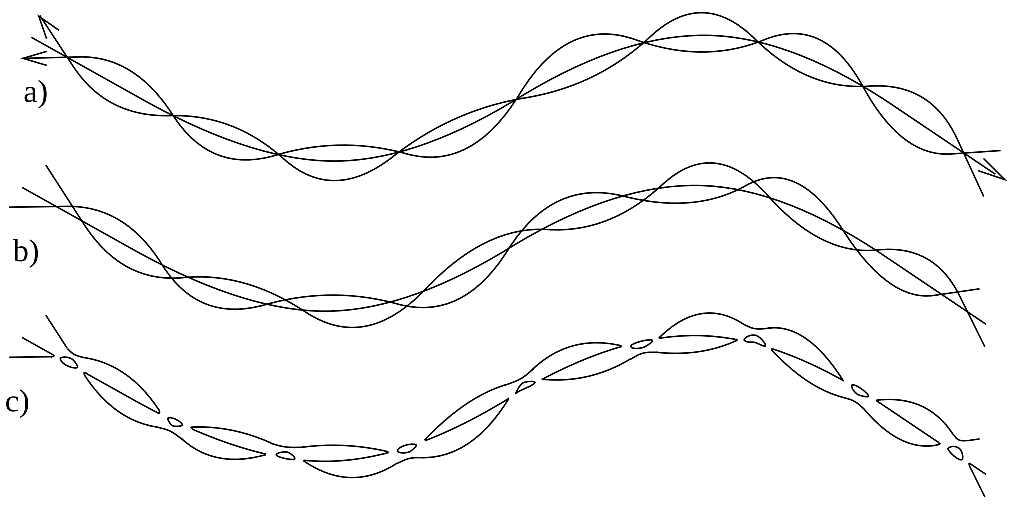

FIGURE 9. Pseudoholomorphic realization of complex scheme (2)

The main interest of Corollary 3.2 is that the complex scheme (2) admits a very simple realization by a real pseudo-holomorphic curve of degree 9. Indeed, let $C = \{f = 0\}$ be a real cubic curve with an oval, and $L = \{l = 0\}$ be the union of three lines, each cutting the pseudo-line of C at three distinct real points. Let $A_{\mathrm{sing}} = \{fg = 0\}$ with $g = (f + \varepsilon l)(f - \varepsilon l)$ and $0 < \varepsilon \ll 1$. Then A_{sing} is a reducible algebraic curve of degree 9 with nine triple points. Its real locus consists of three nested ovals and a union of three pseudolines arranged as shown in Figure 9(a). In the class of real pseudoholomorphic curves, it can be perturbed as in Figure 9(b). If we consider f and l as holomorphic sections of the line bundle $\mathcal{O}_{\mathbb{CP}^2}(3)$ rather than homogeneous polynomials, then the perturbation can be realized by replacing f with $f + h$ where h is a $\mathcal{C}^1$-small smooth (non-analytic) conjugation-invariant section which is complex analytic in some neighborhoods of the triple points. If h is small enough, the obtained curve is analytic near all double points. Finally, we perturb the double points by adding to $(f+h)g$ a yet smaller conjugation-invariant section of $\mathcal{O}_{\mathbb{CP}^2}(9)$ whose signs at the double points are chosen so that the real locus of the resulting curve A is the union of three nested ovals with the curve shown in Figure 9(c). If the complex orientations of the cubics are chosen as in Figure 9(a), the perturbation is coherent with them (see Figure 10), and hence the resulting curve of degree 9 is separating and its complex scheme is (2). The non-analytic part of A is close to A_{sing}, hence A is symplectic.

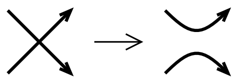

FIGURE 10. A perturbation of transversal intersection according to complex orientations

Note that since (2) is algebraically unrealizable, so is the intermediate nodal curve. This fact however is much easier: it immediately follows from Abel's theorem applied to the divisors cut by any two of the cubic curves on the third one. This observation was the initial hint that (2) is algebraically unrealizable and that Abel's theorem might be used in the proof.

The proof of Theorem 3.1 is based on the Abel Theorem combined with the following result due to Gabard [11]. Let A be a real algebraic curve, i.e., a Riemann surface endowed with an antiholomorphic involution. Let g be the genus of A and r be the number of connected components of $\mathbb{R}A$. Then there exists a real morphism $f : A \to \mathbb{P}^1$ of degree at most $(r + g + 1)/2$ which is *separating*, which means that $f^{-1}(\mathbb{RP}^1) = \mathbb{R}A$.

If one of inequalities (3) does not hold, then, using Poincaré residues, one can construct a real holomorphic 1-form on A which contradicts Abel Theorem applied to the divisor of f (viewed as a meromorphic function on A).

4. Algorithmic recognition of quasipositive braids

As it is pointed out in Section 2.1, pseudoholomorphic realizability of a fiberwise isotopy type is equivalent to the quasipositivity of a certain braids. In general, the algorithmic problem to decide whether a given braid is quasipositive, is open and it seems to be very hard. However, in [P7] it is solved in some particular cases using the Garside Theory (which was founded in [12] and further developed in [3], [6], [7], [13] and in many other papers by different authors).

Let B_n be the group of braids with n strings. Let $e : B_n \to \mathbb{Z}$ be the homomorphism defined by $\sigma_i \mapsto 1$ for each i. Then $e(b)$ is the *algebraic length* of b (called also the *exponent sum* of b).

4.1. Quasipositivity problem for braids with three strings.

The braid group B_2 is isomorphic to $\mathbb{Z}$, thus the recognition of quasipositive braids with two strings is evident. An algorithmic solution of the quasipositivity problem for braids with three strings is given in [P7, §6]. It is as follows. Any braid with three strings can be written in the form

$$b = \Delta^{-p} x_1 \dots x_n, \qquad x_i \in \{\sigma_1, \sigma_2\} \tag{5}$$

where $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

Theorem 4.1. *A braid b given by (5) is quasipositive if and only if some letters can be removed from $x_1 \dots x_n$ so that the resulting word represents the braid Δ^p.*

This theorem provides an algorithm of complexity $O(n^{e(b)+c})$ where c is a constant (1, 2 or 3) depending on a precise definition of the complexity. Notice that in applications to real algebraic or real pseudoholomorphic curves with many ovals (the case which is traditionally considered as most interesting) the algebraic length of the appeared braids is small.

Using a kind of the branch-and-bounds method, this algorithm is improved in [P7, §6]. The resulting algorithm is still exponential in $e(b)$ but the base of exponent is considerably reduced. A **C**-program implementing this algorithm is also presented in [P7, §6].

4.2. Quasipositivity problem for braids of algebraic length 2.

We start this section by introducing some notions from the Garside Theory which are needed to formulate our results. We follow the definitions and notation introduced in [14]. Given elements a, b of a group G, we denote $a^b = b^{-1}ab$, $a^G = \{a^b \mid b \in G\}$, and $a \sim b \Leftrightarrow b \in a^G$.

A *Garside structure* on a group G is a triple $(G, \mathcal{P}, \Delta)$ where $\mathcal{P}$ is a submonoid in G satisfying $\mathcal{P} \cap \mathcal{P}^{-1} = \{1\}$ (called the monoid of *positive elements*) and a special element $\Delta \in \mathcal{P}$ (called the *Garside element*) such that the following properties hold:

- (G1) The partial order $\preccurlyeq$ defined on G by $a \preccurlyeq b \Leftrightarrow a^{-1}b \in \mathcal{P}$ is a lattice order. That is, for every $a, b \in G$ there exist a unique least common multiple $a \vee b$ and a unique greatest common divisor $a \wedge b$ with respect to $\preccurlyeq$.
- (G2) The set $[1, \Delta] = \{a \in G \mid 1 \preccurlyeq a \preccurlyeq \Delta\}$, called the set of *simple elements*, generate G.
- (G3) Conjugation by Δ preserves $\mathcal{P}$. That is, $(X \in \mathcal{P}) \Rightarrow (X^\Delta \in \mathcal{P})$.
- (G4) For all $X \in \mathcal{P} \setminus \{1\}$, one has:

$$\|X\| = \sup\{k \mid \exists a_1, \dots, a_k \in \mathcal{P} \setminus \{1\} \text{ such that } X = a_1 \dots a_k\} < \infty.$$

Similarly to $\preccurlyeq$ we define the order $\succcurlyeq$ by $a \succcurlyeq b \Leftrightarrow ab^{-1} \in \mathcal{P}$. A Garside structure $(G, \mathcal{P}, \Delta)$ is said to be *of finite type* if the set of simple elements $[1, \Delta]$ is finite. An element $a \in \mathcal{P} \setminus \{1\}$ is called an *atom* if $\|a\| = 1$. We denote the set of atoms by $\mathcal{A}$.

The following three notions are introduced in [P7]. A Garside structure $(G, \mathcal{P}, \Delta)$ is called *homogeneous* if for any $X, Y \in \mathcal{P}$ one has $\|XY\| = \|X\| + \|Y\|$. It is called *symmetric* if for any simple elements u, v one has $u \preccurlyeq v \Leftrightarrow v \succcurlyeq u$, and it is called *square free* if there do not exist $U, V \in \mathcal{P}$ and $x \in \mathcal{A}$ such that $Ux^2V \in [1, \Delta]$.

We say that a decomposition $X = \Delta^p \cdot A_1 \cdot \ldots \cdot A_n$, $A_i \in [1, \Delta] \setminus \{1, \Delta\}$, is in *left normal form* if $A_i = \Delta \wedge A_i \dots A_n$ for each i.

Then the *infimum, supremum*, and *canonical length* of X are defined as $\inf X = p$, $\sup X = p + n$, $\ell(X) = n$ respectively. The *initial* and *final factor* of X are $\iota(X) = \Delta^p A_1 \Delta^{-p}$ and $\varphi(X) = A_n$ respectively. The *summit length* and the *super summit set* of X are defined as

$$\ell_s(X) = \min_{Y \in X^G} \ell(Y), \qquad \mathrm{SSS}(X) = \{Y \in X^G \mid \ell(Y) = \ell_s(X)\}$$

respectively. The *cycling* and *cyclic sliding* are the mapping $\mathfrak{s} : G \to G$ and $\mathbf{c} : G \to G$ respectively defined by

$$\mathbf{c}(X) = X^{\iota(X)}, \qquad \mathfrak{s}(X) = X^{\mathfrak{p}(X)}, \qquad \mathfrak{p}(X) = \iota(X) \wedge (\varphi(X)^{-1}\Delta).$$

The *ultra summit set* and the *set of sliding circuits* of X are defined as

$$\mathrm{SC}(X) = \{Y \in \mathrm{SSS}(X) \mid \exists k > 0 \text{ such that } \mathfrak{s}^k(Y) = Y\},$$
$$\mathrm{USS}(X) = \{Y \in \mathrm{SSS}(X) \mid \exists k > 0 \text{ such that } \mathbf{c}^k(Y) = Y\}.$$

It is known (see [14]) that $\mathrm{SC}(X) \subset \mathrm{USS}(X)$. Given $X \in \mathrm{USS}(X)$, the *cycling orbit* of X is defined as $\{\mathbf{c}^k(X) \mid k > 0\}$.

Theorem 4.2. (Theorem 1 in [P7].) *Let $(G, \mathcal{P}, \delta)$ be a symmetric homogeneous square-free Garside structure of finite type with set of atoms $\mathcal{A}$. Let $x, y \in \mathcal{A}$ and $k > 0$, $l \geq 0$ be integers. Suppose that $X \in (x^k)^G (y^l)^G$. Then either $X \in (x_1^k y_1^l)^G$ or any cycling orbit in $\mathrm{SC}(X)$ contains an element whose left normal form is*

$$\delta^{-n} \cdot A_n \cdot \ldots \cdot A_1 \cdot x_1^k \cdot B_1 \cdot \ldots \cdot B_n \cdot y_1^l$$

where $n \geq 1$, $x_1 \in x^G \cap \mathcal{A}$, $y_1 \in y^G \cap \mathcal{A}$, and A_i, B_i are simple elements such that $A_i \delta^{i-1} B_i = \delta^i$.

In particular, this theorem gives an algorithm to decide whether a given braid of algebraic length ≤ 2 is quasipositive. Indeed the Birman-Ko-Lee [3] Garside structure $(B_m, \mathcal{P}, \delta)$ on the braid group B_m satisfies the required properties (i.e., it is symmetric, homogeneous, and square-free). This Garside structure is defined as follows. Let $\sigma_{i,j}$, $1 \leq i < j \leq m$, be the *band generators* $\sigma_{i,j} = a^{-1}\sigma_{j-1}a$, $a = \sigma_{j-1} \ldots \sigma_i$. Then $\mathcal{A} = \{\sigma_{i,j} \mid 1 \leq i < j \leq m\}$, $\mathcal{P}$ is the monoid generated by $\mathcal{A}$, and $\delta = \sigma_{m-1} \ldots \sigma_2 \sigma_1$. The algorithms of computation of all objects involved in Theorem 4.2 can be found in [3].

5. Plane complex curves with small Betti numbers

5.1. Rational cuspidal curves.

In [P4] I study rational cuspidal curves, i.e. rational complex algebraic curves in $\mathbb{CP}^2$ homeomorphic to the 2-sphere. The word ‘cuspidal’ in this context means that all singularities are analytically irreducible (cusps). Using logarithmic Bogomolov-Miyaoka-Yau inequality, Fujita theory of Zariski decompositions on affine surfaces, and computations with resolution graphs (more precisely, with Eisenbud-Neumann splice diagrams), I prove the following results. The *logarithmic Kodaira dimension* $\kappa(V)$ of an affine complex algebraic variety V is the $(K_X + D)$-dimension of X where D is a simple normal crossing divisor on a smooth compact variety X such that $X \setminus D = V$.

Let $\alpha = (3+\sqrt{5})/2$. Let $\phi_0, \phi_1, \ldots$ be Fibonacci numbers indexed so that $\phi_0 = 0$, $\phi_1 = 1$, $\phi_{k+2} = \phi_k + \phi_{k+1}$.

Theorem 5.1. (Theorems A and B in [P4].) *Let C be a rational cuspidal curve of degree d in $\mathbb{C}P^2$. Let m be the maximal multiplicity of its cusps. Let $\bar\kappa$ be the log-Kodaira dimension of $\mathbb{C}P^2 \setminus C$. Then:*

(1) $d < \alpha(m+1) + 1/\sqrt{5}$;
(2) *if* $\bar\kappa = -\infty$, *then* $d < \alpha m$;
(3) $\bar\kappa \neq 0$;
(4) *if* $\bar\kappa = 2$, *then* $d < \alpha(m+1) - 1/\sqrt{5}$.

In the same paper (Theorem C in [P4]) I have also shown that these estimates are sharp in the sense that there do not exist constants α' and β' such that $\alpha' < \alpha$ and the inequality $d \le \alpha' m + \beta'$ holds for any rational cuspidal curve. Namely, for any $j > 0$, $j \not\equiv 2 \mod 4$, I constructed a rational cuspidal curve C_j of degree $d_j = \phi_{j+2}$ which has a single cusp of multiplicity $m_j = \phi_j$, thus $\lim_{j\to\infty} d_j/m_j = \alpha$.

5.2. Lin – Zaidenberg Conjecture.

Lin and Zaidenberg [24, 25] (see also [1; §5]) asked the following questions.

(Q1) *Does there exist a connection between the topology of an irreducible plane affine complex algebraic curve and the number of its irreducible singularities?* (Q2) *Is it true, for example, that the number of irreducible singularities of such a curve A does not exceed $1 + 2b_1(A)$ where $b_1(A)$ is the first Betti number of A?*

Conjecturally, the answer to the both questions is positive. The first and the most fascinating case of this conjecture was proven by Lin and Zaidenberg themselves [45]: if $b_1(A) = 0$, then an automorphism of $\mathbb{C}^2$ transforms A into $x^p = y^q$, in particular, A has at most one singular point. Borodzik and Żołądek [4] proved that the answer to Question (Q2) is positive in one more particular case. Namely, if A is homeomorphic to an annulus, then A has at most three singular points.

If we pass from A to its closure in $\mathbb{CP}^2$, then the number of singular points may only increase whereas the first Betti number may only decrease. Thus a positive answer to (Q1) follows from the analogous conjecture for plane complex projective curves. A particular case of the projective conjecture was proven in [46]: if a projectively rigid curve in $\mathbb{CP}^2$ is homeomorphic to a sphere, then it has at most 9 singular points. Then Tono [36] proved a much stronger result: if a curve in $\mathbb{CP}^2$ is homeomorphic to a Riemann surface of genus g, then it has no more than $(21g + 17)/2$ singular points (thus no more than 8 when $g = 0$).

In my paper [P8] I extended Tono's arguments to the case of an arbitrary plane projective curve and obtained a proof of the projective analog of the conjecture and hence, a positive answer to Question (Q1).

Let us give precise statements. Let C be an algebraic curve in $\mathbb{CP}^2$. A singular point of C is called a *cusp* if C has a single local analytic branch at it. Let s be the number of all singular points of C and c the number of cusps. Let $b_i = b_i(C)$ be the i-th Betti number of C. So, b_2 is the number of irreducible components. Let $g = g(C)$ be the total genus of C, i.e., the sum of the genera of the normalizations of all the irreducible components.

Theorem 5.2. *If* $\bar\kappa(\mathbb{CP}^2 \setminus C) = 2$ (by [41] this is so, for example, when one of irreducible components of C has ≥ 3 singular points), *then* $c \le \frac{9}{2}b_1 + \frac{3}{2}g - 6b_2 + \frac{29}{2}$ *and* $s \le \frac{11}{2}b_1 - \frac{1}{2}g - 5b_2 + \frac{27}{2}$.

Corollary 5.3. *If C is irreducible, then* $c \le \frac{9}{2}b_1 + \frac{3}{2}g + \frac{17}{2} \le \frac{21}{4}b_1 + \frac{17}{2}$ *and* $s \le \frac{11}{2}b_1 - \frac{1}{2}g + \frac{17}{2} \le \frac{11}{2}b_1 + \frac{17}{2}$.

Let $C^{\rm aff}$ be the intersection of C with some fixed affine chart and let $b_i^{\rm aff} = b_i(C^{\rm aff})$. We denote the number of singular points, the number of cusps, and the number of points at infinity of $C^{\rm aff}$ by $s^{\rm aff}$, $c^{\rm aff}$, and p respectively.

Corollary 5.4. *If* $\bar\kappa(\mathbb{CP}^2 \setminus C) = 2$, *then* $c^{\rm aff} \le \frac{9}{2}(b_1^{\rm aff} - b_0^{\rm aff} - p) + \frac{3}{2}(g - b_2) + 19$ *and* $s^{\rm aff} \le \frac{11}{2}(b_1^{\rm aff} - b_0^{\rm aff} - p) + \frac{1}{2}(b_2 - g) + 19$.

Corollary 5.5. *If C is irreducible, then* $c^{\rm aff} \le \frac{9}{2}(b_1^{\rm aff} - p) + \frac{3}{2}g + 13 \le \frac{9}{2}b_1^{\rm aff} + \frac{3}{2}g + \frac{17}{2}$ *and* $s^{\rm aff} \le \frac{11}{2}(b_1^{\rm aff} - p) - \frac{1}{2}g + 14 \le \frac{11}{2}b_1^{\rm aff} - \frac{1}{2}g + \frac{17}{2}$.

In the same paper I observed that [36] easily implies Zaidenberg's conjecture about the finiteness of the number of graphs (considered up to homeomorphism) realized as the dual graph at infinity of a minimal compactification of a $\mathbb{Q}$-acyclic affine algebraic surface of general type. More precisely, the following holds.

Let $D = D_1 + \dots + D_n$ be a reduced curve with simple normal crossings on a smooth algebraic surface V and let Γ be the dual graph of D. We set $\beta(D_i) = D_i(D - D_i)$ (the degree of the corresponding vertex of Γ). If $\beta(D_i) = 1$, we say that D_i is a *tip* of D. We assume that D does not contain any rational (-1)-curve D_i with $\beta(D_i) \le 2$. Zaidenberg [44; p. 16] conjectured that *only finite number of pairwise non-homeomorphic graphs* Γ *can be obtained in this way under the condition that* $\bar\kappa(V \setminus D) = 2$ *and* $b_i(V \setminus D) = 0$ *for* $i > 0$.

Proposition 5.6. (Remark 3 in [P8].) *The number of tips of D is at most 17, hence Zaidenberg's conjecture holds true.*

Papers constituting the thesis

P1. S. Yu. Orevkov, *Link theory and oval arrangements of real algebraic curves*, Topology **38** (1999), 779–810.

P2. S. Yu. Orevkov, *Classification of flexible M-curves of degree* 8 *up to isotopy*, GAFA – Geometric and Functional Analysis **12** (2002), no. 4, 723–755.

P3. S. Yu. Orevkov, E. I. Shustin, *Flexible, algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach*, J. fur die Reine und Angew. Math. **551** (2002), 145–172.

P4. S. Yu. Orevkov, *On rational cuspidal curves I. Sharp estimate for the degree via multiplicities*, Math. Annalen **324** (2002), 657–673.

P5. S. Yu. Orevkov, *Riemann existence theorem and construction of real algebraic curves*, Annales de la Faculté des Sciences de Toulouse. Mathématiques, (6) **12** (2003), no. 4, 517–531.

P6. S. Yu. Orevkov, E. I. Shustin, *Pseudoholomorphic algebraically unrealizable curves*, Moscow Math. J. **3** (2003), 1053–1083.

P7. S. Yu. Orevkov, *Algorithmic recognition of quasipositive braids of algebraic length two*, J. of Algebra **423** (2015), 1080–1108.

P8. S. Yu. Orevkov, *Remark on Tono's theorem about cuspidal curves*, Math. Nachrichten **290** (2017), no. 17–18, 2992–2994.

P9. S. Fiedler-LeTouzé, S. Orevkov, E. Shustin, *Corrigendum to the paper "A flexible affine M-sextic which is algebraically unrealizable"*, J. Alg. Geom. **29** (2020), 109–121.

P10. S. Yu. Orevkov, *Algebraically unrealizable complex orientations of plane real pseudoholomorphic curves*, GAFA – Geom. and Funct. Anal. **31** (2021), 930–947.

References

[1] S. S. Abhyankar and A. Sathaye, *Uniqueness of plane embeddings of special curves*, Proc. Amer. Math. Soc. **124** (1996), 1061–1069.

[2] V. I. Arnold, *The arrangement of the ovals of real plane algebraic curves, involutions of four-dimensional manifolds and the arithmetics of integral quadratic forms*, Funk. anal. i prilozh. **5** (1971), no. 3, 1–9 (Russian); English translation, Funct. Anal. Appl. **5** (1971), 169–175.

[3] J. Birman, K.-H. Ko, S.-J. Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. **139** (1998), 322–353.

[4] M. Borodzik and H. Żołądek, *Number of singular points of an annulus in* $\mathbb{C}^2$, Ann. Inst. Fourier **61** (2011), 1539–1555.

[5] B. Chevallier, *Four M-curves of degree* 8, Funk. anal. i prilozh. **36** (2002), no. 1, 90–93 (Russian); English translation, Funct. Anal. Appl. **36** (2002), 76–78.

[6] P. Dehornoy, *Groupes de Garside*, Ann. Sci. Ecole Norm. Sup. **35** (2002), 267–306.

[7] P. Dehornoy, L. Paris, *Gaussian groups and Garside groups, two generalizations of Artin Groups*, Proc. London Math. Soc. (3) **79** (1999), 569–604.

[8] S. Fiedler-LeTouzé, *Cubics as tools to study the topology of M-curves of degree* 9 *in* $\mathbb{R}P^2$, J. London Math. Soc. (2) **66** (2002), 86–100.

[9] S. Fiedler-LeTouzé, *Pencils of cubics and algebraic curves in the real projective plane*, CRC Press, Boca Raton, FL, 2019.

[10] S. Fiedler-LeTouzé, S. Yu. Orevkov, *A flexible affine M-sextic which is algebraically unrealizable*, J. of Algebraic Geometry **11** (2002), 293–310.

[11] A. Gabard, *Sur la représentation conforme des surfaces de Riemann à bord et une caracérisation des courbes séparantes*, Comment. Math. Helv. **81** (2006), no. 4, 945–964.

[12] F. A. Garside, *The braid group and other groups*, Quart. J. Math. **20** (1969), 235–254.

[13] V. Gebhardt, *A new approach to the conjugacy problem in Garside groups*, J. of Algebra **292** (2005), 282–302.

[14] V. Gebhardt, J. González-Meneses, *The cyclic sliding operation in Garside groups*, Math. Z. **265** (2010), 85–114.

[15] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[16] D.A. Gudkov, *Construction of a new series of M-curves*, Doklady AN SSSR **200** (1971), no. 6, 1289–1272. (Russian)

[17] D. A. Gudkov, G. A. Utkin, *Amer. Math. Soc. Transl. Ser. 2, vol. 112, Amer. Math. Soc.*, 1978.

[18] A. Harnack, *Über die Vielfaltigkeit der ebenen algebraischen Kurven*, Math. Ann. **10** (1876), 189–199.

[19] D. Hilbert, *Über die reellen Züge der algebraischen Kurven*, Math. Ann. **38** (1891), 115–137.

[20] F. Klein, *Ueber eine neue Art von Riemann'schen Flächen*, Math. Ann. **10** (1876), 398–416.

[21] A. B. Korchagin, *New possibilities in Brusotti's method for the constructions of M-curves of degrees* ≥ 8, in: Methods of Qualitative Theory of Differential Equations, Gorki Univ. Press, Gorki, 1978, pp. 149–159. (Russian)

[22] A. B. Korchagin, *The new M-curves of degrees* 8 *and* 9, Soviet Math. Doklady **39** (1989), 569–572.

[23] A. B. Korchagin, E. I. Shustin, *Affine curves of degree 6 and smoothing of non-degenerate six-fold singular points*, Izv. AN SSSR, ser. mat. **52** (1988), no. 6, 1181–1199 (Russian); English transl., Math. USSR-Izvestia **33** (1989), 501–520.

[24] V. Ya. Lin and M. G. Zaidenberg, *On the number of singular points of a plane affine algebraic curve*, in: Linear and complex analysis problem book, Lect. Notes Math., vol. 1043, Springer, 1984, pp. 662–663.

[25] V. Ya. Lin and M. G. Zaidenberg, *On the number of singular points of a plane affine algebraic curve*, in: Linear and complex analysis problem book, Lect. Notes Math., vol. 1574, Springer, 1994, p. 479.

[26] N. M. Mishachev, *Complex orientations of plane M-curves of odd degree*, Funk. anal. i prilozh. **9** (1975), no. 4, 77–78 (Russian); English transl., Funct. Anal. Appl. vol 9 (1975), 342–343.

[27] S. Yu. Orevkov, *A new affine M-sextic*, Funk. anal. i prilozh. **32** (1998), no. 2, 91–94 (Russian); English transl., Funct. Anal. and Appl. **32** (1998), 141–143.

[28] S. Yu. Orevkov, *New M-Curve of Degree 8*, Funk. anal. i prilozh. **36** (2002), no. 3, 90–93 (Russian); English transl., Funct. Anal. and Appl. **36** (2002), 247–249.

[29] S. Yu. Orevkov, *Plane real algebraic curves of odd degree with a deep nest*, J. of Knot Theory and Ramifications **14** (2005), 497–522.

[30] S. Yu. Orevkov, *Complex orientation formulas for M-curves of degree $4d+1$ with 4 nests Annales de la Faculté des Sciences de Toulouse. Mathématiques, (6)* **19** (2010), 13–26.

[31] I. G. Petrovsky, *On the topology of real plane algebraic curves*, Ann. Math. **39** (1938), 187–209.

[32] V. A. Rohlin, *Congruences modulo 16 in Hilbert's sixteenth problem. II*, Funk. anal. i prilozh. **7** (1973), no. 2, 91–92 (Russian); English transl., Funct. Anal. Appl. **7** (1973), 163–164.

[33] V. A. Rokhlin, *Complex topological characteristics of real algebraic curves*, Usp. mat. nauk **33** (1978), no. 5, 77–89 (Russian); English transl., Russian Math. Surv. **33** (1978), no. 5, 85–98.

[34] E.I. Shustin, *New M-curve of degree* 8, Mat. zametki, **42** (1987), no. 2, 180–186 (Russian); English transl., Math. Notes of Acad. Sci. USSR **42** (1987), 606–610.

[35] E. Shustin, *New M- and $(M-1)$-curves of degree* 8, In: Rokhlin seminar/ O. Viro et al, eds., Lecture Notes in Math. vol 1346, Springer Verlag, 1988, pp. 487–493.

[36] K. Tono, *On the number of the cusps of cuspidal plane curves*, Math. Nachr. **278** (2005), 216–221.

[37] O. Ya. Viro, *Curves of degree 7, curves of degree 8, and the Ragsdale conjecture*, Doklady AN SSSR **254** (1980), no. 6, 1306–1310; English translation, Soviet Math. Doklady **22** (1980), 566–570.

[38] O. Ya. Viro, *Real plane curves of degrees 7 and 8: New prohibitions*, Izv. AN SSSR, ser. mat. **47** (1983), no. 5, 1135–1150 (Russian); English transl. Math. USSR, Izv. **23** (1984), 409–422.

[39] O. Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Uspekhi mat. nauk **41** (1986), no. 3(249), 45–67 (Russian); English transl., Russian Math. Surveys **41** (1986), 55–82.

[40] O. Ya. Viro, *Real algebraic plane curves: constructions with controlled topology*, Algebra i analiz **1** (1989), no. 5, 1–73 (Russian); English transl., Leningrad Math. J. **1** (1990), 1059–1134.

[41] I. Wakabayashi, *On the logarithmic Kodaira dimension of the complement of a curve in P^2*, Proc. Japan Acad. Ser A **54** (1978), 157–162.

[42] J.-Y. Welschinger, *J-courbes réelles à nids profonds sur les surfaces règlèes*, Zapiski nauchn. seminarov POMI **267** (2000), 88–118; . Reprinted in:, J. Math. Sci. (N. S.) **113** (2003), no. 6, 777–794.

[43] A. Wiman, *Über die reellen Züge der ebenen algebraischen Kurven*, Math. Ann. **90** (1923), 222–228.

[44] M. Zaidenberg, *in: Open problems on open algebraic varieties*, arXiv:alg-geom/9506006.

[45] M. G. Zaidenberg and V. Ya. Lin, *An irreducible, simply connected algebraic curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve*, Dokl. AN SSSR **271:5** (1983), 1048–1052 (Russian); English Transl., Soviet Math., Doklady **28** (1983), 200–204.

[46] M. G. Zaidenberg and S. Yu. Orevkov, *On rigid rational cuspidal plane curves*, Uspekhi mat. nauk **51:1** (1996), 149–150 (Russian); English Transl., Russian Math. Surveys **51** (1996), 179–180.