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THE OPTIMAL DESIGN OF ELIMINATION TOURNAMENTS WITH A SUPERSTAR

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The optimal design of elimination tournaments with a superstar *

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Abstract

We study single- and double-elimination tournaments with heterogeneous players of two types: regular players and a superstar. Players choose efforts in each match with linear costs, winning with a probability calculated with the Tullock success function. We consider several designer maximization problems: total efforts, probability of winning the strongest player, and a weighted composed function. We show that a double-elimination tournament is less profitable in most cases, except when the tournament organizer is concerned about the probability that the superstar wins the tournament.

JEL Classification: C72, D47, Z20.

Keywords: single-elimination tournament, double-elimination tournament, tournament design, heterogeneous players, superstar.

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1 Introduction

The sport and e-sport industries are rapidly developing and receiving a large amount of resources, both human and capital. Tournaments are an important part of these industries. Since this is the huge industry, tournament organizers aim to find a mechanism to maximize the returns from the event.

One common way to enlarge the entertainment and the performance of the tournament or league is to invite a superstar. These widely-known players attract numerous fans to their matches or the club or league's social media pages. At the same time, the number of superstars in a particular sport or a particular league is limited attracting superstars can be expensive. For example, Cristiano Ronaldo completed a transfer to Al Nassr in 2023. He is now playing in the Saudi Pro League. This deal is valued at 200 million euros per season[1]. Naturally, this transfer has led to the explosive growth of Al Nassr fans[2] which is expected to be converted into fans' spending.

The other powerful instrument for the tournament organization is playing with its format. For example, the cyber-sport discipline Dota 2 has demonstrated several reorganizations during the last few years. First, in 2017, the main championship *The International* was held not with 16 teams, as it had been before, but with 18[3]. Second, in 2016, together with the main Dota 2 tournament *The International*, three smaller tournaments called *majors* were also organized. In each of them, teams receive special points in addition to the cash prize, which helps them qualify for the main tournament. The first two tournaments were held in a round-robin format. In the third *Manila Major*, four teams were additionally invited, and they played in the group stage and then in a double-elimination format for winners.

The examples above highlight the importance of a fundamental preliminary analysis of the consequences stemming from the institutional changes. Two major factors are to be taken into account: who participates in a tournament, and what incentives for players are created. The presence of an extra-strong player, a superstar, is modeled

[1]Optus Sport. Official: Cristiano Ronaldo completes move to Saudi Arabian side Al Nassr. https://sport.optus.com.au/news/transfers/os51755/transfer-news-cristiano-ronaldo-al-nassr-salary-contract-fee-details-latest. (May, 2023)

[2]After the announcement of the transition, the number of subscribers on the club's Instagram increased in three days from 853k to above 10 million. https://www.espn.co.uk/football/story/_/id/38680916/cristiano-ronaldo-social-media-power-boosts-al-nassr-instagram-10m-followers (November, 2023)

[3]In 2016, the prize pool was $20.77 million, and in 2017 it was already $24.78. It is noteworthy that for the main official tournament, the prize fund for the most part (more than 95%) is collected by the spectators of the tournament (community) by buying in-game items. The tournament has been held since 2011.

as the different measure of prize, while the linear costs for efforts are the same. We introduce the setting with a superstar and other players being equal, but the model produces a secondary heterogeneity coming from the seeding. Obviously, the weak player who is seeded in a pair with another weak is in the more advantageous situation than the one who is seeded with a superstar.

The incentives for participants are also created by the rules of a tournament. We consider two knockout formats: with a single-elimination (after the first defeat) and with a double-elimination (where a player gets a second chance in the lower bracket). On the one hand, the single-elimination tournament motivates players to put in high efforts since the defeat is fatal. However, due to the noise in the success probability function, even the superstar may lose. Then, on the other hand, the double-elimination tournament is less subject to an accidental defeat and provides a higher *fairity*, since a double defeat looks almost impossible for a really strong player. But these two formats are hard to compare in terms of their performance because of the very different number of rounds under each.

In our paper, we combine all mentioned factors into the unique model of a tournament with 4 participants, including one strong player. Two formats of tournament, with a single-elimination and a double-elimination, are solved, and the optimal levels of efforts are determined. The tournaments are compared in their performance (total efforts), normalized efforts per match, probability of winning the tournament by the superstar, and finally, the new aggregate criterion weighted benefits from the performance and various costs of organization, i.e. fixed costs of the whole tournament, costs per match and costs of attracting the superstar with a given level. This setting is connected to several directions of tournament studies.

Prize distribution and performance in tournaments

The seminal work by Lazear and Rosen (1981), considering only one match between two players, shows that the total efforts of the players increase with the growth of the prize fund. Moreover, they prove that not the absolute value of the prize, but the difference between prizes generates this result. The allocation of the whole prize fund to the first place was proven to be optimal also for the round-robin tournaments (Krumer, Megidish and Sela, 2017) with similar players as well as for knockout tournaments. Moldovanu and Sela (2001) show that it is a general optimal allocation for the model with linear costs to choose only the first-place prize. We use this insight and consider the tournaments with the first-place price only.

Another related problem of tournament design concerns the estimation of its efficiency. Researchers often choose total efforts as the objective function (Szymanski, 2003). The author reviews articles on competitions in sports and also combines different approaches to team and individual sports.

As an alternative, the tournament designer may care about the competitiveness or fairness of the tournament, the probability of the strongest player winning, or the probability of a final between the two strongest players (Groh et al., 2012). Maximizing each of these indicators entails the problem of the optimal tournament design. Groh with coauthors (2012) focuses on comparing various seedings at the single-elimination tournament for 4 heterogeneous players. They develop how the starting seeding affects each of these indicators and which seeding maximizes different organizer's objective functions. We borrow the findings of Groh et al. (2012) and adopt formulas to the setting with the unique superstar in the single-elimination tournament. In our setting all seedings are equivalent, but the players inside the seeding are not, and thus, having the same valuation of the final prize, put different efforts in equilibrium.

Comparison of tournament formats

While the single-elimination tournaments are more elaborated theoretically, their alternative with the elimination after a second defeat is analyzed poorly. Huang (2016) considers 4-player double-elimination knockout tournaments with homogeneous participants. He compares the two formats (SET and DET) in terms of the sum of all-round efforts, and concludes that the double-elimination tournament has a higher sum. In our paper, we extend this conclusion and show that, in a double-elimination tournament with heterogeneous players, the total efforts are higher. However, these two formats need a different number of matches for the same number of players, which is associated with higher tournament costs for the double-elimination format.

Deck and Kimbrough (2015) explore single- and double-elimination tournaments using the all-pay auction as a probability of winning the match. They demonstrate that only the final rounds, in which players put in non-zero efforts, are significant. With the help of experiments, they confirm their results and show that reality is not far from the theory. In our model with the Tullock success function, the importance of the final round for players also remains, and the efforts there are the largest. Additionally, we also show that the players put in non-zero efforts in all previous rounds.

Chen, Jiang and Wang (2021) explore the influence of “psychological momentum” in single- and double-elimination tournaments with a Tullock success function. Psychological momentum is the state of the player after the previous game; it can be positive after a victory and negative after a defeat. The state can be expressed in the degree of motivation for the next game, the will to win, etc. They show that, for negative and positive momentum, the average amount of efforts per match is greater in a single-elimination tournament. In our work, we confirm this conclusion for players without psychological momentum.

Influence of superstars

Usually, the presence of a superstar is determined by a large dispersion in incomes in the labor market. Such a disproportion arises due to absolute substitutability and increasing returns on talent (Rosen, 1981). In the entertainment industry, regardless of the number of consumers, the cost of a superstar is about the same. Superstars have been shown in the sports industry to increase revenue (Hausman and Leonard, 1997). At the same time, both the attractiveness of a superstar (Mullin and Dunn, 2002) and her performance (Berri, Scmidt, and Brook, 2004; Berri and Schmidt, 2006) can affect profitability. Brandes, Franck and Nüesch (2008) show that superstars attract fans to the stadiums of their players in German soccer, while local teams attract local fans with their popularity.

Brief presentation of results

In our analysis, we deduce that, in a double-elimination tournament with 4 players, the total efforts are greater than in a single-elimination tournament if the superstar's strength is not too large. The main reason is the greater number of matches, which compensates the lower per-match efforts. When one looks at the per-match efforts, players in the single-elimination tournament demonstrate considerably greater performance.

An interesting observation is that if the superstar is overqualified, i.e. her valuation of prize is more than 2.1 times greater than that of regular players, then even the total efforts in double-elimination tournament are lower. This dramatic difference in strengths, together with the rules giving the superstar a second chance, makes her victory almost deterministic, such that the further growth of efforts doesn't make sense.

The new criterion of tournament efficiency includes the cost of attracting the superstar and is considered as the zero-stage optimization problem of choosing the optimal level of superstar strength for a given format. Numerical examples demonstrate that the distribution of weights across different costs and the view of attracting function influence the optimal choice of knockout tournament and the optimal qualification of the superstar. Thus, inviting a too qualified superstar is never optimal for any format, but the exact optimal level of the superstar depends on the format.

Since the double-elimination tournament requires more matches than the single-elimination one for 4 players, we study an alternative tournament with a comparable number of matches: with 8 players and single-elimination. We show that this may be a good alternative only when the designer is concerned only about maximizing total efforts. Otherwise, the formats with 4 players are more profitable under all other objective functions.

Summarizing, our study elaborates a comprehensive analysis of two formats of knockout tournaments and provides a base for the reasonable choice of the optimal format for the organizer. Different criteria make the result useful for organizers with different objectives and various "brightness" of the superstar in the tournament. We believe that even small improvement of the performance, because of the large budgets on sports events, adds the valuable contribution into the economics of tournaments.

Sections 2 and 3 present the equilibrium analysis of single- and double-elimination knockout tournaments with 4 players. Section 4 deals with various designer objective functions and compares two formats. Section 5 considers the option of attracting more participants. In section 6 we briefly discuss the potential application of this kind of results.

2 The model of single-elimination tournament

Consider a single-elimination tournament (SET) for 4 players (Figure 1). There are 3 matches in a 4-player tournament. After the first defeat, the player is out. The player who loses in the final takes second place. Players who are eliminated in the first rounds take 3-4 places. Sometimes the organizer of the tournament arranges a match for 3rd place, but here we do not allow this.

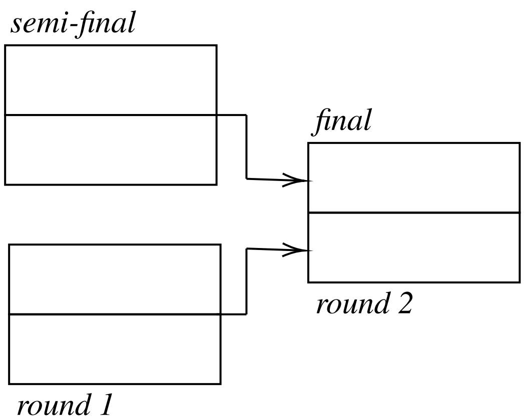


Figure 1 – Tournament structure for single-elimination tournament for 4 players

Remark 1 (Timing). Matches during the tournament can take place simultaneously or sequentially. Equal matches can be played at the same time. For example, semi-final or quarter-final matches in which the number of matches is strictly greater than 1, i.e. at least 4 players participate in this part of the tournament. First, we will consider classical tournaments (see Huang, 2016), in which the matches take place simultaneously if there are at least 2 matches in this part of the tournament. Argumentation of Huang (2016) is based on the players' homogeneity, which leads to the fact that the order of the matches doesn't matter. Simultaneous matches can also be considered in terms of disclosure. That is, matches can be played sequentially, but the players do not know which opponent they can hit or do not have time to change the equilibrium level of efforts when they find out. In this case, the matches are numbered according to the scheme from top to bottom and from right to left during the tournament, that is, first the matches within one round are numbered, and then the matches of the next round are numbered. The scheduling issue with sequential matches will be dealt with separately at the end of this section.

We consider players who are heterogeneous in their abilities or talents. Their difference is modeled by different prize evaluations. Here we have the first place prize only, then without loss of generality we can rearrange prize evaluation in the following way: $V_1 \geq V_2 \geq ... \geq V_n$, where player 1 is the strongest while player n is the weakest. V_i is referred as the *strength* of player i.

There are two types of players in the tournament:

- A *superstar* is a player with the strength $V_1 = \alpha > 1$, he is unique.
- A *regular player* is a player with the strength $V_i = 1$, $i =$ 2,3,4. All players except the superstar are regular, they are weaker.

Note that the setting also allows considering $\alpha \leq 1$, and this situation may be interpreted as the presence of one weak player, or newbie.

A single match organization

Consider a match between player i and player j. Each player simultaneously and independently chooses an efforts level. Let's denote by e_{it} the level of efforts of the player i against player j in round t. Let P_{it} be the probability of winning match t by player i against player j that is represented by the Tullock success function:

$$P_{it} = \frac{e_{it}}{e_{it} + e_{jt}},$$

where reciprocally e_{jt} is the level of efforts that player j exerts against player i.

Efforts generate costs for players. Assume that the costs are represented by a linear function $c(e) = e$. Finally, player i solves the following maximization problem in match t:

$$\max_{e_{it}} V_{it} = \frac{e_{it}}{e_{it} + e_{jt}} V^*_{it'} + \frac{e_{jt}}{e_{it} + e_{jt}} V^*_{it''} - e_{it} \tag{1}$$

where $V^*_{it'}$ and $V^*_{it''}$ is the maximum expected value that player i can get if he wins or loses the match, respectively. From Chen, Jiang and Wang (2021), one learns the equilibrium efforts.

Proposition 1 (Chen, Jiang and Wang (2021)). The equilibrium efforts in match t between players i and j who simultaneously and independently choose efforts in a tournament with the Tullock success function and the linear cost function are given by:

$$\begin{cases} e_{it}^{*} = \dfrac{V_{jt'}^{*} - V_{jt''}^{*}}{\left(1 + \frac{V_{jt'}^{*} - V_{jt''}^{*}}{V_{it'}^{*} - V_{it''}^{*}}\right)^2} \\ e_{jt}^{*} = \dfrac{V_{it'}^{*} - V_{it''}^{*}}{\left(1 + \frac{V_{it'}^{*} - V_{it''}^{*}}{V_{jt'}^{*} - V_{jt''}^{*}}\right)^2} \end{cases} \tag{2}$$

This proposition holds also for asymmetric players, with $V_{it'}^{*}$ may not be equal $V_{jt'}^{*}$.

3-matches SET

The multistage model of the entire tournament is solved by the backward induction. The solution begins with the single final match. After this, we can re-estimate the expected payoff for the players in the previous step accounting for the expected (not guaranteed) prize at the final, and recursively solve the entire tournament.

Final

In the final, two types of situations can be met superstar vs regular and regular vs regular. The table 1 shows the equilibrium parameters for different types of finals. For all $\alpha > 1$, the total equilibrium efforts are greater for a superstar-versus-regular final. In matches where the total strength of the players is higher, they spend more efforts, which produces a more spectacular final.

	superstar (i) vs regular (j)	regular vs regular
e_i	$\frac{\alpha^2}{(1+\alpha)^2}$	$\frac{1}{4}$
e_j	$\frac{\alpha}{(1+\alpha)^2}$	$\frac{1}{4}$
TE[a]	$\frac{\alpha}{(1+\alpha)}$	$\frac{1}{2}$
P_i	$\frac{\alpha}{(1+\alpha)}$	$\frac{1}{2}$
V_i	$\frac{\alpha^3}{(1+\alpha)^2}$	$\frac{1}{4}$
V_j	$\frac{1}{(1+\alpha)^2}$	$\frac{1}{4}$

Table 1 – Single-elimination tournament (4 players): final

[a]Total efforts (TE) is the sum of all equilibrium efforts in the match or the tournament.

One can observe that the stronger player chooses α times more efforts. This pattern is always traced, the ratio of equilibrium efforts is equal to the ratio of strength levels. In particular, it immediately follows from Proposition 1, that

$$\frac{e^*_{it}}{e^*_{jt}} = \frac{V^*_{it'} - V^*_{it''}}{V^*_{jt'} - V^*_{jt''}} \tag{3}$$

The next step is to consider the semi-finals for different types of players. Assume that the semi-finals take place simultaneously. This adds a new, *endogenous*, source of heterogeneity among regular players depending on the tournament seeding: obviously, the regular player who has to play with the superstar in the semifinal is in a worse position in comparison with two other regular players.

Semifinal with a superstar

From the final, we learn two facts: (1) if the superstar wins, she will face a regular player and will get $\frac{\alpha^3}{(1+\alpha)^2}$ in expectation. (2) if the regular player wins, she will face the other regular player and will get $\frac{1}{4}$ in expectation. Because the expected gains from the final have changed and do not coincide with the prize for the first place, we may redefine the strengths of the players in the semifinal. Thus, we deduce that with the growth of α, the relation of strengths between the superstar and the regular opponent grows as $\frac{4\alpha^3}{(1+\alpha)^2}$. Figure 2 illustrates that the difference in strength becomes higher in the semifinal compared to the final.

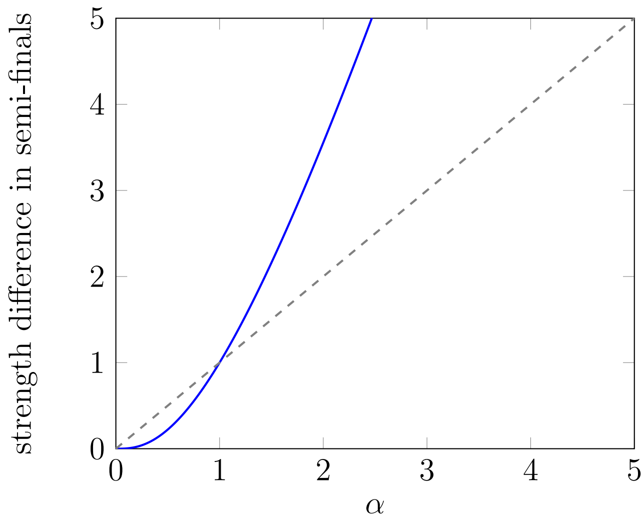


Figure 2 – The difference in strength in the semi-final of SET in the match between a regular player and the superstar

Remark 2. The reverse way to represent the same idea is to claim that players become more close in strength at the end of the tournament. For example, if player X is twice as good as player Y at the beginning of SET, it means that player X is only 1.44 times better at the end of the tournament. So, this may explain why matches become more spectacular towards the end of a tournament.

Denote $\frac{4\alpha^3}{(1+\alpha)^2} = \beta$. Then the equilibrium efforts are given by

$$e_s = \frac{\beta^2}{4(1+\beta)^2},$$
$$e_r = \frac{\beta}{4(1+\beta)^2},$$

where index s means the superstar, while r means the regular player. The winning probability for the superstar in this semi-final is equal to

$$P_s = \frac{\beta}{1+\beta}.$$

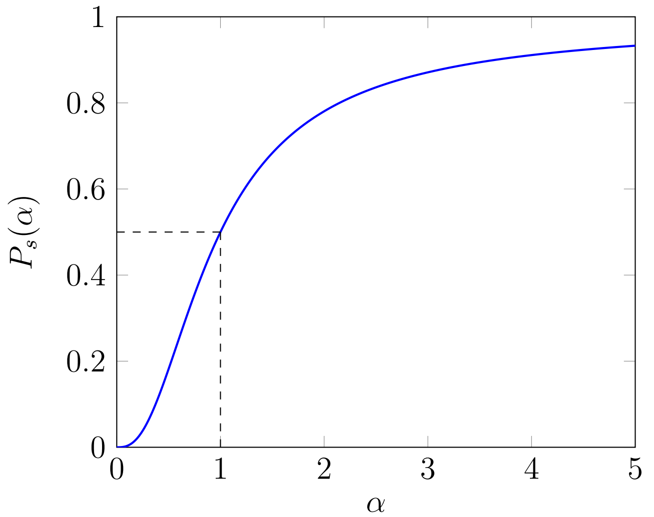


Figure 3 – The winning probability of the superstar in the final of SET

The probability P_s grows quite quickly, as it is shown in Figure 3. But at the same time, it cannot be said that the chance of winning is 1 even for a difference in strength equal to 4, although with increasing α it tends to 1. For example, for a difference in strength equal to 2 or 1.44, the probability of winning for a superstar is 0.78 or 0.67, respectively. This means that the superstar has a fairly high (more than 10%) chance of losing in the first round. Also, if we look at the overall odds of winning a tournament, it will be even smaller, around 0.52 and 0.45 respectively. These values are still higher than 25% (random distribution of the prize), but in the first case it is almost 50%, and in the second it is even less.

Semifinal without a superstar

Both participants of this semifinal face the superstar at the final with the probability $\frac{\beta}{1+\beta}$. The expected prize for each of them is

$$W := \frac{\beta}{1+\beta} \cdot \frac{1}{(1+\alpha)^2} + \frac{1}{4}\left(1 - \frac{\beta}{1+\beta}\right), \tag{4}$$

where $\beta = \frac{4\alpha^3}{(1+\alpha)^2}$

Therefore, we obtain the equilibrium efforts $e_i = e_j = \frac{W}{4}$.

Equilibrium performance in SET

As the initial difference in strengths α increases, the probability of winning the entire tournament for the superstar increases, while for all other players it decreases. At the same time, for the player who was unlucky to be seeded in the semifinal with the superstar, this probability decreases faster. This is summarized in the Table 2.

α	superstar	regular seeded with superstar	regular — with regular
0.9	0.21	0.28	0.26
1	0.25	0.25	0.250
1.44	0.45	0.17	0.194
2	0.52	0.11	0.185

Table 2 – Probability of winning SET for different types of players.

Therefore, the non-random seeding of players creates the possibility of manipulating the probability of winning for the participants. The tournament organizer can worsen the conditions for the player whom she likes less. This can lead to the dissatisfaction of both players and spectators.

The total efforts increase monotonically with the growth of the superstar's strength, while the growth rate slows down (see Figure 4). This means that it is profitable to invite a superstar to the tournament, but as the strength of the superstar increases, the returns decrease.

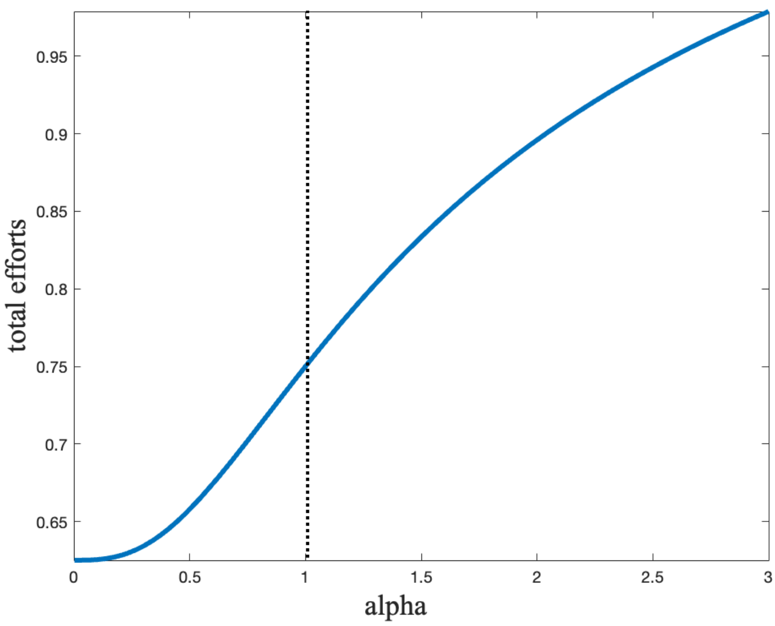


Figure 4 – Total efforts in SET

This motivates us to consider the normalized total efforts, which are the total efforts divided by the sum of all strengths $3 + \alpha$. Figure 5 demonstrates that the normalized total efforts decrease fast starting from $\alpha \approx 1.5$. Therefore, there exists an optimal balance between the strengths and the performance, that should be accounted by the tournament organizer, i.e. it may not be profitable to invite a "super" superstar. We come back to this problem in Section 4 where the tournament designer problem will be discussed in more detail.

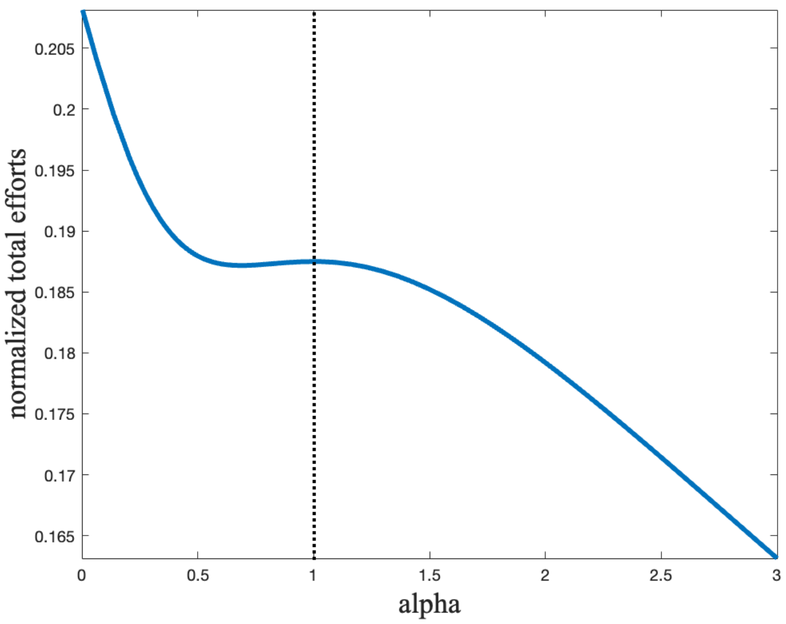


Figure 5 – Strength-normalized total efforts in SET

Schedule equivalence

Let's look more carefully at different scheduling options. The first version of the schedule is when two matches of the semi-finals are played simultaneously, it is considered above. The second version organizes the semi-final with a superstar before the semi-final with two regular players. In the third version, the semi-final with the superstar is played after the semi-final with the regular players.

Proposition 2. In the 4-player SET with a superstar, where players have the linear costs and the Tullock success function, the following statements hold for a risk-neutral tournament organizer

1. tournament versions 1 and 3 are equivalent,

2. tournament versions 1 and 2 are equivalent.

Proof. For (1), the equivalence is due to tournament structure and disclosures. In the first round, the regular player wins, but this information is available to the other pair during simultaneous games since the same players play in the first round. It turns out that in the first round no new information is disclosed. On the other hand, for

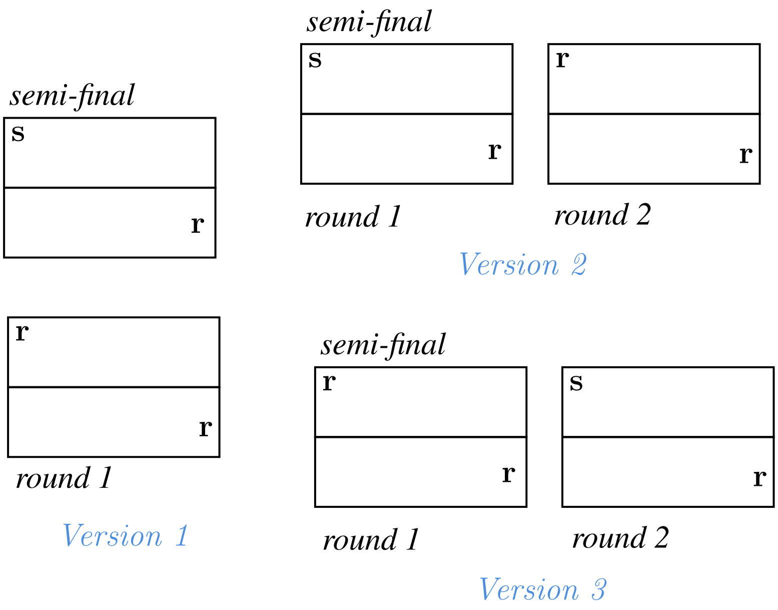


Figure 6 – Scheduling for semi-final at SET4

the participants of the first round it is still not known who their opponent will be. Therefore, as in the case of simultaneous matches, they weigh their expected payoff by probabilities. It turns out that the behavior of the first couple does not change, since they play first. And the behavior of the second pair does not change, since they do not receive any new information. The final match does not depend on the order of the semi-finals, so the actions of the players do not change in it either.

For (2), nothing changes in the first match of SET, as the players do not receive new information. On the other hand, the behavior of the players in the second round changes. Denote by EV_{sr0} the expected payoff that the regular player (r) expects to receive in the semi-finals (s) if the superstar lost (0). An index 1 means that the superstar has won. Similarly, we introduce equilibrium efforts for regular players e_{sr0}. Denote by P_{ss} the probability of the superstar winning the semi-final. Then the equilibrium parameters are given in the Table 3.

	Sequentially (Version 2)	Simultaneously
EV_{sr0}	EV_{fr0}	$P_{ss}EV_{fr1} + (1 - P_{ss})EV_{fr0}$
EV_{sr1}	EV_{fr1}	$P_{ss}EV_{fr1} + (1 - P_{ss})EV_{fr0}$
e_{sr0}	$\frac{EV_{sr0}}{4}$	$\frac{P_{ss}EV_{fr1}+(1-P_{ss})EV_{fr0}}{4}$
e_{sr1}	$\frac{EV_{sr1}}{4}$	$\frac{P_{ss}EV_{fr1}+(1-P_{ss})EV_{fr0}}{4}$

Table 3 – Payoffs and efforts in the semi-finals in SET with schedules 2 and 1

We see that, in version 2, there are two scenarios of the first semi-final: when the superstar wins his match and when he loses. This leads to the existence of two different cost options depending on the previous outcome. For a risk-neutral tournament organizer, the total expected efforts will be unchanged, and for this match, they will be equal to $\frac{P_{ss}EV_{fr1}+(1-P_{ss})EV_{fr0}}{4} \times 2$. Thus, it turns out that for the tournament organizer, versions 1 and 2 are equivalent, while the efforts chosen by the players will be different. The behavior of the players in the final will not change. $\square$

The risk-neutrality of the tournament organizer was crucial for Proposition 2. However, there are situations in which the attitude of the tournament organizer to risk is different. For example, if she wants to get some kind of sponsorship contract, then she may prefer the first option of the tournament, as it is more stable in terms of overall efforts, which can serve as a proxy for the entertainment of the tournament. Or if the tournament organizer doesn't run many tournaments (which is usually the case), she may also want more stable scenarios.

3 The model of double-elimination tournament

The double-elimination tournament (DET) consists of the upper and lower brackets. We consider the classic (Huang, 2016) version with 4 players (Figure 7), in which all players are in an equal position and start from the upper bracket. In any knockout tournament, the player is eliminated from the tournament after the second defeat. After the first defeat, he still continues participating in the tournament in the lower bracket. The games are played in the order shown in Figure 3. That is, we believe that in the first round, the games go in parallel and the players do not know in advance which opponent they can meet in the next round.

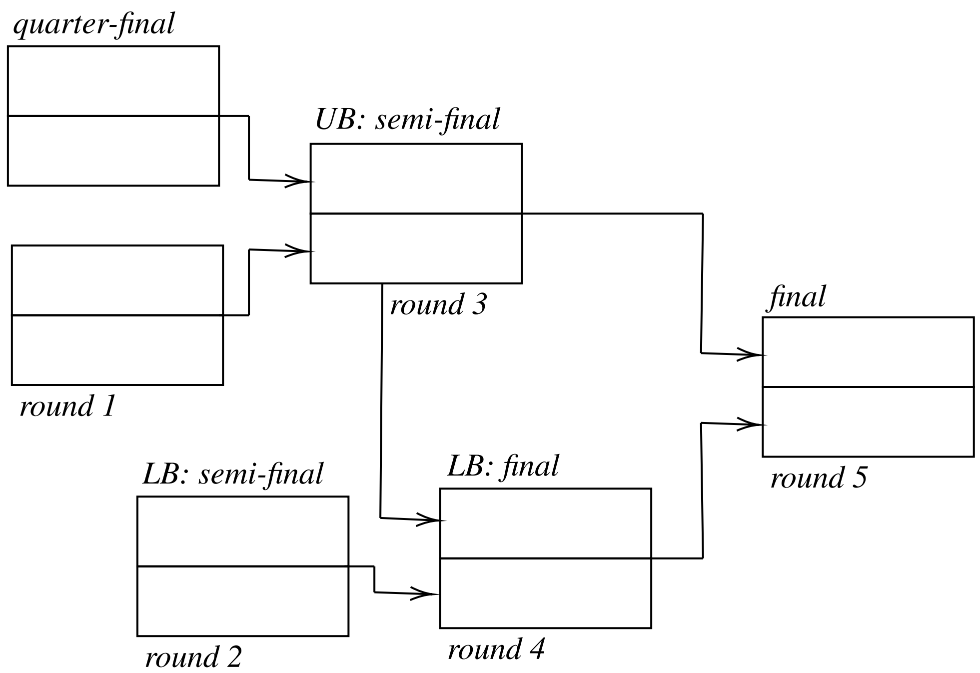


Figure 7 – Tournament structure for double-elimination tournament for 4 players

Equilibrium analysis in DET

The final of DET is similar to that in SET. However, the probability of meeting the superstar in the final is different (Table 4).

α	SET	DET
0.9	0.45	0.39
1	0.5	0.5
1.44	0.67	0.79
2	0.78	0.91

Table 4 – Probability to meet the superstar at the final for different types of tournaments.

DET raises the likelihood of meeting the superstar in the final compared to SET. This is a possible argument in favor of DET if the organizer cares about reducing the chance of a strong player loss or dropping out of the tournament before the final. On the other hand, if the probability of meeting the superstar in the final becomes too high (greater than about 0.9), then it may be less interesting to watch the starting matches of the tournament.

The round immediately preceding the final is *the lower bracket final*, and the winner of this match gets the opportunity to participate in the upper bracket final (hereinafter referred to as the final). The structure of this match is similar to the semi-finals in a single-elimination knockout tournament for 4 players. However, now it is worth considering three options for possible events: (i) the superstar plays with the regular player, (ii) the regular player against another regular player, and in the final the winner of this pair waits for the superstar, and (iii) the regular player against another regular player, and in the final the winner of this pair waits for the regular player. The implementation of one of the options depends on how the players play in the previous matches. In addition, since the rounds are played sequentially, the players have already known whom they meet in the final.

In the third round, the semi-final of the upper bracket takes place. Now the match is slightly different, that is, in case of a defeat, the player does not receive a zero, as happens in the case of the final and all matches of the lower bracket (as well as in all matches of the single-elimination tournament). The prize for losing is now equal to the player's expected payoff in the lower bracket final. However, as we know from Leaser and Rosen (1981), only the difference between the prizes matters to the players.

We assume that the semifinal of the lower bracket is organized before the semi-final of the upper bracket. Tournament organizers usually try to play matches consecutively so as not to split the spectators. However, the upper bracket quarter-finals are assumed to be played simultaneously. This reflects that the players do not know their future opponents. Consideration of the situation with fully sequential games will be similar to SET.

It follows from figure 8, that for a regular player who was not seeded with a superstar at the beginning of the tournament, it makes no sense to deliberately lose in the first round since she will receive a smaller expected payoff in this case at any α.

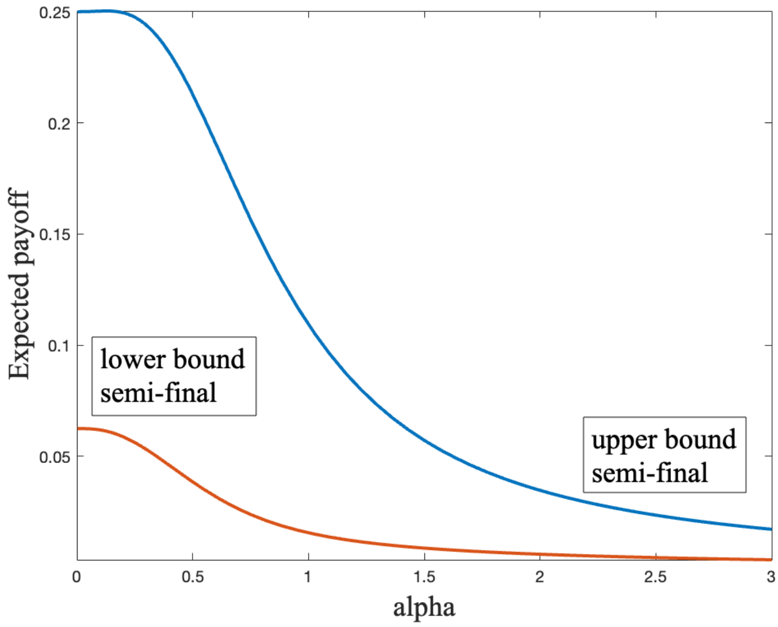


Figure 8 – Expected payoff that regular player can get at upper bound and lower bound semi-finals.

Remark 3. In the 4-player DET with a superstar, where players have the linear costs and the Tullock success function, the risk-neutral tournament organizer is indifferent if the quarter-finals will be held simultaneously or sequentially.

Proof. The proof repeats the proof of Proposition 2. □

4 Comparison of different tournament types

Consider different objective functions of a tournament designer: the probability for the strongest player to win the tournament, the total efforts, and the average total efforts.The first two types of objective functions were discussed in Groh, Moldovanu, Sela and Sundle (2012).

The first indicator is responsible for the "fairness" of the tournament. It is quite obvious that the strongest player should win the tournament with a higher probability than her weaker opponents. This metric is reasonable for a tournament organizer whose main goal is to select the strongest player through the tournament mechanism.

The total combined efforts of the players can be a proxy for the tournament designer's revenue. However, looking at the overall efforts separately sometimes does not reflect the true picture, as different tournaments have different numbers of matches. So we introduce a third metric, the average total efforts, which shows how much efforts are generated on average in each match. This normalization, in particular, allows comparing the tournaments with the different numbers of participants, and, as a result, of different lengths. For the same number of players, it adds additional understanding of the performance for different types of eliminating tournaments, requiring a different number of matches.

The total efforts (TE) and the average total efforts (ATE)

The total efforts in DET occur to be greater than in SET if the superstar strength doesn't exceed the strength of regular players dramatically. In this case, the fact that the number of matches in DET is larger explains the cumulative effect of large efforts. But with a superstar power level growth above 2.153, the situation changes (see Figure 9). The large difference in strengths leads to demotivating players to put in large efforts, since the chance of winning twice against the superstar is too low.

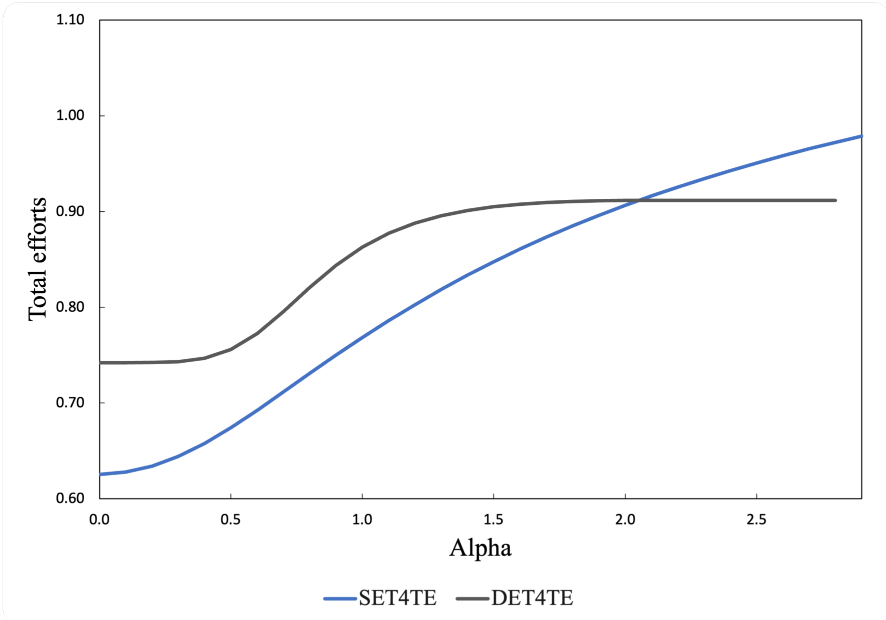


Figure 9 – The total efforts in SET and DET

From this figure, one can note, that total efforts in DET are only a bit greater than in SET (when they do), while the number of matches in DET is twice greater. This explains that the average total efforts, accounting for the length of the tournament, are always greater in SET, i.e. on average SET is more spectacular (Figure 10).

As we showed above, the matches closer to the final are more spectacular. From this point of view, since SET is shorter, all its matches are closer to the final in comparison

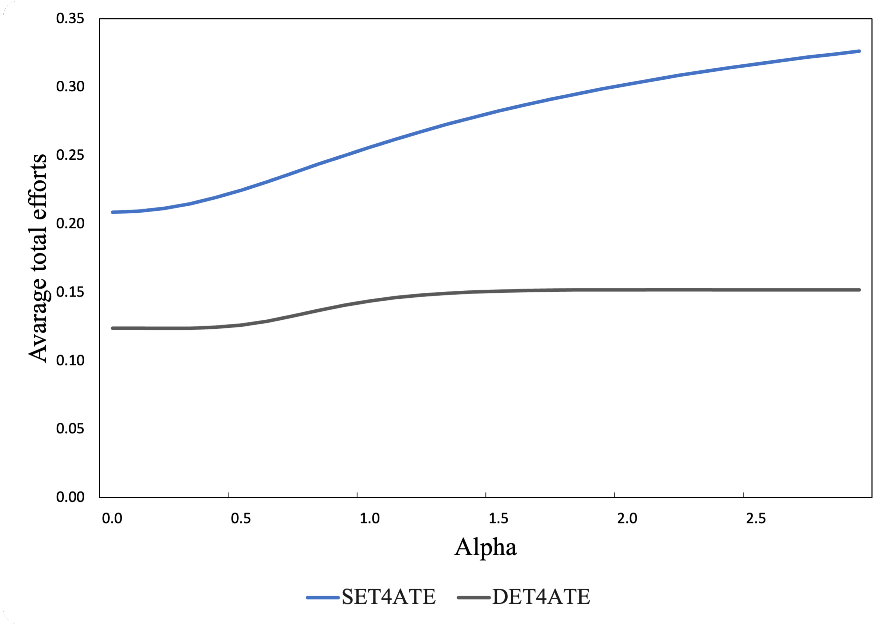


Figure 10 – The average total efforts in SET and DET

with DET.

The results demonstrate that the choice of the tournament format is not so obvious and in some cases the designer who cares about the total efforts may prefer DET even if it is less spectacular in separate matches.

Probability of the superstar winning a tournament

DET provides a second chance to every player who defeats once, this is one of the arguments behind DET. This makes the double failure for the superstar almost impossible in DET, and, thus, her probability of winning the tournament is strictly greater in DET than in SET.

However, as one can see from Figure 11, the probability of winning by the superstar is far from 1 even in DET. This is due to lower efforts in the case of too strong leader, i.e. players balance their efforts and do not strive too tough as *in equilibrium* this is meaningless.

Tournament designer problem

When the organizer chooses one of the metrics above, she ignores the relative expenditures of the tournament organizer arising under the given format of the tournament. There may exist a competition for the superstar attraction, which makes it costly, and the stronger the superstar is, the greater the fee is. Also, every match claims organization costs, together with the costs of the whole tournament, for instance, on the extended advertising company.

Consider the following objective function for the organizer. Suppose the tournament

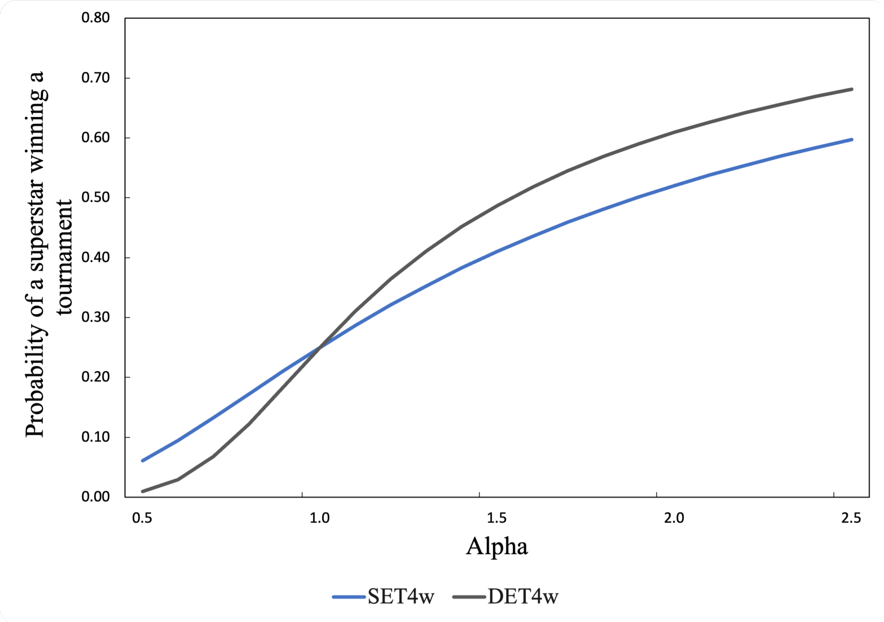


Figure 11 – Probability of the superstar winning SET and DET

organizer maximizes the expected profit proportional to the performance measured by the total efforts. The revenue is given by $TR = A \times TE_T(\alpha)$, where $TE_T(\alpha)$ is the expected total efforts, in a particular type of tournament, and $A > 0$ is a parameter reflecting the relative importance of revenue with concerning costs.

The first kind of costs are the costs of holding each match m, then they will work out for all matches $N_T \times m$, where N_T is the number of matches in the particular tournament. The second kind of costs are the fixed costs for the tournament F. The third kind are the costs of attracting the superstar $c(\alpha)$. It is natural to assume that $c(\alpha)$ is a strictly increasing and convex function, i.e. $c'(\alpha) > 0$ and $c''(\alpha) > 0$ for all $\alpha > 1$.

Then the objective function of the tournament organizer is given by

$$\Pi(\alpha) = A \cdot TE_T(\alpha) - c(\alpha) - N_T m - F \tag{5}$$

This is equivalent to solving the following maximization problem:

$$\max_{T,\alpha} \Pi(\alpha) = \max_{T,\alpha} \left\{ TE_T(\alpha) - \frac{c(\alpha) - N_T \times m - F}{A} \right\}. \tag{6}$$

If the weight A of the efforts importance is large enough, then this function reaches the maximum at some level of the superstar talents.

Example

Consider the following example of the designer problem. Let the fixed costs be zero. Assume that the relative costs of the superstar attraction are given by the quadratic function $\frac{c(\alpha)}{A} = 0.1\alpha + 0.025\alpha^2$. Consider two cases: when the match organization is costless and when these costs are positive, $\frac{m}{A} = 0.05$.

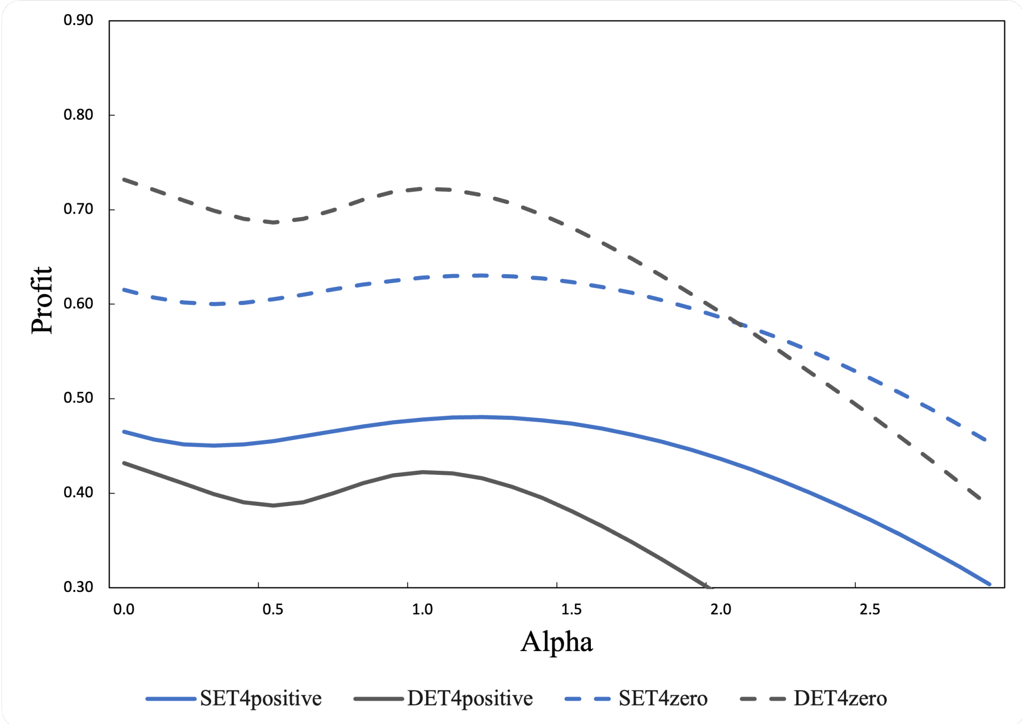


Figure 12 – Profit for positive ($\frac{m}{A} = 0.05$) and zero costs ($\frac{m}{A} = 0.00$).

Figure 12 demonstrates that different elimination formats are differently profitable for the organizer. When the match organization is costly, the designer minimizes the number of matches and prefers SET. Moreover, the maximums in α under SET and DET do not coincide, and under SET the designer attracts more strong superstar since the gain from additional efforts prevails over the losses from the attraction.

If the match organization is free of charge, then DET becomes a more preferable format, since it generates larger total efforts under the same level of the superstar. However, accounting for the attraction costs, the organizer will invite the less productive superstar in DET than in SET, since the economy on attraction costs is larger than the gain from a bit greater performance.

5 Discussion: inviting more players

As one can see, in SET and DET, the organizer may reach different performances of the tournament, but organizing different numbers of matches, 3 versus 6. The DET is twice as long as SET and requires more time slots for organization. If the time for the tournament is limited by external reasons, then the schedule in SET may look too sparse in comparison with DET if the designer wants to fill in the all available slots. In this case, the organizer may consider inviting more regular players to enrich the schedule.

SET with 8 players requires 7 matches (see Figure 13), and therefore may be a more competitive alternative for 4-player DET. Again, assume that there is the unique superstar of the strength α and 7 symmetric regular players. All quarter-finals and semi-finals take place simultaneously.

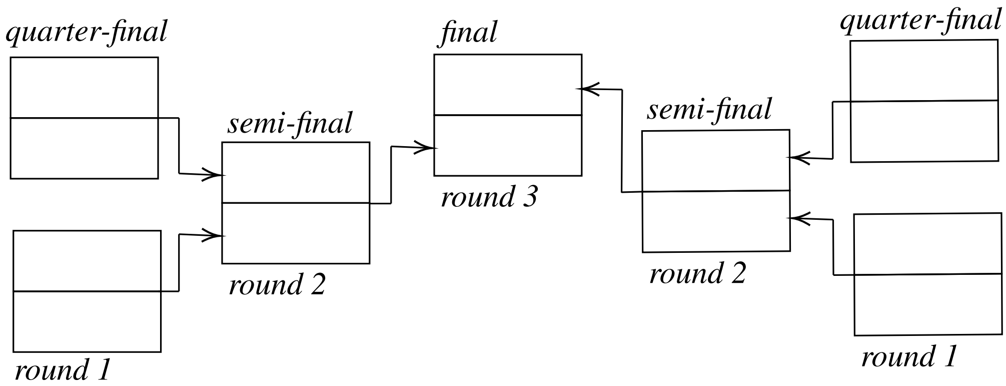


Figure 13 – SET structure with 8 players

Final

The final is similar to SET with 4 players, but the probability of meeting the superstar in the final is lower. The reason is that, to reach the final, the superstar needs to win not only the semi-final (and these probabilities are equal in SETs with 4 and 8 players), but also to win the additional match, i.e. the quarter-final. Generally, the longer the tournament, the smaller the chance to meet the superstar in the final.

Proposition 3. In SET with $n \geq 2$ players with the superstar, the probability that the superstar will play in the final is greater than for the same tournament with $2n$ players.

Proof. The tournaments with n players have the $\log_2 n$ rounds. The tournaments with 2n players have $\log_2 2n = \log_2 2 + \log_2 n$ rounds, which is greater by one than for tournaments with n players. The first $\log_2 n$ rounds are identical, therefore, the superstar needs to play one more round in a tournament with $2n$ players, and the probability to win this round is strictly less than 1 (it is equal to 1 only if all others evaluate the prize as 0). Thus, the total probability of winning a longer tournament is lower. $\square$

Semi-finals

In the semi-finals, two situations can arise: when only regular players survive in the tournament, or one of the semi-finals will be held with the superstar. The first type is similar to the tournament with 4 homogeneous players and is solved by Huang (2016). The second one is SET with 4 players, discussed in Section 2. The probability of meeting the second option is equal to the probability of winning the first round by the superstar. As the strength of the superstar increases, the probability of meeting her in the semi-finals also grows.

Quarter-finals

We assume that 4 quarter-finals are taking place simultaneously. The heterogeneity of players arises as a result of seeding. In the quarter-finals, players can be divided into 4 types: (s) the superstar, (r1) the regular player who plays with the superstar now, (r2) the regular player who plays with a regular player but can face the superstar in the semi-final, and (r3) the regular players who only have a chance to face a superstar in the final. The probabilities for each type of winning the tournament are presented in Figure 14.

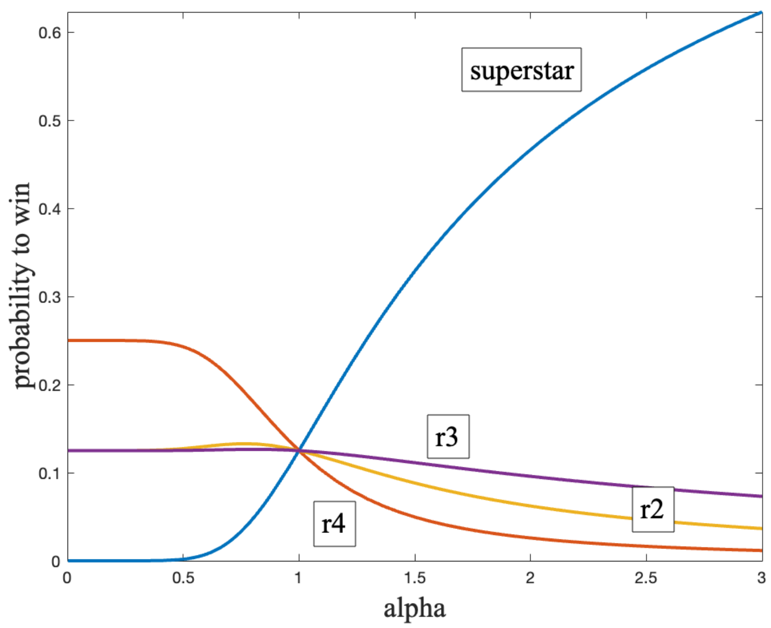


Figure 14 – Probability of winning a SET-8 for different types of players

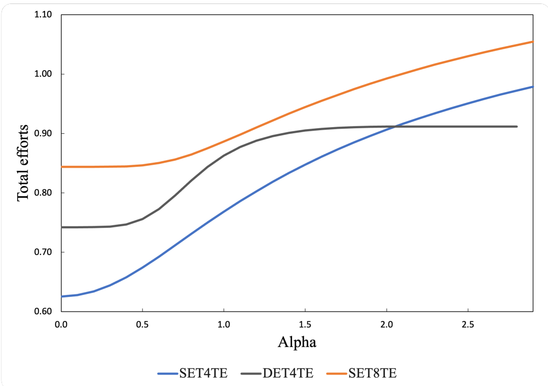
In the worst situation is the player who has to play with a superstar in the first round, and she defeats immediately with high probability. In the best position are the players who are seeded into the opposite branch of the tournament and have a chance to face the superstar only in the final. The intermediate position is for the players who could meet the superstar in the semi-finals. It can be concluded that the more distant the player from the superstar in the tournament bracket, the better are her chances.

Comparison of SET-8 with 4-player tournaments

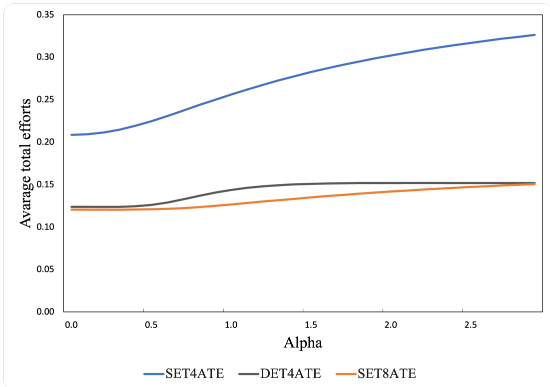
It is quite clear, that because of the larger number of matches, the total efforts in SET-8 are greater than in SET-4 and DET-4. The reverse relation holds for the average total efforts. The probability of winning the tournament for the superstar is lower in DET-8 than in two other formats.

The situation becomes more interesting when the designer cares about the organizational costs. If we turn back to the Example from Section 4, it occurs that, for costly matches (Figure 15), inviting 8 players is beneficial only if the level of the superstar is very high and the organizer cannot invite a less productive superstar. When the optimal choice of α is also allowed together with the tournament format, SET-4 is the best option.

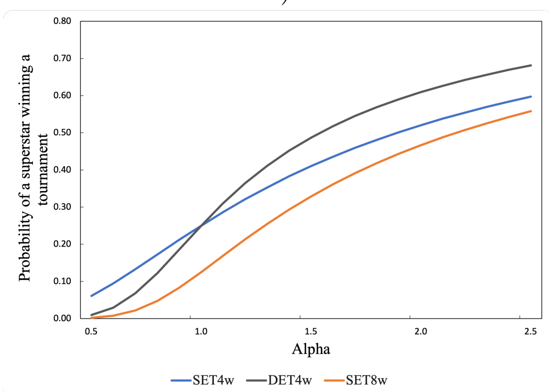
When the costs are limited only to the attraction costs (Figure 15), then SET-8 becomes the most preferable format for the organizer. Indeed, the costs of attraction do not depend on the length of the tournament, while the performance is greater with 8 players. The problem is that the attraction costs produce bad incentives for the organizer, and inviting the player equal to others becomes more profitable than overpaying for the superstar.



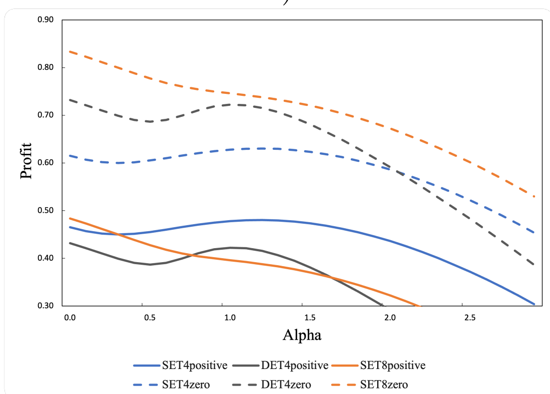
a)



b)



c)



d)

Figure 15 – Comparison of SET-8 with 4-player tournaments: a) total efforts, b) average total efforts, c) probability of the superstar winning the tournament, d) profit.

Thus, the solution with inviting a larger number of participants occurs to be less successful in comparison with playing with the format of elimination and the accurate choice of the superstar qualification.

6 Conclusion

We investigated two formats of elimination tournaments and studied the equilibrium behavior of participants. The presence of a superstar provides a prior heterogeneity in the tournament, but we find even more heterogeneity stemmed from the seeding and the format. Accounting for all these factors, like seeding, format, and strength, makes the analysis non-trivial and the intuition about the advantages of different formats ex-ante unclear, though their deep understanding is required for the tournament designer.

We take two well-known optimization criteria for the organizer and suggest the new one, including the cost of the match conducting and the attraction of a superstar. We see that the optimal organizer choice varies with the strength of the superstar. Under the criteria based on the average efforts, SET format is generally more preferable, while caring about the superstar winning recommends choosing DET. The developed models also allow estimating the optimal level of the attracted superstar, since too strong player is too costly for the organizers and demotivates other participants to put on competitive efforts.

The limited number of tournament participants is enough to model the final part of a tournament, which is the most important and entertaining. That is why for the designer, the optimization problem arises sharply, and our recommendations are useful for the rational choice and for stimulating higher performance. This is not the area where the blind experiments are possible and painless because of the large budgets and visibility of the events. The strong analytic theory behind the policy decisions in this area is what may improve the championships for all sides: participants, spectators, organizers, and sponsors.

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