Set-alternating schemes: A new class of large Condorcet domains

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Condorcet domains

What is the largest size of a set of linear preference orders for n alternatives such that majority voting is transitive when each voter chooses his preferences from this set?

The latest lower bound on the size of maximum Condorcet domains 2.1890^n was investigated by (Karpov, Slinko, 2023).

We present a new method of constructing large Condorcet domains that leads to a new lower bound.

Condorcet domains

Let a finite set $X = [n] = \{1, \ldots, n\}$ be the set of alternatives. Let $L(X)$ be the set of all linear orders over X. Each agent $i \in N$ has a preference order P_i over X (each preference order is a linear order). Let $L(X)$ be the set of all linear orders over X. For brevity, we will write preference order as a string, e.g. $12 \ldots n$ means 1 is the best alternative, n is the worst.

A subset of preference orders $D \subseteq L(X)$ is called a *domain* of preference orders. A domain D is a *Condorcet domain* if whenever the preferences of all agents belong to the domain, the majority relation of any preference profile with an odd number of agents is transitive.

Peak-pit domains

Each Condorcet domain restriction of the domain to each triple of alternatives satisfies a never condition iNj, $i, j \in [3]$. iNj means that i^{th} alternative from the triple according to ascending order does not fill in j^{th} place within this triple in each order from the domain.

A domain D is a *peak-pit* domain if, for each triple of alternatives, the restriction of the domain to this triple is either single-peaked ($iN3$), or single-dipped ($iN1$).

Peak-pit domains

For example restriction *abc* to triple $a, b, c \in [n]$, $a < b < c$ satisfies never conditions $1N2, 1N3, 2N1, 2N3, 3N1, 3N2$, but violates never conditions $1N1, 2N2, 3N3$.

$1N1$: bac,bca,cab,cba (single-dipped domain);
$1N2$: abc,acb,bca,cba (group-separable domain);
$1N3$: abc, acb,bac,cab (single-peaked domain).

Maximal width domains

Danilov, Karzanov, Koshevoy (2012) investigated Condorcet domains of tilling type.

Danilov, Karzanov, Koshevoy (2013) investigated symmetric Condorcet domains.

Both types of domains contain orders $12\dots n$, and $n,\dots 21$. This property is called *maximal width*.

We do not consider domains with maximal width.

Generalized Fishburn alternating scheme

Definition 1

Starting with a subset $A \subseteq [n]$ we consider the following never conditions on triples $L(X)$: For $i < j < k$ with $j \in A$ assign the never condition $2N3$. For $i < j < k$ with $j \notin A$ we assign the never condition $2N1$. This is the *generalized Fishburn domain* generated by A.

Fishburn domain

If A consist of even number, then we obtain Fishburn's domain. aving the natural ordering of alternatives we obtain the following maximal Fishburn's domains:
$F_3 = \{123, 213, 231, 321\}$,
$F_4 = \{1234, 1243, 2134, 2143, 2413, 2431, 4213, 4231, 4321\}$,
The reversed domains are also Fishburn's domains. Galambos, Reiner (2008) gave the exact formula for the cardinality of F_m:

$$|F_m| = (m+3)2^{m-3} - \begin{cases} (m - \frac{3}{2})\binom{m-2}{\frac{m}{2}-1} & \text{for even } m; \\ (\frac{m-1}{2})\binom{m-1}{\frac{m-1}{2}} & \text{for odd } m. \end{cases}$$

Set-alternating scheme

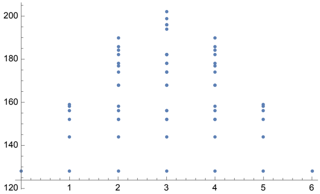
Definition 2

Starting with a subset $A \subseteq [n]$ we consider the following never conditions on triples $L(X)$: For $i < j < k$ with $j \in A$ assign the never condition $1N3$. For $i < j < k$ with $j \notin A$ we assign the never condition $3N1$. This is the *set-alternating scheme* generated by A.

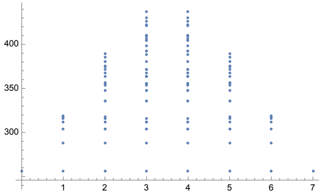
We let $D_X(A)$ denote the Condorcet domain which is generated by this scheme and let $f_n(A)$ denote the cardinality of $D_X(A)$.

If $A = [2, \ldots, n-1]$, then all triples are assigned the $1N3$ never condition. This gives an Arrow's single-peaked domain, but it is not Black's single-peaked since the domain does not contain two mutually reversed orders.

Set-alternating scheme



(a) $n = 8$



(b) $n = 9$

Figure: Domain size and set size for all subsets of $\{2, \ldots, n-1\}$

Properties of set-alternating schemes

Proposition 1

Each domain defined by set-alternating scheme and generalized Fishburn alternating scheme is a copious peak-pit maximal Condorcet domain.

Proposition 2

Each domain defined by set-alternating scheme and generalized Fishburn alternating scheme is connected.

Recursive properties of set-alternating schemes

We let $f_n(A)$ denote the cardinality of $D_{[n]}(A)$, w - the maximal element in A, $A' = A \setminus \{w\}$

Proposition 3

If $w = n - 1$, then $f_n(A) = 2f_{n-1}(A')$.

Proposition 4

If $1 < w < n - 1$, then $f_n(A) = 2f_{n-1}(A) + f_{n-1}(A') - S$, where $S = \sum_{j=w-1}^{n-2} f_j(A')$.

If $n - 1$ is a member of A then $A_1 = A \setminus n - 1$ generates a domain which is at least as large as that for A.
If 2 is not a member of A then $A_2 = A \cup \{2\}$ generates a domain which is at least as large as that for A.

Some combinatorial results

Proposition 5

For $k = 1$*, we have* $f_n(\{n-k\}) = 2^{n-1}$ *and* $f_n(\{k\}) = 2^{n-1}$*. For* $k > 1$ *and* $n \leq k+1$*, we have* $f_n(\{n-k\}) = 2^{n-1}$ *and* $f_n(\{k\}) = 2^{n-1}$*. For* $k > 1$ *and* $n > k+1$*, we have* $f_n(\{n-k\}) = 5 \cdot 2^{n-3} - 2^{n-k-2}$ *and* $f_n(\{k\}) = 5 \cdot 2^{n-3} - 2^{k-2}$*.*

Proposition 6

The size of the domain on n alternatives which for each triple satisfies never conditions 1N3 and 3N1, is the $(n+1)^{th}$ *Fibonacci number.*

New schemes

Definition 3

$D_X(A_n)$ is the result of the *even* $1N33N1$*-alternating scheme* if

$$A_n = \{2, 4, 6, \ldots, n - 2 + p_n\},$$

where $p_n = (n \bmod 2)$.

Definition 4

$D_X(B_n)$ is the domain given by the *odd* $1N33N1$- *alternating scheme* if

$$B_n = \{2, 3, 5, \ldots, n - 3 + p_n\},$$

where $p_n = (n \bmod 2)$.

$n = 6$ example

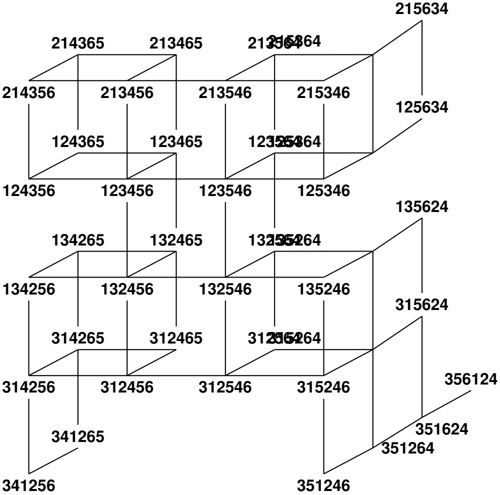


Figure: The median graph for the even $1N3, 3N1$-alternating scheme for $n = 6$.

Conjectures

Regarding the maximum domain size we conjecture the following.
For each $n \geq 5$, the maximum size of set-alternating domain is the size of the odd $1N33N1$-alternating domain. This conjecture has been verified computationally for $n \leq 24$.
We believe that the conjecture holds in a more general form as well.
For each $n \geq 16$, the largest unitary domain D produced by any N_1, N_2 set-alternating scheme, using general pairs of never conditions N_1, N_2, occurs if and only if D is given either by the odd $1N33N1$-alternating scheme or its reverse complement. This conjecture has been verified computationally for $n \leq 20$.
For $n < 16$ the maximum cardinality occurs of the Fishburn's alternating scheme.

Main result

We partition all orders in $D_X(A_n)$ on orders that start from set $\{1,2\}$, orders that start from set $\{1,2,3,4\}$, but not from set $\{1,2\}$, orders that start from set $\{1,2,3,4,5,6\}$, but not from set $\{1,2,3,4\}$, etc. Orders from part k start from $[2k]$, but not $[2(k-1)]$.

Orders from the first part start from 12, 21. There are $2a(n-2)$ such orders. Orders from the second part start from 1324, 1342, 3124, 3142, 3412. There are $5a(n-4)$ such orders.

Lemma 5

In all orders from k^{th} part of $D_X(A_n)$ alternatives from A_{2k} are in ascending order.

Lemma 6

In all orders from k^{th} part of $D_X(A_n)$ alternatives from $\overline{A_{2k}}$ are in ascending order.

Dyck words

Definition 7

A sequence $a_1 a_2 \ldots a_{2k}$ of k elements u and k elements d such that for all $1 \leq j \leq 2k$ we have $|i \in [j] | a_i = u\}| \geq |i \in [j] | a_i = d\}|$ is a *Dyck word*.

Proposition 7

(Deutsch, 1999) The number of Dyck words of size $2k$ is C_k, where C_k is the k^{th} Catalan number.

Bijection

The first element in Dyck words is u. It has no correspondence in orders. Each consequent element in top $2k$ elements segment of an order from k^{th} part of $D_X(A_n)$ corresponds with consequent element in Dyck word: if the element belongs to A_{2k} then d, if not, then u. The last element in Dyck word is d. It has no correspondence in orders.

Top 4 elements	Dyck word
1324	*ududud*
1342	*uduudd*
3124	*uuddud*
3142	*uududd*
3412	*uuuddd*

Main result

For $m = n/2$, even n we define $w(m) = a(n)$. From bijection we have

$$w(m) = \sum_{k=1}^{m} C_{k+1} w(m-k), \qquad (1)$$

where C_{k+1} is the $k+1$ Catalan number. Solving the recurrence we get

Proposition 8

For even n we have

$$a(n) \sim \frac{\sqrt{2}}{4} \left(\sqrt{2 + 2\sqrt{2}} \right)^n,$$

for odd n we have $a(n) \sim \frac{\sqrt{\sqrt{2}-1}}{2} \left(\sqrt{2 + 2\sqrt{2}} \right)^n$.

Conclusions

We show that the domain size for sufficiently high n exceeds 2.1973^n, improving the previous record 2.1890^n.

Details can be found in

Karpov, A., Markström, K., Riis, S., Zhou, B. 2023.
Set-alternating schemes: A new class of large Condorcet domains. Arxiv preprint arXiv:2308.02817.

Finally

Any comments will be greatly appreciated.

Thanks for your attention!