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Coherent states for quantum models with non-Lie symmetry algebras

Summary of thesis

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INTRODUCTION

Goal and objectives of research. Formulation of the problem.

The study is aimed at studying non-Lie algebras with finitely many generators, describing their irreducible representations, and constructing a spectral theory for these algebras.

From the algebraic point of view, there are two main questions in this work. First, how to construct coherent states and the corresponding irreducible representations of algebras with non-Lie permutation relations. Secondly, how to relate these quantum representations to some classical symplectic leaves in a Poisson manifold. An additional task is to establish a connection between irreducible representations, coherent states, and special functions that arise during their construction.

In terms of physics, the main task of the study is the study of algebras that arise naturally (as symmetry algebras) in various quantum mechanical models. Irreducible representations and coherent states of these algebras play a decisive role in the spectral analysis of quantum problems. At the same time there is a task of a development of an algebraic approach, which consists in the successive application of the methods of operator averaging and coherent transform. This approach becomes the key one for studying quantum models with a strong degeneracy of the spectrum of the leading part of the operator (for example, due to resonance), since the standard perturbation theory does not work here.

Relevance and degree of development of the problem

Interest in studying algebras with non-Lie permutation relations is mostly based on examples in the q-deformation theory and in the "quantum inverse scattering problem"; see [1, 2, 3, 4, 5, 6], as well as some interesting examples in [7]. Another very general direction in which non-Lie permutation relations appear is the quantum version of the Marsden–Weinstein–Lie–Cartan reduction; see, for instance, [8, 9, 10] and [11] for more details.

The systematic consideration of quantum algebras with nonlinear relations was started by the schools of V.P. Maslov and L.D. Faddeev [7, 12, 13, 14, 6, 15, 3, 11] in the 70s-80s, although the first attempts to use such algebras were made by physicists much earlier.

There are several directions of studying non-Lie permutation relations and quantization of general nonlinear Poisson brackets. The *first direction* deals with computation of the product in the enveloping algebra via the operators of regular representation (or via the "generalized shift" or Delsart operators); for the basic facts about the generalized shift operators, see [17], for applications to non-Lie relations, see some general formulas in [18, 19] and the review around this topic in the book [11]. This direction is related to the noncommutative geometry due to A. Connes [20].

The *second direction* is based on the theory of deformation quantization proposed in [21, 22]. It is a fundamental basis for the quantum group theory that related the quantum Yang–Baxter equation and the quadratic Faddeev–Zamolodchikov algebras; see, [23, 19, 24, 25].

The *third direction* of studying is the "semiclassical" approach that makes it possible to derive all essential objects related to a non-Lie algebra, but only approximately, with an accuracy $O(\hbar^\infty)$ with respect to the Plank "constant" $\hbar \to 0$; see [18, 14, 26] and detailed proofs in [11]. This is the theory of asymptotic quantization.

The *fourth direction* is based on the notion of coherent states, initiated in the earliest times of the quantum mechanics by Erwin Schrödinger [27] and Werner Heisenberg [28], in optics by R. Glauber [29, 30] (where the name "coherent states" was first introduced), and defined in general form by J. Klauder [31, 32] and F. Berezin [33]. In fact, a version of coherent states was also developed in the theory of holomorphic functions under the name "reproducing kernels," see [34, 35, 36].

From the viewpoint of the representation theory, the main property of coherent states is the possibility to write out an irreducible representation of a given algebra via differential operators acting on the space of parameters of the states. For the Heisenberg algebra, this fact was understood already by F. A. Fock [37] and P. Dirac [38], and was generalized for a wide class of Lie algebras [39, 40] and for certain q-analogs (see, for instance, in [41]). In the framework of general quantization process, coherent states first were used in [33, 42, 32] and were deeply involved into this theory [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53] especially in the context of the fundamental geometric quantization due to B. Kostant and J.-M. Souriau and the representation theory [54, 55, 56, 57, 58, 8, 59, 60, 61, 62]. This list of references is very far from being complete but shows the variety of approaches to this area. We also stress that there are many crucial questions in this theory still being open from the 70s.

For a long time, the list of physical systems, where the algebras with nonlinear relations play a significant role in the description of spectrum and dynamics, had been confined to infinite-dimensional field systems and spin chains; see the list of references in [15]. Examples of non-Lie algebras with finitely many generators, whose properties manifest themselves in fundamental effects of quantum mechanics (the Zeeman and Zeeman–Stark effects), were discovered and studied in detail in [63, 9, 64, 65]. The algebras are quantum algebras, that is, deformations of some classical Poisson algebras (with a polynomial Poisson tensor).

Another series of examples is the "resonant algebras" corresponding to the multifrequency quantum oscillator. They were found in [66, 67] and studied in detail in [68, 69, 70, 71]. These non-Lie algebras with finitely many generators also belong to the class of quantum algebras, and for them it is also possible to construct a complete theory of irreducible representations. The application of resonant algebras covers a wide range of basic models of wave optics and quantum physics, since states localized near a stable equilibrium position play a fundamental role in them. The harmonic part of such systems, the "oscillator", defines the main component of the movement, while the anharmonic part represents perturbation. After the quantum averaging procedure, this perturbation begins to commute with the harmonic part, i.e., it is an element from its symmetry algebra. If the frequencies of the harmonic part are in resonance, then the symmetry algebra is noncommutative. In the general case, this is an algebra with polynomial commutation relations.

Thus, the modern quantum physics and quantum mathematics demonstrate the importance of algebras with nonlinear commutation relations.

As for coherent states, interest in their study in mathematical physics and applied mathematics steadily increases. Applications that use coherent states range from quantization to signal and image processing. Over the past almost hundred years (since 1926, when coherent states were introduced by E. Schrödinger in [27]), not only their numerous generalizations and modifications have appeared (see, for example, [40, 72, 73, 76, 77]), but also significant changes in the very definition of coherent states; see [74] about this. At the beginning, the defining property was the ability of coherent states to minimize the product of variances in the Heisenberg uncertainty relation. But later this property turned out to be optional.

The modern definition of coherent states is based on the four Gaso-Klauder axioms [78]. The first two basic axioms are general and obligatory for all types of coherent states. They were formulated by J. Klauder in [79] in 1963 and rewritten almost forty years later in [74]. They postulate the completeness of the family of coherent states and the continuity of the overlap function in the parameters [1]. The other two (special) axioms refer to the special case where coherent states are constructed for a given Hamiltonian. The main thing here is the property of temporal stability: the time-evolution of each coherent state always remains a coherent state. In this case, the parameters of coherent states are usually associated with coordinates in the corresponding phase space, and their evolution should correspond to the classical behavior of these coordinates. These properties are needed for physical applications; see, for example, [75].

[1] Coherent states form an overfull system of vectors in the Hilbert space; their pairwise inner products depend on the parameters numbering the vectors of this family and define the overlap function.

In mathematical papers [16, 68, 70, 71] coherent states are constructed not for a given Hamiltonian, but for a given algebra. Such coherent states are used as the kernel of an integral transform from the space of one representation of an algebra to the space of another representation of it. Here the property of coherent states to intertwine algebra representations in Hilbert spaces becomes the main one. Thus, for algebras, including those with nonlinear commutation relations, the defining properties of coherent states are their completeness, the continuity of the overlapping function in parameters, and the intertwining property.

Note that the question of the construction of coherent states for algebras with non-Lie commutation relations remains open. It is solved only for some special cases of non-Lie algebras and some special classes of non-Lie algebras.

Personal contribution of the author to the development of the problem

In the works of the author of the dissertation research, several classes of non-Lie algebras are identified that allow the construction of a complete theory of irreducible representations. For these algebras, Casimir elements are found, irreducible representations are constructed in spaces of antiholomorphic functions, coherent states corresponding to them, reproducing kernels, reproducing measures; correspondences of the constructed quantum representations with classical symplectic leaves in a Poisson manifold are established, complex structures are built on them; connection between irreducible representations, coherent states and hypergeometric or elliptic functions is revealed.

For some basic quantum models (in particular, for the hydrogen atom and for the Dirac monopole in uniform magnetic and inhomogeneous electric fields, for Penning traps of various configurations), non-Lie symmetry algebras have been identified and studied in detail. For them, irreducible representations and families of coherent states are constructed, which are then used to calculate the asymptotics of the eigenvalues and construct an integral representation of asymptotics of the eigenfunctions of the corresponding spectral problems.

Most of the results of the dissertation work were published in joint works with M. V. Karasev (see below for a list of published articles with the results of the dissertation); some results were published in collaboration with E. V. Vyborny and O. V. Blagodyreva (see the same list). Results belonging to the co-authors, for example, M. V. Karasev's formulas for the Weyl and Wick products, E. V. Vyborny's formulas for tunneling spectrum splitting, are not included in the list of results submitted for defense. The exceptions are the formulations of problems proposed by M. V. Karasev related to the physical models studied in these works, the idea of consistent application of quantum averaging and coherent transform, as well as the idea of reduction (averaging) of coherent states.

The main results submitted for defense, in particular, the construction of coherent states, belong to the author.

Research methods

One of the main methods that makes it possible to study quantum non-integrable systems near equilibrium points or invariant subspaces is *operator averaging followed by reduction into the algebra of integrals of motion* of the model leading part of the Hamiltonian.

The quantum models considered in the dissertation are characterized by the presence of a rich symmetry algebra in the higher part of the Hamiltonian. Moreover, the relationships in it are usually nontrivial. This algebra is of non-Lie type, that is, it cannot be represented as a finite-dimensional Lie algebra; its natural generators satisfy nonlinear (e.g., polynomial) commutation relations. To analyze this type of algebra, in this study, new *methods for constructing irreducible representations and coherent states* were developed and applied to write the averaged Hamiltonian in the form of a differential operator (in the space of antiholomorphic functions over the corresponding symplectic leaf). Note that, in some of the quantum systems under consideration, the degeneracy of the spectrum of the main term of the Hamiltonian is not completely removed in the subprincipal term of perturbation theory, and then it is also necessary to consider a secondary symmetry algebra, also of a non-Lie type, for which it is again necessary to construct all the required objects of the theory of representations and coherent states.

The analysis of reduced Hamiltonians on a given algebra is carried out using *coherent transforms* over a quantum leaf or geometric coherent transforms over a Lagrangian submanifold in a leaf. Last method gives a geometrically invariant description of the semiclassical approximation in spaces with a general symplectic structure, i.e., generalizes the well-known Maslov canonical operator method [80, 81, 82].

The quantum methods (on which this research is based) follow the classical methods of mechanics, associated with normal forms and averaging; see [83, 84, 85]. The concepts of noncommutative algebras and quantization theory used follow the methods presented, for example, in the books [15, 11].

In this study, the algebraic averaging technique is used to work in commutator algebras with non-Lie relations. In this case, all calculations are performed in parallel both in the quantum version and in the classical approximation (at the level of Poisson brackets instead of commutators). This circumstance is fundamentally important from the point of view of the applicability of the semiclassical approximation to reduced the Hamiltonians.

For systems with multi-frequency resonance, the thesis proposes a new *approach to calculating the coefficients of the averaged Hamiltonian using a twisted product on the symbol space* of differential operators with polynomial coefficients.

Irreducible representations and coherent states of non-Lie symmetry algebras can be used to calculate the semiclassical asymptotics of the spectrum and eigenstates of the original Hamiltonian *through geometric objects* (Kähler form, reproducing measure, trajectories of an averaged or doubly averaged Hamiltonian system on quantum symplectic leaves).

The thesis also proposes a new *approach to solving spectral problems with a continuous spectrum.* It consists in using a coherent transform, the integral kernel of which is not ordinary coherent states, but coherent distributions that have all the key properties of coherent states, but do not have a finite norm in the Hilbert space.

Main results submitted for defense

1. For a special "basic" class of algebras generated by non-Lie commutation relations and possessing a "creation-annihilation" structure, a method was developed for constructing irreducible representations (in Hilbert spaces of antiholomorphic distributions), coherent states and reproducing kernels. In the case of regular commutation relations, a correspondence is established between irreducible representations of quantum algebra and symplectic leaves of the Poisson algebra. Relations between irreducible representations and hypergeometric functions are obtained.

2. The method for constructing irreducible representations, coherent states and reproducing kernels has also been developed for several different generalizations of the basic class of algebras with non-Lie commutation relations. Generalizations concern the complication of commutation relations, as well as an increase in the number of algebra generators.

3. A class of non-Lie algebras whose symplectic leaves are a cylinder or a torus is identified and studied. For such algebras, we constructed coherent transforms and irreducible representations corresponding to complex structures on the cylinder and torus. The reproducing kernels of Hilbert spaces (in which irreducible representations are realized) and the coherent transforms are represented through the Riemann theta function and its modifications. The corresponding reproducing measures are found.

4. We identified a class of non-Lie commutation relations that can be represented by point operators (that is, operators whose integral kernels are generalized functions with point supports). All operator irreducible representations are constructed for such relations. They are realized by point operators in Hilbert spaces of antiholomorphic functions. It is shown that the reproducing kernels of these spaces are expressed through hypergeometric series, the theta function, as well as their modifications. We constructed coherent states that intertwine abstract representations of relations with irreducible ones.

5. The developed methods for constructing irreducible representations and coherent states are applied to a number of well-known algebras: the simplest Lie algebras, quadratic algebras of the Zeeman effect and degenerate Sklyanin–Faddeev algebra. Using the developed reduction method, we calculated the coherent states of the eight-dimensional quadratic algebra arising from the Kustaanheimo spinor regularization of the hydrogen atom problem.

6. Algebras with polynomial commutation relations were identified and studied for a quantum particle in electric and magnetic fields. Namely, the following quantum models were studied: a charged particle in the Coulomb–Dirac field, the Zeeman effect in the Coulomb–Dirac field, the Zeeman–Stark effect for the hydrogen atom. For these systems, we performed quantum averaging with reduction into a polynomial symmetry algebra of the principal or subprincipal part of the Hamiltonian and we studied the averaged Hamiltonians. The asymptotic expressions of the eigenvalues and the asymptotic eigenfunctions are found.

7. We identified and studied "resonant" algebras describing irreducible (with pairwise coprime frequencies) elliptic resonance. For them, we found a finite set of generators subject to polynomial commutation relations and we constructed all irreducible representations and the corresponding coherent states and reproducing kernels, as well as reproducing measures.

8. A complete theory of irreducible representations, families of coherent states, reproducing kernels and reproducing measures were constructed for algebras describing the three-frequency reducible resonance in the elliptic and hyperbolic cases of resonance.

9. Resonant algebras with polynomial commutation relations were identified and studied for quantum models with resonance in the principal part of the Hamiltonian. Namely, traps of charged particles with partial (two-frequency) and full (three-frequency) hyperbolic resonance were studied: a cubic Penning-Ioffe trap and planar Penning and Penning–Ioffe traps with round and rectangular electrodes. For the listed systems, we performed quantum averaging with reduction into the algebra of hyperbolic resonance and we studied the averaged Hamiltonians. The asymptotic expressions of the eigenvalues and the asymptotic eigenfunctions are found.

10. A new approach was developed to the procedure of quantum averaging of the Hamiltonian of a resonant harmonic oscillator perturbed by a differential operator with polynomial coefficients. This approach is applied to the spectral problem for a cylindrical Penning trap.

11. A coherent transform was constructed, the integral kernel of which is the family of coherent Schwartz distributions of the Heisenberg algebra. This transform is applied to the spectral problem of an inverted oscillator.

Scientific novelty

In the dissertation work, basic and completely new constructions of irreducible representations and coherent states of some classes of algebras with non-Lie commutation relations having the 'creation-annihilation" structure, as well as some generalizations of these classes of algebras, were obtained.

New original results are obtained concerning the general properties of non-Lie resonant symmetry algebras arising from frequency resonances in the elliptic and hyperbolic cases.

A completely new method for constructing irreducible representations and coherent states of resonant algebras has been developed.

An averaging method followed by reduction into a symmetry algebra of the leading part of the system operator and further study of the reduced operator using the theory of representations of algebras with non-Lie commutation relations is developed.

For the first time, algebras with polynomial commutation relations, which arise as symmetry algebras in basic models of mathematical physics that describe the motion of a charged particle in a Coulomb or Coulomb–Dirac field and weak external electric and magnetic fields of various configurations, are identified.

For a charged particle in a Coulomb or Coulomb–Dirac field and weak external electric and magnetic fields (of various configurations), the averaged Hamiltonian was obtained and studied for the first time, presented as a function of the symmetry algebra generators of the highest part of the Hamiltonian. Formulas for the asymptotic behavior of eigenvalues and asymptotic eigenfunctions, written as integrals of coherent states, are also new.

Similar new results were obtained for spectral problems associated with resonant Penning and Penning–Ioffe traps. These basic trap-type quantum nanosystems cannot be treated by traditional approaches due to the infinite degeneracy of the spectrum of the leading part. For these models, resonant symmetry algebras with polynomial commutation relations are identified for the first time. Expressions for the averaged Hamiltonian are obtained through the generators of these algebras. New formulas are derived for the asymptotic behavior of eigenvalues and asymptotic eigenfunctions in terms of coherent states of resonant algebras.

Theoretical and practical significance

The theoretical significance of the work lies in the creation of new approaches to the study of algebras of non-Lie type, the development for them of new constructions of irreducible representations and coherent states.

The practical significance is that the developed technique of irreducible representations and coherent states, as well as the developed technique of quantum averaging, can be successfully applied to the study of quantum models with strong degeneracy of the spectrum of the higher part of the operator, when standard approaches do not work.

Degree of reliability

All results that are submitted for defense are presented with detailed proofs in 30 articles, of which
– 2 articles were published in the AMS Translations series of books published by the American
Mathematical Society,
– 21 articles were published in peer-reviewed journals, cited in the WoS and Scopus databases, including 11 articles in quartiles Q1–Q2,
– 7 articles in the journal from the Higher Attestation Commission list.

List of articles with the results of the dissertation

The list of published thirty articles, which reflect the main scientific results of the thesis, is divided into two parts.

Main list of articles with dissertation results

1a. M.V. Karasev, E.M. Novikova
Non-Lie permutation representations, coherent states, and quantum embedding.
In: “Coherent transform, Quantization, and Poisson Geometry (M.V. Karasev editor)”, AMS Translations, AMS, Providence, 1998, **187**, 1–202.

2a. M.V. Karasev, E.M. Novikova
Algebras with polynomial commutation relations for a quantum particle in electric and magnetic fields.
In: “Quantum Algebras and Poisson Geometry in Mathematical Physics (M.V. Karasev editor)”. Advances in Modern Mathematics. AMS, Providence, 2005, **216**, 19–135.

3a. E.M. Novikova
Minimal basis of the symmetry algebra for three-frequency resonance.
Russian Journal of Mathematical Physics, 2009,**16**(4), 518–528.

4a. M.V. Karasev, E.M. Novikova
Algebra and quantum geometry of multifrequency resonance.
Izvestiya: Mathematics, 2010, **74**(6), 1155–1204.

5a. O.V. Blagodyreva, M.V. Karasev, E.M. Novikova
Cubic Algebra and Averaged Hamiltonian for the Resonance 3:(-1) Penning-Ioffe Trap.
Russian Journal of Mathematical Physics, 2012, **19**(4), 441–450.

6a. M.V. Karasev, E.M. Novikova
Secondary Resonances in Penning Traps. Non-Lie Symmetry Algebras and Quantum States.
Russian Journal of Mathematical Physics, 2013, **20**(1), 283–294.

7a. M.V. Karasev, E.M. Novikova
Planar Penning trap with combined resonance and top dynamics on quadratic algebra.
Russian Journal of Mathematical Physics, 2015, **22**(4), 463–468.

8a. M.V. Karasev, E.M. Novikova, E.V. Vybornyi
Bi-states and 2-level systems in rectangular Penning traps.
Russian Journal of Mathematical Physics, 2017, **24**(4), 454–464.

9a. M.V. Karasev, E.M. Novikova
Algebra of Symmetries of Three-Frequency Resonance: Reduction of a Reducible Case to an Irreducible Case.
Mathematical notes, 2018, **104**(5-6), 833–847.

10a. E.M. Novikova
Algebra of Symmetries of Three-Frequency Hyperbolic Resonance.
Mathematical notes, 2019, **106**(6), 940–956.

11a. E.M. Novikova
On calculating the coefficients in the quantum averaging procedure for the Hamiltonian of the resonance harmonic oscillator perturbed by a differential operator with polynomial coefficients.
Russian Journal of Mathematical Physics, 2021, **28**(3), 406–410.

12a. E.M. Novikova
New Approach to the Procedure of Quantum Averaging for the Hamiltonian of a Resonance Harmonic Oscillator with Polynomial Perturbation for the Example of the Spectral Problem for the Cylindrical Penning Trap.
Mathematical Notes, 2021, **109**(5), Pages 777–793.

13a. E.M. Novikova
Coherent Schwartz distributions of the Heisenberg algebra and inverted oscillator.
Journal of Mathematical Physics, 2022, **63**, 123507.

Additional list of articles with dissertation results

14b. M.V. Karasev, E.M. Novikova
Coherent transform of the spectral problem and algebras with nonlinear commutation relations.
Journal of Mathematical Sciences, 1999, **95**(6), 2703–2798.

15b. M.V. Karasev, E.M. Novikova
Coherent Transforms and Irreducible Representations Corresponding to Complex Structures on a Cylinder and on a Torus.
Mathematical Notes, 2001, **70**(6), 779–797.

16b. M.V. Karasev, E.M. Novikova
Nonlinear Commutation Relations: Representations by Point-Supported Operators.
Mathematical Notes, 2002, **72**(1), 48–65.

17b. M.V. Karasev, E.M. Novikova
Algebra with Quadratic Commutation Relations for an Axially Perturbed Coulomb-Dirac Field.
Theoretical and Mathematical Physics, 2004, **141**(3), 1698–1724.

18b. M.V. Karasev, E.M. Novikova
Algebra with polynomial commutation relations for the Zeeman effect in the Coulomb-Dirac field.
Theoretical and Mathematical Physics, 2005, **142**(1), 109–127.

19b. M.V. Karasev, E.M. Novikova
Algebra with polynomial commutation relations for the Zeeman-Stark effect in the hydrogen atom.
Theoretical and Mathematical Physics, 2005,**142**(3), Pages 447–469.

20b. M.V. Karasev, E.M. Novikova
Eigenstates of the quantum Penning-Ioffe nanotrap at resonance.
Theoretical and Mathematical Physics, 2014, **179**(3), 729–746.

21b. M.V. Karasev, E.M. Novikova
Inserted perturbations generating asymptotical integrability.
Mathematical notes, 2014, **96**(6), 965–970.

22b. M.V. Karasev, E.M. Novikova, E.V. Vybornyi
Non-Lie Top tunneling and Quantum bilocalization in Planar Penning Trap.
Mathematical notes, 2016, **100**(6), 807–819.

23b. M.V. Karasev, E.M. Novikova, E.V. Vybornyi
Instantons via breaking geometric symmetry in hyperbolic traps.
Mathematical notes, 2017, **102**(5-6), 776–786.

24c. E.M. Novikova
Algebraic modeling of observables and states for hydrogen-lyke center. I. Quadratic algebra
Nanostuctures. Mathematical physics and Modelling, 2012, **7**(1), 107–124.

25c. E.M. Novikova
Coherent states of cubic non-Lie algebra and spectral problem for hydrogen atom in resonance Zeeman-Stark effect.
Nanostuctures. Mathematical physics and Modelling, 2012, **7**(2), 59–86.

26c. E.M. Novikova
Algebraic modeling of observables and states for hydrogen-lyke center. II. Coherent states.
Nanostuctures. Mathematical physics and Modelling, 2012, **7**(2), 87–102.

27c. O.V. Blagodyreva, M.V. Karasev, E.M. Novikova
Integral representation of eigenstates for 3:(-1) resonance Penning nanotrap.
Nanostuctures. Mathematical physics and Modelling, 2013, **9**(1), 5–18.

28c. M.V. Karasev, E.M. Novikova
Stable two-dimensional tori in Penning trap under a combined frequency resonance.
Nanostuctures. Mathematical physics and Modelling, 2015, **13**(2), 55–92.

29c. E.M. Novikova
Spectral clasters in planar Penning trap with resonance breaking of axial symmetry.
Nanostuctures. Mathematical physics and Modelling, 2016, **15**(2), 75–98.

30c. E.M. Novikova
Resonance planar Penning trap with rectangular electrodes.
Nanostuctures. Mathematical physics and Modelling, 2017, **16**(2), 69–88.

Structure and scope of the dissertation

The dissertation consists of an introduction, six chapters, a conclusion and a list of references (160 items). The volume of the dissertation is 275 pages.

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The content of the work

In the introduction, we describe the goals and objectives of the research, the degree of development of the problem, the relevance of the dissertation work, we indicate the author's personal contribution to the development of the problem, we describe the research methods, scientific novelty, theoretical and practical significance of the work, and formulate the main results submitted for defense. The introduction also contains a list of the author's publications on the research topic.

Chapter 1 of the dissertation consists of eight sections. Here we consider nonlinear permutation relations with the structure "creation-annihilation".

In section 1.1 (paper [1a], part I) we study the "basic" nonlinear commutation relations with one creation operator $\mathbf{B}$ and one annihilation operator $\mathbf{C}$:

$$\begin{gathered}[\mathbf{C},\mathbf{B}] = f(\mathbf{A}), \qquad \mathbf{CA} = \varphi(\mathbf{A})\mathbf{C}, \qquad \mathbf{AB} = \mathbf{B}\varphi(\mathbf{A}),\\ [\mathbf{A}_\mu,\mathbf{A}_\nu] = 0 \qquad (\mu,\nu = 1,\dots,k).\end{gathered} \tag{1}$$

Structure functions $f : \mathbb{R}^k \to \mathbb{R}$ and $\varphi_\mu : \mathbb{R}^k \to \mathbb{R}$ $(\mu = 1,\dots,k)$, for simplicity, are considered polynomial.

Note that the permutation relations (1) (with various structural polynomials) describe symmetry algebras in the problems of the hydrogen atom and the Dirac monopole in weak external electric and magnetic fields and are used to study these quantum models in the dissertation work (in Chapter 3). Therefore, we will dwell on them in more detail than on the other classes of permutation relations studied in Chapter 1.

For algebra (1), we construct all (up to equivalent) irreducible representations satisfying the Hermitian conditions

$$\mathbf{B}^* = \mathbf{C}, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \qquad (\mu = 1,\dots,k) \tag{2}$$

in Hilbert spaces of antiholomorphic generalized functions with a vacuum vector

$$\mathbf{C}\mathfrak{P}_0 = 0, \qquad \mathbf{A}_\mu\mathfrak{P}_0 = a_\mu\mathfrak{P}_0 \quad (\mu = 1,\dots,k); \qquad \|\mathfrak{P}_0\| = 1; \tag{3}$$

we obtain the corresponding coherent states and reproducing kernels of spaces of irreducible representations; we find orthonormal bases in spaces of irreducible representations and defin coherent transforms that intertwine abstract representations with irreducible ones.

All of the above designs use the following basic functions:

$$\mathcal{A}_a(n) \stackrel{\text{def}}{=} \underbrace{\varphi\big(\dots\big(\varphi\big(\varphi}_{n}(a)\big)\big)\big), \qquad \mathcal{F}_a(n) \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{j=0}^{n} f\big(\mathcal{A}_a(j)\big). \tag{4}$$

Theorem 1. (a) *There is a one-to-one correspondence between the set of irreducible Hermitian representations of algebra* (1) *that possess a vacuum vector* (3) *and the following subset* $\mathcal{R} \subset \mathbb{R}^k$:

$$a \in \mathcal{R} \;\Leftrightarrow\; \{\mathcal{F}_a(n) > 0 \textit{ for } n \in \mathbb{Z}_+ \textit{ or } \\ \exists N \in \mathbb{Z}_+ : \mathcal{F}_a(n) > 0 \textit{ for } 0 \le n < N \textit{ and } \mathcal{F}_a(N) = 0 \text{ (property (*))}\}. \tag{5}$$

Such a representation corresponding to $a \in \mathcal{R}$ *is finite-dimensional if and only if there exists an integer* $N = N_a$ *satisfying the property* (*) *in* (5); *in this case, the number* $N+1$ *is the dimension of the representation.*

(b) *Let* $a \in \mathcal{R}$. *Let functions* $\mathcal{B}_a$ *and* $\mathcal{C}_a$ *(real or complex) factorize the basic function* $\mathcal{F}_a$:

$$\mathcal{F}_a(n) = \mathcal{B}_a(n)\mathcal{C}_a(n), \qquad n \ge 0, \tag{6}$$

with the single condition $\mathcal{B}_a(N) = 0$ *in the case* (*) *in* (5). *Then the operators*

$$\overset{\circ}{B} = \overline{z}\,\mathcal{B}_a\Big(\overline{z}\frac{d}{d\overline{z}}\Big), \qquad \overset{\circ}{C} = \mathcal{C}_a\Big(\overline{z}\frac{d}{d\overline{z}}\Big)\frac{d}{d\overline{z}}, \qquad \overset{\circ}{A} = \mathcal{A}_a\Big(\overline{z}\frac{d}{d\overline{z}}\Big) \tag{7}$$

define an Hermitian representation of the algebra (1) *in the Hilbert space of of anti-holomorphic distributions that that can be represented by power series* $g(\overline{z}) = \sum_{n=0}^{\infty} g_n \overline{z}^n$ *with the inner product*

$$(g, g')_{\mathcal{P}_{s(a)}} \overset{\text{def}}{=} \sum_{n=0}^{\infty} s_n(a)\, g_n\, \overline{g'_n}, \tag{8}$$

where

$$\begin{aligned} s_0(a) &= 1, \\ s_n(a) &= \frac{n!\,\mathcal{F}_a(n-1)\dots\mathcal{F}_a(0)}{|\mathcal{B}_a(n-1)|^2\dots|\mathcal{B}_a(0)|^2} \qquad &&\textit{for} \quad 1 \le n \le N, \\ s_n(a) &= \infty &&\textit{for} \quad n \ge N+1. \end{aligned}$$

The representation (7) *is irreducible and possesses the vacuum* 1 *in the Hilbert space* $\mathcal{P}_{s(a)}$.

(c) *Representations* (7) *assigned to different vectors* $a \in \mathcal{R}$ *are not equivalent, but for each chosen* $a \in \mathcal{R}$, *representations assigned to different factorizations* (6) *are equivalent.*

(d) *An abstract Hermitian representation of algebra* (1), (2) *in a Hilbert space* H_a *with the vacuum vector* $\mathfrak{P}_0 = \mathfrak{P}_0(a)$, *satisfying* (3) *for some* $a \in \mathbb{R}^k$, *can be intertwined with representation* (7) *by means of the following generalized coherent states*:

$$\mathfrak{P}_z = \mathfrak{P}_o + \sum_{n\ge1} \frac{1}{n!\,\overline{\mathcal{C}}_a(n-1)\dots\overline{\mathcal{C}}_a(0)} (z\mathbf{B})^n\,\mathfrak{P}_0. \tag{9}$$

The generalized "reproducing kernel," corresponding to the (9)

$$\mathcal{K}_{s(a)}(z,\overline{w}) = (\mathfrak{P}_z, \mathfrak{P}_w)_{H_a}, \tag{10}$$

is the kernel of the unity operator in $\mathcal{P}_{s(a)}$ *and is performed by the following distribution over* $\mathbb{R}^2 \times \mathbb{R}^2$:

$$\mathcal{K}_{s(a)}(z,\overline{w}) = \sum_{n\ge 0} \frac{(z\overline{w})^n}{s_n(a)}. \tag{11}$$

(g) *The following orthonormal bases*

$$\left\{ \frac{\overline{z}^n}{\sqrt{s_n(a)}} \,\middle|\, n \ge 0 \right\} \text{ в } \mathcal{P}_{s(a)} \quad \textit{and} \quad \{\mathfrak{P}_0\} \cup \left\{ \frac{1}{\sqrt{s_n(a)\mathcal{B}_a(n-1)\dots\mathcal{B}_a(0)}} \mathbf{B}^n \mathfrak{P}_0 \,\middle|\, n \ge 1 \right\} \text{ в } H_a$$

correspond to each other under the coherent transform

$$j : g \mapsto \mathfrak{p}, \qquad (\mathfrak{p}, \mathfrak{p}')_{H_a} = \big(g, (\mathfrak{p}', \mathfrak{P})_{H_a}\big)_{\mathcal{P}_{s(a)}}, \qquad \forall\, \mathfrak{p}' \in H_a,$$

$$j^{-1} : \mathfrak{p} \mapsto g, \qquad g = (\mathfrak{p}, \mathfrak{P})_{H_a}.$$

They are eigenbases for the commuting operators $\overset{\circ}{A}_1, \dots, \overset{\circ}{A}_k, \overset{\circ}{B}\overset{\circ}{C}$ (*in the space* $\mathcal{P}_{s(a)}$) *or* $\mathbf{A}_1, \dots, \mathbf{A}_k, \mathbf{BC}$ (*in the space* H_a); *the corresponding eigenvalues are equal to* $(\mathcal{A}_a)_1(n), \dots, (\mathcal{A}_a)_k(n), n\mathcal{F}_a(n),\ 0 \le n \le N$.

For commutation relations (1), the thesis also describes two types of Casimir elements. If a function κ on $\mathbb{R}^k$ is preserved by the mapping φ, то $\kappa(\mathbf{A})$, then $\kappa(\mathbf{A})$ is a Casimir element of algebra (1); if a function ρ on $\mathbb{R}^k$ satisfies the equation

$$\rho\big(\varphi(A)\big) - \rho(A) = f(A), \qquad A \in \mathbb{R}^k, \tag{12}$$

then

$$\mathbf{K} = \mathbf{BC} - \rho(\mathbf{A}) \tag{13}$$

is a Casimir element of algebra (1).

Section 1.2 (paper [1a], part I) examines the "Floquet generalization"

$$\begin{gathered} \mathbf{CB} = \mathbf{B}\,\omega(\mathbf{A})\,\mathbf{C} + f(\mathbf{A}), \\ \mathbf{CA} = \varphi(\mathbf{A})\,\mathbf{C} + \psi(\mathbf{A}), \qquad \mathbf{AB} = \mathbf{B}\,\varphi(\mathbf{A}) + \psi(\mathbf{A}), \\ [\mathbf{A}_\mu, \mathbf{A}_\nu] = 0 \qquad (\mu, \nu = 1, \dots, k), \\ \mathbf{B}^* = \mathbf{C}, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \qquad (\mu = 1, \dots, k) \end{gathered} \tag{14}$$

of basic relations (1), (2) with real structure functions $\omega : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R}^k \to \mathbb{R}$, $\varphi : \mathbb{R}^k \to \mathbb{R}^k$, $\psi : \mathbb{R}^k \to \mathbb{R}^k$, satisfying the generalized Jacobi identities (see [7, 13] or [11])

$$\begin{aligned} &\text{(a)} \qquad \big(\varphi_\mu(A) - A_\mu\big)\,\psi_\nu(A) = \big(\varphi_\nu(A) - A_\nu\big)\,\psi_\mu(A), \\ &\text{(b)} \qquad \psi_\mu\big(\varphi(A)\big) = \omega(A)\,\langle \psi(A), \delta\varphi_\mu\big(\varphi(A), A\big)\rangle, \end{aligned}$$

where the vector-valued difference differentiation δ is defined by the formula $\delta F(A, A') \stackrel{\text{def}}{=} \int_0^1 \partial F/\partial A\big(\tau A + (1-\tau)A'\big)\,d\tau$, and $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors in $\mathbb{R}^k$.

For relations (14), as well as for basic relations (1), (2), all irreducible representations in Hilbert spaces of antiholomorphic functions with a vacuum vector, the corresponding coherent states and reproducing kernels are constructed, and the Casimir elements are indicated.

In section 1.3 (article [1a], part I) for the basic relations (1), (2) we formulated conditions for the structural polynomials f and φ_μ ($\mu = 1, \ldots, k$), for which relations (1) admit irreducible representations by differential (rather than pseudodifferential) operators. It is shown that coherent states and reproducing kernels corresponding to such representations are expressed through hypergeometric functions. The parameters of hypergeometric functions are the roots of the basic functions $\mathcal{F}_a$ (4).

In the same section 1.3, we formulated conditions on the structure functions of basic relations (1), (2), as well as on the structure functions of their Floquet generalization (14), under which these relations admit irreducible representations of q-differentialdifferential - operators, i.e. polynomials in q-differentiation and q^{-1}-differentiation operators with polynomial coefficients. It is shown that coherent states and reproducing kernels corresponding to irreducible representations by q-differential operators are expressed through q-hypergeometric functions. In this case, the parameters of q-hypergeometric functions are related to the roots of the base function $\mathcal{F}_a$ (4).

In section 1.4 (article [1a], part I) from the base class of nonlinear relations (1) we identified a subclass of "regular" relations

$$[\mathbf{C}, \mathbf{B}] = \rho\big(\varphi^\hbar(\mathbf{A})\big) - \rho(\mathbf{A}), \qquad \mathbf{C}\mathbf{A} = \varphi^\hbar(\mathbf{A})\mathbf{C}, \qquad \mathbf{A}\mathbf{B} = \mathbf{B}\varphi^\hbar(\mathbf{A}). \tag{15}$$

Compared to relations (1), here, firstly, the semiclassical parameter $\hbar$ is introduced; secondly, the mapping $\varphi^\hbar$ is defined as the shift in time $\hbar$ along the trajectories of the vector field $v = \sum_{\mu=1}^k v_\mu(A)\ ,\partial/\partial A_\mu$ on $\mathbb{R}^k$, fibering $\mathbb{R}^k$; thirdly, equation (12) was used. For regular relations (15) (in addition to the main ones), a number of additional results were obtained. Let us describe the main one - the construction of a reproducing measure for the irreducible representation (7).

For relations (15), the basic functions (4) have the form $\mathcal{A}_a(n) = \mathfrak{A}_a(\hbar n)$, $\mathcal{F}_a(n) = \rho\big(\mathfrak{A}_a(\hbar n)\big) - \rho(a)$, where $\mathfrak{A}_A(t)$ denotes the trajectory of the field v, passing through the point $A \in \mathbb{R}^k$:

$$\frac{d}{dt}\mathfrak{A}_A = v(\mathfrak{A}_A), \qquad \mathfrak{A}_A\Big|_{t=0} = A.$$

Therefore, the factors $\mathcal{B}_a$ and $\mathcal{C}_a$, which factorize the base function $\mathcal{F}_a$ into (6), can be chosen in the form $\mathcal{B}_a(n) = \mathcal{D}_a(\mathfrak{A}_a(\hbar(n+1)))$, $\mathcal{C}_a(n) = \mathcal{E}_a(\mathfrak{A}_a(\hbar(n+1)))/(n+1)$, where

$$\mathcal{D}_a(A)\,\mathcal{E}_a(A) = \rho(A) - \rho(a), \qquad \mathcal{E}_a(a) = 0, \qquad \mathcal{D}_a\big(\mathfrak{A}_a(\hbar(N+1))\big) = 0$$

(the number N is defined in (5)). In this case, the reproducing kernel (11) will be written as follows:

$$\mathcal{K}(z,\overline{w}) = 1 + \sum_{n=1}^{N} \frac{(z\overline{w})^n}{\mathcal{H}(\hbar n)\dots\mathcal{H}(\hbar)}, \qquad \text{where} \quad \mathcal{H}(t) \stackrel{def}{=} \frac{\overline{\mathcal{E}_a}(\mathfrak{A}_a(t))}{\mathcal{D}_a(\mathfrak{A}_a(t))}.$$

Note that by choosing the factors $\mathcal{D}_a$ and $\mathcal{E}_a$ it is possible to ensure a fairly rapid growth of the numerical sequence $\mathcal{H}(\hbar n)$ (with the help of which the coefficients of the power law are specified series for $\mathcal{K}(z,\overline{w})$). Therefore, without loss of generality, we can assume that the function $\mathcal{K}(z,\overline{w})$ is analytic in z and $\overline{w}$ on the entire plane $\mathbb{C}$.

Theorem 2. (a) *Let ℓ be a smooth solution of the equation*

$$\mathcal{H}\Big(-\hbar x\frac{d}{dx}\Big)\ell(x) = x\ell(x), \qquad x > 0,$$

such that $\int_0^\infty x^n|\ell(x)|\,dx < \infty$ for $0 \le n \le N$. We normalize ℓ by the following condition: $\frac{1}{\hbar}\int_0^\infty \ell(x)\,dx = 1$. Then for any generalized functions g, $g' \in \mathcal{P}$ the scalar product (8) *can be written in integral form*

$$(g,g')_{\mathcal{P}_{s(a)}} = \frac{1}{2\pi\hbar}\int_{\mathbb{C}} g(\overline{z})\,\overline{g'(\overline{z})}\,\ell(|z|^2)\,dzd\overline{z},$$

where $dzd\overline{z} = dxd\varphi$ u $z = \sqrt{x}\exp\{i\varphi\}$. Therefore, in this case, all elements from $\mathcal{P}$ are regular L^2-functions on the plane.

(b) *If the function ℓ from item* (a) *exists, then the reproducing kernel $\mathcal{K}(z,\overline{w})$ satisfies the reproducing property* :

$$\frac{1}{2\pi\hbar}\int_{C} \mathcal{K}(z,\overline{z}')\,\mathcal{K}(z',\overline{z})\,\ell(|z'|^2)dz'd\overline{z}' = \mathcal{K}(z,\overline{z}),$$

i.e., there is a decomposition of unity

$$\frac{1}{2\pi\hbar}\int_\Omega \frac{\mathcal{K}(z,\overline{z}')\mathcal{K}(z',\overline{z})}{\mathcal{K}(z,\overline{z})\mathcal{K}(z',\overline{z}')}\,dm(z',\overline{z}')=1$$

with reproducing measure $dm(z,\overline{z})=M(|z|^2)dzd\overline{z}$*, where* $M(x)\overset{def}{=}k(x)\ell(x)$.

Further, in the thesis it is shown that for relations (15) there is a Casimir element $\mathbf{K}$ (13), and there are also independent Casimir elements $\kappa_1(\mathbf{A}),\dots,\kappa_{k-1}(\mathbf{A})$ (since there are independent real smooth functions $\kappa_1,\dots,\kappa_{k-1}$ on $\mathbb{R}^k$, which are preserved under the mapping $\varphi^\hbar$). In the irreducible representation (7), the Casimir operators are scalar. More precisely, conditions (3) imply the equalities

$$\mathbf{K}=-\rho(a)\cdot\mathbf{I},\qquad \kappa_j(\mathbf{A})=\kappa_j(a)\cdot\mathbf{I}\quad (j=1,\dots,k-1).$$

Using Hermitian generators $\mathbf{S}_1=(\mathbf{B}+\mathbf{C})/2$, $\mathbf{S}_2=i(\mathbf{B}-\mathbf{C})/2$, these equalities can be rewritten in the form

$$\mathbf{S}_1^2+\mathbf{S}_2^2=\frac{1}{2}\Big(\rho(\mathbf{A})+\rho\big(\varphi^\hbar(\mathbf{A})\big)\Big)-\rho(a)\cdot\mathbf{I},\quad \kappa_j(\mathbf{A})=\kappa_j(a)\cdot\mathbf{I}\quad (j=1,\dots,k-1)$$

and interpreted as the equations of the symplectic leaf corresponding to irreducible representation (7) associated with $a\in\mathcal{R}$. This leaf is a surface (of revolution) embedded in the space $\mathbb{R}^{k+2}$ with classical coordinates $S_1,S_2,A_1,\dots,A_k$.

For regular relations (15), we established a correspondence between quantum objects (irreducible representations, coherent states and reproducing kernels) and classical objects (symplectic leaves of the corresponding Poisson algebra). It is shown that the symplectic leaves corresponding to the constructed irreducible representations with the vacuum vector (3) are surfaces of revolution. In addition, in the regular case, operators of complex structure are obtained, and also the semiclassical asymptotics of the quantum K?hler potential and the density of reproducing measures were calculated.

In sections 1.5 and 1.6 we study generalizations of relations (1) and (14) to the case of several pairs of creation-annihilation operators:

$$\begin{gathered}
\mathbf{C}^q\mathbf{B}_p=\sum_{r,s}\mathbf{B}_r\omega_{sp}^{qr}(\mathbf{A})\mathbf{C}^s+f_p^q(\mathbf{A})\qquad (p,q=1,\dots,d),\\
\mathbf{C}^p\mathbf{A}=\varphi_p(\mathbf{A})\mathbf{C}^p,\qquad \mathbf{A}\mathbf{B}_p=\mathbf{B}_p\varphi_p(\mathbf{A})\qquad (p=1,\dots,d),\\
[\mathbf{A}_\mu,\mathbf{A}_\nu]=0\qquad (\mu,\nu=1,\dots,k),\\
[\mathbf{B}_p,\mathbf{B}_q]=0,\quad [\mathbf{C}^p,\mathbf{C}^q]=0\qquad (p,q=1,\dots,d),\\
\mathbf{B}_p^*=\mathbf{C}^p\quad (p=1,\dots,d),\qquad \mathbf{A}_\mu^*=\mathbf{A}_\mu\quad (\mu=1,\dots,k).
\end{gathered}\tag{16}$$

Here the functions φ_p are vector-valued. We assume that all structure functions are real and satisfy the generalized Jacobi conditions:

$$
\begin{aligned}
&\text{(a)}\quad \varphi_p\big(\varphi_q(A)\big) = \varphi_q\big(\varphi_p(A)\big),\\
&\text{(b)}\quad f_p^q(A)\big(\varphi_p(A) - \varphi_q(A)\big) = 0,\\
&\text{(c)}\quad \omega_{sp}^{qr}(A)\big(\,\varphi_q\big(\varphi_r(A)\big) - \varphi_p\big(\varphi_s(A)\big)\,\big) = 0,\\
&\text{(d)}\quad \sum_{s=1}^{d} \omega_{sp}^{qr}(A) f_t^s(A) + \delta_t^r f_p^q\big(\varphi_t(A)\big) = \sum_{s=1}^{d} \omega_{st}^{qr}(A) f_p^s(A) + \delta_p^r f_t^q\big(\varphi_p(A)\big),\\
&\text{(e)}\quad \sum_{s=1}^{d} \omega_{sp}^{qr}\big(\varphi_t(A)\big)\omega_{uv}^{st}(A) + \sum_{s=1}^{d} \omega_{sp}^{qt}\big(\varphi_r(A)\big)\omega_{uv}^{sr}(A)\\
&\qquad\quad = \sum_{s=1}^{d} \omega_{sv}^{qr}\big(\varphi_t(A)\big)\omega_{up}^{st}(A) + \sum_{s=1}^{d} \omega_{sv}^{qt}\big(\varphi_r(A)\big)\omega_{up}^{sr}(A).
\end{aligned}
\tag{17}
$$

For simplicity, we consider the case when all ω_{sp}^{qr}, f_p^q and φ_p are polynomials.

From identities (17b), (17c) it follows that (by renumbering the generators B and synchronously renumbering the generators C) matrix $F = (\!(f_s^r)\!)$ and matrices $\Omega_p^q = (\!(\omega_{sp}^{qr})\!)$ can be reduced to block-diagonal form:

$$
f_s^r = 0 \;\text{ for }\; \varphi_r \neq \varphi_s; \qquad \omega_{sp}^{qr} = 0 \;\text{ for }\; \varphi_p = \varphi_q, \quad \varphi_r \neq \varphi_s.
$$

In Section 1.5 we study the case of two one-dimensional blocks (i.e. $d = 2$, $\varphi_1 \neq \varphi_2$):

$$
\begin{aligned}
&\mathbf{C}^1\mathbf{B}_1 = \mathbf{B}_1\alpha_1^1(\mathbf{A})\mathbf{C}^1 + \mathbf{B}_2\alpha_1^2(\mathbf{A})\mathbf{C}^2 + f_1^1(\mathbf{A}),\\
&\mathbf{C}^2\mathbf{B}_2 = \mathbf{B}_2\alpha_2^2(\mathbf{A})\mathbf{C}^2 + \mathbf{B}_1\alpha_2^1(\mathbf{A})\mathbf{C}^1 + f_2^2(\mathbf{A}),\\
&\mathbf{C}^1\mathbf{B}_2 = \mathbf{B}_2\theta(\mathbf{A})\mathbf{C}^1, \qquad \mathbf{C}^2\mathbf{B}_1 = \mathbf{B}_1\theta(\mathbf{A})\mathbf{C}^2,\\
&\mathbf{C}^1\mathbf{A} = \varphi_1(\mathbf{A})\mathbf{C}^1, \qquad \mathbf{A}\mathbf{B}_1 = \mathbf{B}_1\varphi_1(\mathbf{A}),\\
&\mathbf{C}^2\mathbf{A} = \varphi_2(\mathbf{A})\mathbf{C}^2, \qquad \mathbf{A}\mathbf{B}_2 = \mathbf{B}_2\varphi_2(\mathbf{A}),\\
&[\mathbf{A}_\mu, \mathbf{A}_\nu] = 0, \qquad [\mathbf{B}_1, \mathbf{B}_2] = 0, \qquad [\mathbf{C}^1, \mathbf{C}^2] = 0,\\
&\mathbf{B}_1^* = \mathbf{C}^1, \qquad \mathbf{B}_2^* = \mathbf{C}^2, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \quad (\mu = 1, \dots, k).
\end{aligned}
\tag{16$'$}
$$

For relations (16′) in this work we constructed all irreducible Hermitian representations (up to equivalent ones) in Hilbert spaces of antiholomorphic functions over $\mathbb{R}^4$ with a vacuum vector, obtained the corresponding coherent states and reproducing kernels of spaces of irreducible representations; found orthonormal bases in spaces of irreducible representations and defined coherent transforms that intertwine abstract representations with irreducible ones. As in the one-dimensional case, the construction uses some basic functions, which are specified by explicit formulas through the structure functions of the relations (16′).

In the case when the relations ($16'$) are regular, Casimir elements are found for them. In an irreducible representation, these elements are scalar and define the equations of symplectic leaves. It is shown that the symplectic leaves, corresponding to the constructed irreducible representations, are 4-dimensional birotation surfaces, i.e. these leaves are 4x-dimensional submanifolds in $\mathbb{R}^{k+4}$ fibered by tori.

In Section 1.6 relations (16) are first considered in the case of one multidimensional block (i.e. in the case of $d \geq 2$, $\varphi_1 = \cdots = \varphi_k$) under the condition $\omega_{sp}^{qr} = \omega \cdot \delta_s^q \cdot \delta_p^r$:

$$\begin{aligned}
&\mathbf{C}^q\mathbf{B}_p = \mathbf{B}_p\omega(A)\mathbf{C}^q + f_p^q(\mathbf{A}),\\
&\mathbf{C}^p\mathbf{A} = \varphi(\mathbf{A})\mathbf{C}^p, \qquad \mathbf{A}\mathbf{B}_p = \mathbf{B}_p\varphi(\mathbf{A}),\\
&[\mathbf{A}_\mu, \mathbf{A}_\nu] = 0, \qquad [\mathbf{B}_p, \mathbf{B}_q] = 0, \qquad [\mathbf{C}^p, \mathbf{C}^q] = 0,\\
&\mathbf{B}_p^* = \mathbf{C}^p, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \qquad (p, q = 1, \ldots, d, \quad \mu, \nu = 1, \ldots, k).
\end{aligned} \tag{16''}$$

where the structure functions ω, φ are real-valued, matrix $F(A) = \big(\!\big(f_p^q(A)\big)\!\big) = F^*(A)$ Hermitian, $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible mapping, and the Jacobi identity holds: $F\big(\varphi(A)\big) = \omega(A) \cdot F(A)$.

For the relations ($16''$) in the work we described the Casimir elements. In the regular case, the symplectic leaves $\Omega = \Omega^{2d}$ algebras ($16''$) are embedded in $\mathbb{R}^{k+2d}$ as hypersurfaces.

If the structure function ω does not vanish, then the algebra ($16''$) can be reduced to the Heisenberg algebra. Therefore, the most interesting case is when ω has zeros on $\mathbb{R}^k$.

If $d = 2$, then the symplectic leaves are four-dimensional. In this case, we constructed all (up to equivalent) irreducible Hermitian representations of relations ($16''$) in Hilbert spaces of antiholomorphic generalized functions over $\mathbb{R}^4$ with a vacuum vector, obtained the corresponding coherent states, the reproducing kernels of spaces of irreducible representations and orthonormal bases in them, and also determined coherent transforms, intertwining abstract representations with irreducible ones.

The design uses a basic matrix, which is specified by an explicit formula through the structure functions of the relations ($16''$). Reproducing kernels are expressed through a function that is introduced in this work and generalizes the hypergeometric function to the case of several variables. To define this function, in Section 1.6 we introduced the concept of a matrix factorial, which generalizes the gamma function to the case of matrices.

At the end of Section 1.6, relations (16) are considered in the case of one two-dimensional block, when the matrix ω_p^q is not scalar:

$$d = 2, \qquad \varphi_p(A) = \varphi(A), \qquad \omega_{sp}^{qr}(A) = \delta_p^r \cdot \delta_s^q \cdot \omega_p^q(A) \qquad (p, q, r, s = 1, 2).$$

Then relations (16) take the form

$$\begin{gathered}
\mathbf{C}^q\mathbf{B}_p = \mathbf{B}_p\,\omega_p^q(\mathbf{A})\,\mathbf{C}^q + f_p^q(\mathbf{A}),\\
\mathbf{C}^p\mathbf{A} = \varphi(\mathbf{A})\mathbf{C}^p, \qquad \mathbf{A}\mathbf{B}_p = \mathbf{B}_p\varphi(\mathbf{A}),\\
[\mathbf{A}_\mu, \mathbf{A}_\nu] = 0, \qquad [\mathbf{B}_p, \mathbf{B}_q] = 0, \qquad [\mathbf{C}^p, \mathbf{C}^q] = 0,\\
\mathbf{B}_p^* = \mathbf{C}^p, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \qquad (p, q = 1, 2, \quad \mu, \nu = 1, \dots, k)
\end{gathered} \tag{16'''}$$

with the following Jacobian conditions on structure functions:

$$\begin{gathered}
\overline{\varphi} = \varphi, \qquad \overline{\omega_p^q} = \omega_p^q = \omega_q^p, \qquad \overline{f_p^q} = f_q^p,\\
\omega_p^q\big(\varphi(A)\big)\omega_t^q(A) = \omega_t^q\big(\varphi(A)\big)\omega_p^q(A),\\
f_p^q\big(\varphi(A)\big) = \omega_t^q(A)\cdot f_p^q(A), \qquad t \neq p,\\
\omega_1^1 \neq \omega_2^2 \quad \Longrightarrow \quad f_2^1 = f_1^2 = 0.
\end{gathered}$$

For the relations (16'''), the construction of Hermitian representations with a vacuum vector is described and an interesting formula for coherent states is obtained. In this formula, the exponential of the linear combination of creation operators $\mathbf{B}_p$ is applied to the vacuum vector, and the coefficients of this linear combination are some differential operators acting on the unit function according to the complex parameters of the family of coherent states.

In the mentioned sections 1.1 - 1.4 we assume that in the Hilbert representation space, the spectrum of the operator **BC** contains a zero eigenvalue. The corresponding eigenvector is vacuum. And **in section 1.7** (paper [15b]) we study the case when neither the operator **BC** nor the operator **CB** has a zero eigenvalue. The corresponding symplectic leaves are a cylinder or a torus.

More precisely, **in section 1.7** for the relations

$$\begin{gathered}
\mathbf{C}\cdot\mathbf{B} = \varphi_\hbar^0(\mathbf{BC}, \mathbf{A}), \qquad \mathbf{C}\cdot\mathbf{A} = \varphi_\hbar(\mathbf{BC}, \mathbf{A})\cdot\mathbf{C}, \qquad \mathbf{A}_j\cdot\mathbf{A}_l = \mathbf{A}_l\cdot\mathbf{A}_j,\\
\mathbf{B}^* = \mathbf{C}, \qquad \mathbf{A}_j^* = \mathbf{A}_j \qquad (j, l = 1, \dots, k)
\end{gathered} \tag{17}$$

we found the Casimir elements, constructed all operator-irreducible representations in which the operators **BC** and **CB** do not have a zero eigenvalue, constructed complex structures on the cylinder and torus, obtained coherent states in Hilbert spaces (without a vacuum vector) and reproducing kernels, and also found reproducing measures. Coherent states and reproducing kernels are here expressed through the theta function.

In section 1.8 (article [16b]) we identified a special special case of commutation relations (14):

$$\begin{gathered}
\mathbf{C}\mathbf{B} = q\mathbf{B}\,\mathbf{C} + Q(\mathbf{A}_1, \dots, \mathbf{A}_k),\\
\mathbf{A}_\mu \mathbf{B} = \mathbf{B}\,(q_\mu \mathbf{A}_\mu + Q_\mu(\mathbf{A}_{\mu+1}, \dots, \mathbf{A}_k)),\\
\mathbf{C}\mathbf{A}_\mu = (q_\mu \mathbf{A}_\mu + Q_\mu(\mathbf{A}_{\mu+1}, \dots, \mathbf{A}_k))\,\mathbf{C},\\
\mathbf{A}_\mu \mathbf{A}_\nu = \mathbf{A}_\nu \mathbf{A}_\mu,\\
\mathbf{B}^* = \mathbf{C}, \qquad \mathbf{A}_\mu^* = \mathbf{A}_\mu \qquad (\mu,\, \nu \in \{1, \dots, k\}),
\end{gathered} \tag{18}$$

in which q, $q_\mu \in \mathbb{R}$ are non-zero constants, $Q : \mathbb{R}^k \to \mathbb{R}$, $Q_\mu : \mathbb{R}^{k-\mu} \to \mathbb{R}$ are polynomials, and $Q_k = const$.

This special case has the following remarkable property: every operator irreducible representation in which the commutative sublgebra $\mathbf{BC}$, $\mathbf{A}_1, \dots, \mathbf{A}_k$ has a non-empty point spectrum, can be realized in the Hilbert space of antiholomorphic functions by point operators rather than pseudodifferential ones general position operators. (A linear integral operator is called pointwise if its kernel is a generalized function with pointwise support.) The work classifies and describes all possible series of such representations and shows that the corresponding coherent states and reproducing kernels can be represented through hypergeometric series, the theta function, as well as their modifications.

Chapter 2 of the dissertation consists of seven sections, in which the developed methods for constructing irreducible representations and coherent states are applied to a number of well-known algebras: the simplest Lie algebras, the quadratic algebra of the Zeeman effect and the degenerate Sklyanin-Faddeev algebra. Using the developed reduction method, we also constructed coherent states of the eight-dimensional quadratic algebra arising from the Kustaanheimo spinor regularization of the hydrogen atom problem.

In section 2.1 (paper [1a], part II) we showed how the developed methods work in the case of Lie algebras su(2):

$$[\mathbf{S}_1, \mathbf{S}_2] = i\hbar\,\mathbf{S}_3, \quad [\mathbf{S}_2, \mathbf{S}_3] = i\hbar\,\mathbf{S}_1, \quad [\mathbf{S}_3, \mathbf{S}_1] = i\hbar\,\mathbf{S}_2, \quad \mathbf{S}_j^* = \mathbf{S}_j \ (j = 1, 2, 3) \tag{19}$$

and su(1,1):

$$[\mathbf{S}_1, \mathbf{S}_2] = i\hbar\,\mathbf{S}_3, \quad [\mathbf{S}_2, \mathbf{S}_3] = -i\hbar\,\mathbf{S}_1, \quad [\mathbf{S}_3, \mathbf{S}_1] = -i\hbar\,\mathbf{S}_2, \quad \mathbf{S}_j^* = \mathbf{S}_j \ (j = 1, 2, 3). \tag{20}$$

Previously, instead of Hermitian generators, we defined the operators of creation $\mathbf{B} = \mathbf{S}_1 - i\mathbf{S}_2$, annihilation $\mathbf{C} = \mathbf{S}_1 + i\mathbf{S}_2$ and the operator $\mathbf{A} = \mathbf{S}_3$. Then relations (19) and (20) are reduced to the form (1), (2), and then the constructions described in Chapter 1 are used for them. As a result, we obtained irreducible representations, coherent states, reproducing kernels, reproducing measures, coherent transforms and complex structure operators.

It is interesting that using the described method for the algebra su(1,1) it is possible to construct two (equivalent) versions of irreducible representations by differential operators: a standard representation by first-order operators and another representation in which the creation operator is of zero order and the annihilation operator is of second order. For this representation, in contrast to the standard one, the reproducing measure exists for all possible eigenvalues of the operator $\mathbf{A}$ on the vacuum vector. The work also constructs a coherent transform that intertwineы two variants of representations of the algebra su(1,1).

In Section 2.2 (paper [1a], part II; paper [14b]) we studied quadratic algebra related to the Zeeman effect. This algebra was discovered in [63]. It is defined as an algebra of operators commuting simultaneously with the regularized Hamiltonian of the hydrogen atom

$$\mathbf{S}_0 = |\mathbf{q}|\Big(\frac{1}{4}+\mathbf{p}^2\Big), \qquad \text{где} \quad \mathbf{q}=q, \quad \mathbf{p}=-i\hbar\frac{\partial}{\partial q}, \qquad q\in\mathbb{R}^3$$

and with the angular momentum component $\mathbf{M}_3$ commuting with it $\mathbf{M}=\mathbf{q}\times\mathbf{p}$. These operators are self-adjoint in the Hilbert space $L^2_-(\mathbb{R}^3)$ with the norm $\|\varphi\|_- = \big((\pi/4)\int_{\mathbb{R}^3}|\varphi(q)|^2dq/|q|\big)^{1/2}$.
The algebra of their joint symmetries is formed by four generators, satisfying the following relations [63]

$$\begin{aligned}
&[\mathbf{T}_1,\mathbf{T}_2] = i\hbar\,\mathbf{T}_0\mathbf{T}_3, && [\mathbf{T}_0,\mathbf{T}_1] = 2i\hbar\,\mathbf{T}_2,\\
&[\mathbf{T}_2,\mathbf{T}_3] = -\frac{i\hbar}{2}\big(\mathbf{T}_0\mathbf{T}_1+\mathbf{T}_1\mathbf{T}_0\big), && [\mathbf{T}_0,\mathbf{T}_2] = -2i\hbar\,\mathbf{T}_1,\\
&[\mathbf{T}_3,\mathbf{T}_1] = -\frac{i\hbar}{2}\big(\mathbf{T}_0\mathbf{T}_2+\mathbf{T}_2\mathbf{T}_0\big), && [\mathbf{T}_0,\mathbf{T}_3] = 0.\\
&\qquad\qquad \mathbf{T}_j^* = \mathbf{T}_j \qquad (j=0,1,2,3).
\end{aligned} \tag{21}$$

First of all, for this algebra we considered those values of the Casimir operators that are implemented in this quantum model. For such values, in [14b] not only irreducible representations, coherent states, reproducing kernels and reproducing measures in spaces of antiholomorphic functions were constructed, but also representations and coherent states over Lagrangian curves on symplectic leaves. The representation operators for such geometric coherent states have a simple geometric meaning (see Proposition 3.1 in [14b]), due to which they are very convenient to use for constructing the semiclassical asymptotics of the eigenvalues and eigenfunctions of the Hamiltonian.

In [1a] for algebra (21), we studied all possible values of the Casimir elements corresponding to representations with a vacuum vector (and not just those values that are realized for the corresponding physical model). For this purpose, the structure “creation-annihilation” is introduced in algebra (21) $\mathbf{B}=\mathbf{T}_1-i\mathbf{T}_2$, $\mathbf{C}=\mathbf{T}_1+i\mathbf{T}_2$, $\mathbf{A}_1=-\mathbf{T}_0$, $\mathbf{A}_2=\mathbf{T}_3$, and algebra (21) is presented as a special case of algebra (1) with $k=2$, $f(A)=-2\hbar A_1A_2$, $\varphi_1(A)=A_1+2\hbar$, $\varphi_2(A)=A_2-\hbar A_1-\hbar^2$. All irreducible Hermitian representations with a vacuum vector in spaces of antiholomorphic functions are constructed. Moreover, for each set of values of Casimir elements, all possible variants of (equivalent) representations by differential (rather than pseudodifferential) operators are written out. For each such representation, coherent states and reproducing kernels are constructed, and the question of the existence of a reproducing measure is investigated and, in cases where this is possible, the measure is also obtained. Coherent states, reproducing kernels and measures are expressed here through hypergeometric functions.

In addition, the corresponding classical algebra is studied and a complete description of its symplectic leaves, as well as complex structures on symplectic leaves corresponding to irreducible representations, is given.

In Section 2.3 (paper [1a], part II) the degenerate case of the Faddeev–Sklyanin algebra is studied. This is an algebra with four generators $\mathbf{S}_0$, $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ with quadratic commutation relations

$$\begin{aligned} &[\mathbf{S}_0, \mathbf{S}_1] = i\mu(\mathbf{S}_2\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_2), &&[\mathbf{S}_1, \mathbf{S}_2] = i\hbar(\mathbf{S}_0\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_0),\\ &[\mathbf{S}_0, \mathbf{S}_2] = -i\mu(\mathbf{S}_1\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_1), &&[\mathbf{S}_2, \mathbf{S}_3] = i\mu(\mathbf{S}_0\mathbf{S}_1 + \mathbf{S}_1\mathbf{S}_0),\\ &[\mathbf{S}_0, \mathbf{S}_3] = 0, &&[\mathbf{S}_3, \mathbf{S}_1] = i\mu(\mathbf{S}_0\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_0),\\ &\qquad\qquad \mathbf{S}_j^* = \mathbf{S}_j, \qquad j = 0, 1, 2, 3. \end{aligned} \tag{22}$$

Here μ and $\hbar$ are parameters, $1 > \mu > 0$, $\hbar > 0$. Note that the relations (22) cannot be represented by differential operators.

In algebra (22) we introduced the structure "creation–annihilation" $\mathbf{B} = \mathbf{S}_1 - i\mathbf{S}_2$, $\mathbf{C} = \mathbf{S}_1 + i\mathbf{S}_2$, $\mathbf{A}_1 = \mathbf{S}_3 + \mathbf{S}_0$, $\mathbf{A}_2 = \mathbf{S}_3 - \mathbf{S}_0$, and wrote down relations (22) to the form (1), (2) с $k = 2$, $f(A) = \hbar(A_1^2 - A_2^2)$, $\varphi_1(A) = qA_1$, $\varphi_2(A) = A_2/q$, where $q = (1-\mu)/(1+\mu)$ is a new parameter, $0 < q < 1$.

In this work, representations with discrete spectrum $\mathbf{S}_0$ and $\mathbf{S}_3$ are studied. For such representations there is a vacuum vector (3). Therefore, such representations can be searched using the developed scheme.

For algebra (22), Casimir elements are found and all irreducible Hermitian representations with a vacuum vector in spaces of antiholomorphic functions are constructed. Moreover, for each set of values of Casimir elements, three variants of (equivalent) representations by q-differential operators are written. For each such representation, coherent states and reproducing kernels are constructed, complex structure operators are written, and reproducing measures are found for one of the three variants considered. Coherent states and reproducing kernels are here expressed through q-hypergeometric series.

In addition, the corresponding classical algebra is studied and a complete description of its symplectic leaves is given, as well as complex structures on symplectic leaves corresponding to irreducible representations.

In section 2.4 (article [1a], part II) we consider weakly nonlinear commutation relations

$$
\begin{gathered}
[\mathbf{C}^q,\mathbf{B}_p]=\sum_{\alpha=1}^{\ell} f^{\alpha q}_{\ \ p}\mathbf{R}_\alpha+f^q_p(\mathbf{A}),\\
[\mathbf{R}_\alpha,\mathbf{B}_p]=\sum_{r=1}^{d}\mathbf{B}_r\psi_{\alpha p}^{\ \ r},\qquad [\mathbf{C}^q,\mathbf{R}_\alpha]=\sum_{r=1}^{d}\psi_{\alpha r}^{\ \ q}\mathbf{C}^r,\\
\mathbf{A}\mathbf{B}_p=\mathbf{B}_p\varphi(\mathbf{A}),\qquad\qquad \mathbf{C}^p\mathbf{A}=\varphi(\mathbf{A})\mathbf{C}^p,\\
[\mathbf{R}_\alpha,\mathbf{R}_\beta]=\sum_{\gamma=1}^{\ell}\chi^\gamma_{\alpha\beta}\mathbf{R}_\gamma,\qquad [\mathbf{A}_\mu,\mathbf{R}_\alpha]=0,\\
[\mathbf{A}_\mu,\mathbf{A}_\nu]=0,\qquad [\mathbf{B}_p,\mathbf{B}_q]=0,\qquad [\mathbf{C}^p,\mathbf{C}^q]=0,\\
\mathbf{B}_p^*=\mathbf{C}^p,\qquad \mathbf{R}_\alpha^*=\mathbf{R}_\alpha,\qquad \mathbf{A}_\mu^*=\mathbf{A}_\mu\\
\qquad\qquad p,q=1,\dots,d,\quad \mu,\nu=1,\dots,k,\quad \alpha,\beta=1,\dots,\ell.
\end{gathered}
\tag{23}
$$

where $F^\alpha=(\!(f^{\alpha q}_{\ \ p})\!)$, $\Psi_\alpha=(\!(\psi_{\alpha p}^{\ \ q})\!)$, $X^\alpha=(\!(\chi^\alpha_{\beta\gamma})\!)$ are matrices whose elements are either constant or are φ-invariant functions of A, i.e.

$$
f^{\alpha q}_{\ \ p}\big(\varphi(A)\big)=f^{\alpha q}_{\ \ p}(A),\qquad \psi_{\alpha p}^{\ \ q}\big(\varphi(A)\big)=\psi_{\alpha p}^{\ \ q}(A),\qquad \chi^\alpha_{\beta\gamma}\big(\varphi(A)\big)=\chi^\alpha_{\beta\gamma}(A).
$$

It is assumed that the number of pairs of creation-annihilation operators is $d>1$, and that the structure constants and structure functions in (23) correspond to the Hermitian conditions $\varphi=\overline{\varphi}$, $F=F^*$, $F^\alpha=(F^\alpha)^*$, $\Psi_\alpha=\Psi_\alpha^*$, $X^\alpha=-\overline{X^\alpha}=(X^\alpha)^*$, where $F=(\!(f^q_p)\!)$, and also the following generalized Jacobi conditions are satisfied:

$$
\begin{gathered}
\sum_{\varepsilon=1}^{\ell}(\chi^\varepsilon_{\alpha\beta}\chi^\delta_{\varepsilon\gamma}+\chi^\varepsilon_{\beta\gamma}\chi^\delta_{\varepsilon\alpha}+\chi^\varepsilon_{\gamma\alpha}\chi^\delta_{\varepsilon\beta})=0\qquad(\alpha,\beta,\gamma,\delta=1,\dots,\ell);\\
[\Psi_\alpha,\Psi_\beta]=\sum_\gamma\chi^\gamma_{\alpha\beta}\Psi_\gamma\qquad(\alpha,\beta=1,\dots,\ell);\\
\sum_{\alpha=1}^{\ell}(f^{\alpha q}_{\ \ r}\psi_{\alpha p}^{\ \ s}-f^{\alpha q}_{\ \ p}\psi_{\alpha s}^{\ \ r})=\delta^s_r\big(f^q_p(\varphi(A))-f^q_p(A)\big)-\delta^s_p\big(f^q_r(\varphi(A))-f^q_r(A)\big)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(p,q,r,s=1,\dots,d);
\end{gathered}
$$

$$
\begin{gathered}
[F^\alpha,\Psi_\beta]=\sum_{\gamma=1}^{\ell}\chi^\alpha_{\beta\gamma}F^\gamma\qquad(\alpha,\beta=1,\dots,\ell);\\
[F(A),\Psi_\alpha]=0\qquad(\alpha=1,\dots,\ell).
\end{gathered}
$$

Under such assumptions for relations (23), we constructed representations in spaces of antiholomorphic functions (of d complex variables) with a vacuum vector $\mathfrak{P}_0$, subject to the conditions

$$\begin{gathered}\mathbf{A}_\mu \mathfrak{P}_0 = a_\mu \mathfrak{P}_0 \quad (\mu = 1, \dots, k), \quad \mathbf{R}_\alpha \mathfrak{P}_0 = 0 \quad (\alpha = 1, \dots, \ell), \\ \mathbf{C}^q \mathfrak{P}_0 = 0 \quad (q = 1, \dots, d), \qquad \|\mathfrak{P}_0\| = 1.\end{gathered} \tag{24}$$

In addition, we obtained coherent states that intertwine abstract representations with a vacuum vector with the constructed anti-holomorphic representations.

Under some additional assumptions, in the space of the Hermitian representation with a vacuum vector there is a second vacuum vector satisfying conditions (24). In this case, coherent states with two vacuum vectors are constructed for algebra (23). The corresponding representation space is decomposed into a direct sum of two irreducible components, each of which is generated from its own vacuum vector.

In Section 2.5 (paper [1a], part II) we consider an algebra with eight generators subject to the following quadratic relations

$$\begin{gathered}[\boldsymbol{\rho}_p, \boldsymbol{\rho}_q] = 0, \quad [\boldsymbol{\rho}_p, \boldsymbol{\sigma}_q] = -i\hbar\,(\delta_{pq}\boldsymbol{\rho}^2 - \boldsymbol{\rho}_p\boldsymbol{\rho}_q), \quad [\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_q] = -i\hbar\,(\boldsymbol{\sigma}_p\boldsymbol{\rho}_q - \boldsymbol{\sigma}_q\boldsymbol{\rho}_p), \\ \boldsymbol{\rho}_p^* = \boldsymbol{\rho}_p, \quad \boldsymbol{\sigma}_p^* = \boldsymbol{\sigma}_p \qquad\qquad (p, q = 0, 1, 2, 3).\end{gathered} \tag{25}$$

Here $\rho^2 \stackrel{def}{=} \sum_{p=0}^3 \rho_p^2$, $\hbar > 0$. The Jacobian conditions for these relations are satisfied automatically.

The algebra (25) has two Casimir elements $\mathbf{K}_1 = \boldsymbol{\rho}^2$, $\mathbf{K}_2 = \big(\langle \boldsymbol{\rho}, \boldsymbol{\sigma}\rangle + \langle \boldsymbol{\sigma}, \boldsymbol{\rho}\rangle\big)/2$, where $\langle \rho, \sigma\rangle \stackrel{def}{=} \sum_{p=0}^3 \rho_p\sigma_p$. In our thesis we constructed a representation of algebra (25), in which $\mathbf{K}_1 = \mathbf{I}$, $\mathbf{K}_2 = 0$. These equations can be understood as the equations of the symplectic leaf corresponding to the irreducible representation of relations (25), i.e. like equations $\{\rho^2 = 1, \langle \rho, \sigma\rangle = 0\}$ of surface embedded in space $\mathbb{R}^8$ with classical coordinates ρ, σ. This surface is diffeomorphic to $T^*\mathbb{S}^3$.

First, with the help of some transform, relations (25) are reduced to weakly nonlinear relations (23), in which $d = 3$, $k = 1$, $f^{rq}_{\ p} = -2i\hbar\,\varepsilon_{pqr}$, $f^q_p(A) = 2\hbar\,\delta^q_p A$, $\psi_{pq}^{\ r} = i\hbar\,\varepsilon_{pqr}$, $\chi_{pq}^{\ r} = i\hbar\,\varepsilon_{pqr}$, $\varphi(A) = A + \hbar$, and for such relations (using the construction from Section 2.4) an anti-holomorphic representation with two vacuum vectors is constructed, as well as coherent states, a reproducing kernel and a reproducing measure. Note that the coherent states and the reproducing kernel are here expressed through the zero-order Bessel function, and the density of the reproducing measure through the zero-order Macdonald function. Then, using the inverse transform, we derive the antiholomorphic representation, coherent states, reproducing kernel and reproducing measure for the original quadratic algebra (25).

In section 2.6 (article [1a], part II) we considered an example of reduction of coherent states by the symmetry group. This approach was proposed in the work of [8]. It was with the help of this approach that representations of quadratic algebras (21) and (25) were first obtained, as well as the corresponding coherent states in the works [63], [64], [**?**], [10].

The reduction starts with the usual Heisenberg algebra and standard Gaussian coherent states over $\mathbb{R}^n$. In this case, two types of reduction are possible.

The reduction of the first type is performed in the space of an irreducible representation of operator algebra. With such a reduction, the coherent states of a given algebra are projected onto the eigenspace of some of its elements (called a reduction generator). The result is a new family of coherent states corresponding to subalgebra of operators commuting with the reduction generator.

Reduction of the second type is performed in the space of parameters of coherent states, more precisely, in the space antiholomorphic representation of operator algebra. In this case, we consider both the reduction generator and its symbol, which is element of the corresponding Poisson algebra. "New" coherent states are obtained from "old" coherent states by averaging (over parameters) along the trajectories of the Hamiltonian field of this symbol.

We use an operator of type "action" as a reduction generator. This operator, divided by the semiclassical parameter $\hbar$, has an integer spectrum, and its symbol defines a 2π-periodic Hamiltonian flow. Under such conditions, the reduction (of the first type) of coherent states in the space of an irreducible representation is equivalent to the reduction (of the second type) in terms of parameters. It is convenient to keep in mind both of these types of reduction and study some properties using the first type reduction, and others using the second type of reduction.

The two types of reduction described in Section 2.6 apply to Gaussian coherent states of the Heisenberg algebra for dimension $n = 4$. The operator associated with the Kustaanheimo spinor regularization is taken as a reduction generator. As a result of the reduction, we obtain operators of the antiholomorphic representation, coherent states, a reproducing kernel, and a reproducing measure for algebra (25). In the semiclassical limit, the constructed quantum objects correspond to a symplectic leaf of algebra (25), diffeomorphic to $T^*\mathbb{S}^3$.

The reduction method described in Section 2.6 is used in Chapter 4 when calculating reproducing measures for symmetry algebras of multi-frequency resonance.

In section 2.7 (paper [15b]) we give two examples of constructing irreducible representations and coherent states, corresponding to the cylinder and torus. The scheme from section 1.7 is applied here.

The first example is the algebra $su(1, 1)$. For it in Section 2.1 we constructed representations corresponding to symplectic leaves diffeomorphic to the plane; in the space of each such representation there is a vacuum vector annulled by the annihilation operator.

And in Section 2.7, for this algebra, representations corresponding to symplectic leaves diffeomorphic to the cylinder are constructed; these representations are without a vacuum vector. The coherent states and reproducing kernels of such representations are expressed through the theta function.

The second example is the degenerate Sklyanin-Faddeev algebra

$$
\begin{aligned}
&[\mathbf{S}_1, \mathbf{S}_2] = i(\mathbf{S}_0\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_0), \qquad && [\mathbf{S}_0, \mathbf{S}_1] = -i\mu^2(\mathbf{S}_2\mathbf{S}_3 + \mathbf{S}_3\mathbf{S}_2),\\
&[\mathbf{S}_2, \mathbf{S}_3] = i(\mathbf{S}_0\mathbf{S}_1 + \mathbf{S}_1\mathbf{S}_0), \qquad && [\mathbf{S}_0, \mathbf{S}_2] = i\mu^2(\mathbf{S}_3\mathbf{S}_1 + \mathbf{S}_1\mathbf{S}_3),\\
&[\mathbf{S}_3, \mathbf{S}_1] = i(\mathbf{S}_0\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_0), \qquad && [\mathbf{S}_0, \mathbf{S}_3] = 0,
\end{aligned}
\tag{26}
$$

where $\mu > 0$ (the structure constants of this algebra are different from the structure constants of the algebra (22)).

By replacing the structure constant μ with a new constant

$$
q = \frac{1+i\mu}{1-i\mu} = e^{i\varphi}, \qquad \text{where} \quad \mu = \operatorname{tg}\frac{\varphi}{2}, \tag{27}
$$

and introducing new generators $\mathbf{A} = \sqrt{\mu}\mathbf{S}_3 + i\mathbf{S}_0/\sqrt{\mu}$ $\mathbf{B} = \mathbf{S}_1 - i\mathbf{S}_2$, $\mathbf{C} = \mathbf{S}_1 + i\mathbf{S}_2$ (here, to simplify the notation, the non-Hermitian generator $\mathbf{A}$ is used instead of its real and imaginary parts $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$), algebra (26) is reduced to the algebra

$$
\begin{gathered}
[\mathbf{C}, \mathbf{B}] = -i\big(\mathbf{A}^2 - (\mathbf{A}^*)^2\big), \qquad [\mathbf{A}, \mathbf{A}^*] = 0,\\
\mathbf{CA} = q\mathbf{AC}, \qquad \mathbf{AB} = q\mathbf{BA}, \qquad \mathbf{B}^* = \mathbf{C}
\end{gathered}
$$

of type (17), where $\hbar = 1$ and

$$
\varphi_t^0(A_0, A) = A_0 + \frac{\overline{q}(q^{2t}-1)A^2 + q(\overline{q}^{2t}-1)\overline{A}^2}{i(q-\overline{q})}, \qquad \varphi_t(A_0, A) = q^t A.
$$

For this algebra, according to the scheme from Section 1.7, all operator irreducible representations, coherent states, reproducing kernels and reproducing measures are constructed. The corresponding symplectic leaves are embedded in $\mathbb{R}^4$ as a torus.

We separately studied the case when the structure constant q (27) is the Nth root of 1. In this case, in addition to the “classical” Casimir elements

$$
\mathbf{K} = \mathbf{BC} - \frac{\overline{q}\mathbf{A}^2 + q(\mathbf{A}^*)^2}{i(q-\overline{q})}, \qquad \varkappa = \mathbf{AA}^*,
$$

there are “nonclassical” Casimir elements: $\mathbf{B}^N$, $\mathbf{C}^N$, $\mathbf{A}^N$ and $(\mathbf{A}^*)^N$. The corresponding operator irreducible representations are finite-dimensional.

If $q^N \neq 1$ for no $N \in \mathbb{N}$, then operator irreducible representations are infinite-dimensional (although they correspond to compact symplectic leaves).

Chapter 3 of the thesis consists of six sections and contains a description of a series of works [2a], [14b], [17b], [18b], [19b] on the motion of a charged particle in the Coulomb-Dirac field, which is perturbed by electric and (or) uniform magnetic fields.

The first three sections of Chapter 3 are supporting sections.

Section 3.1 (articles [2a], [17b]) contains the definition of the Hamiltonian $\mathbf{H}_0$ of a particle in the Coulomb-Dirac field, the regularization of this Hamiltonian, i.e. reducing it to the operator $\mathbf{S}_0$ with equidistant spectrum, solving the spectral problem for $\mathbf{S}_0$ (on the negative part of the spectrum), as well as describing the algebra $\mathcal{F}_{quant}$ of quantum integrals of motion of the regularized operator $\mathbf{S}_0$.

In Section 3.2 (articles [2a], [14b], [17b]) for the perturbed operator $\mathbf{S}_0+\varepsilon\mathbf{S}_1$, where $\mathbf{S}_0$ is an operator with equidistant spectrum, ε is a small parameter, we described a scheme of quantum averaging method with reduction into the algebra $\mathcal{F}_{quant}$ of symmetries of the leading part of $\mathbf{S}_0$. Using the above scheme, we can find a unitary transform that reduces the operator $\mathbf{S}_0 + \varepsilon\mathbf{S}_1$ to the form $\mathbf{S}_0 + \varepsilon\underline{\mathbf{T}}$, where the new perturbing operator $\underline{\mathbf{T}} = \varepsilon\underline{\mathbf{S}_1} + \varepsilon^2\underline{\mathbf{S}_2} + \ldots$ (it is called averaged) commutes (in all orders in ε) with the leading part of $\mathbf{S}_0$, i.e. is an element of the algebra $\mathcal{F}_{quant}$ of its symmetries.

Note that if both operators $\mathbf{S}_0$ and $\mathbf{S}_1$ commute with some operator $\mathbf{G}$, then the averaged operator $\underline{\mathbf{T}}$ is element of the subalgebra $\mathcal{G}_{quant}$ of symmetries $\mathbf{G}$ in the algebra $\mathcal{F}_{quant}$.

In Section 3.3 (papers [2a], [14b], [17b]) we described a coherent transform method that allows us to write the averaged perturbation $\underline{\mathbf{T}}$ in an irreducible representation of the algebra $\mathcal{F}_{quant}$ (or its subalgebras $\mathcal{G}_{quant}$) in the form of some operator $\overset{\circ}{\underline{T}}$ in the space of an irreducible representation. In this case, the eigenfunctions of the averaged perturbation $\underline{\mathbf{T}}$ are represented as an integral of the eigenfunctions of the operator $\overset{\circ}{\underline{T}}$ and the coherent states of the algebra $\mathcal{F}_{quant}$ (or its subalgebras $\mathcal{G}_{quant}$).

The next three sections contain solutions of three spectral problems.

In Section 3.4 (articles [2a], [17b]) the problem of a particle in an axially perturbed Coulomb-Dirac field is studied. The presence of axial symmetry leads to the existence of the operator $\mathbf{G}$ (this is the corresponding component of angular momentum), which commutes with the higher part (i.e. the regularized Hamiltonian $\mathbf{S}_0$ of the particle in the Coulomb-Dirac field) and with the perturbation $\mathbf{S}_1$ (describing the external electric potential). In the algebra $\mathcal{F}_{quant}$ of symmetries $\mathbf{S}_0$, the subalgebra $\mathcal{G}_{quant}$ of symmetries of the operator $\mathbf{G}$ is given by quadratic commutation relations

$$\begin{aligned}
&[\mathbf{B}_1,\mathbf{B}_2] = i\hbar(\mathbf{B}_0\mathbf{B}_3+\mu\mathbf{B}_4), && [\mathbf{B}_0,\mathbf{B}_1] = 2i\hbar\,\mathbf{B}_2,\\
&[\mathbf{B}_2,\mathbf{B}_3] = -\frac{i\hbar}{2}\big(\mathbf{B}_0\mathbf{B}_1+\mathbf{B}_1\mathbf{B}_0\big), && [\mathbf{B}_0,\mathbf{B}_2] = -2i\hbar\,\mathbf{B}_1,\\
&[\mathbf{B}_3,\mathbf{B}_1] = -\frac{i\hbar}{2}\big(\mathbf{B}_0\mathbf{B}_2+\mathbf{B}_2\mathbf{B}_0\big), && [\mathbf{B}_0,\mathbf{B}_3] = 0,
\end{aligned}$$
$$[\mathbf{B}_4,\mathbf{B}_0] = [\mathbf{B}_4,\mathbf{B}_1] = [\mathbf{B}_4,\mathbf{B}_2] = [\mathbf{B}_4,\mathbf{B}_3] = 0.$$

For this subalgebra, in the work we constructed all irreducible representations (corresponding to the possible values of the Casimir operators in a given physical model), coherent states, reproducing kernels and reproducing measures. The averaged Hamiltonian is explicitly expressed as a function of the generators of the subalgebra $\mathcal{G}_{quant}$, and its highest term (in terms of the external field value) is written in an irreducible representation in the form of the ordinary differential Hein operator [96] . For the eigenfunctions of the original problem, an integral representation is constructed through polynomial solutions of the Hein equation and coherent states of the subalgebra $\mathcal{G}_{quant}$.

In section 3.5 (articles [2a], [18b]) for a particle in a Coulomb-Dirac field perturbed by a uniform magnetic field, it is shown that the degeneracy of the higher part of the spectrum can be removed only in distant (quadratic or higher) terms of perturbation theory in the magnetic field.

Removal of degeneracy is controlled by dynamic algebra with polynomial commutation relations

$$\begin{aligned}
&[\mathbf{A}_i,\mathbf{A}_j] = 0, && [\mathbf{C},\mathbf{B}] = f(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\mathbf{A}_4),\\
&\mathbf{A}_1\mathbf{B} = \mathbf{B}(\mathbf{A}_1-r\hbar), && \mathbf{C}\mathbf{A}_1 = (\mathbf{A}_1-r\hbar)\mathbf{C},\\
&\mathbf{A}_2\mathbf{B} = \mathbf{B}(\mathbf{A}_2+l\hbar), && \mathbf{C}\mathbf{A}_2 = (\mathbf{A}_2+l\hbar)\mathbf{C},\\
&\mathbf{A}_3\mathbf{B} = \mathbf{B}(\mathbf{A}_3+2r\hbar\mathbf{A}_1-r^2\hbar^2), && \mathbf{C}\mathbf{A}_3 = (\mathbf{A}_3+2r\hbar\mathbf{A}_1-r^2\hbar^2)\mathbf{C},\\
&\mathbf{A}_4\mathbf{B} = \mathbf{B}(\mathbf{A}_4-2l\hbar\mathbf{A}_2-l^2\hbar^2), && \mathbf{C}\mathbf{A}_4 = (\mathbf{A}_4-2l\hbar\mathbf{A}_2-l^2\hbar^2)\mathbf{C},
\end{aligned} \tag{28}$$

where the polynomial f in four variables is defined by the formula

$$\begin{aligned}
&f(A_1,A_2,A_3,A_4) \stackrel{def}{=} \hbar\bigg(r\prod_{q=0}^{l-1}\big(A_4+(2q+1)\hbar A_2-q(q+1)\hbar^2\big)\\
&\times\sum_{j=0}^{r-1}\big(2A_1+(r-1-2j)\hbar\big)\prod_{p=j+1-r}^{j-1}\big(A_3+(2p+1)\hbar A_1-p(p+1)\hbar^2\big)\\
&-l\prod_{p=0}^{r-1}\big(A_3+(2p+1)\hbar A_1-p(p+1)\hbar^2\big)\\
&\times\sum_{j=0}^{l-1}\big(2A_2+(l-1-2j)\hbar\big)\prod_{q=j+1-l}^{j-1}\big(A_4+(2q+1)\hbar A_2-q(q+1)\hbar^2\big)\bigg).
\end{aligned}$$

The structure of algebra (28) is determined by the arithmetic proportion

$$\frac{n+1+\frac{|k|}{2}+\frac{k}{2}}{n+1+\frac{|k|}{2}-\frac{k}{2}}=\frac{l}{r}$$

between the principal quantum number n and quantum number k of magnetic charge. Irreducible representations of dynamic algebra are realized in the space of polynomials and define model differential equations with polynomial solutions that determine the leading term of asymptotics of eigenfunctions in the Zeeman effect.

In section 3.6 (articles [2a], [19b]) we studied the Zeeman-Stark effect in the hydrogen atom (in addition, a non-uniform electric potential may be present). It is shown that in the spectrum of a hydrogen atom, which is placed in crossed electric and magnetic fields (values ε), It is possible that resonant clusters may appear in which the eigenvalues are located at a distance of $O(\varepsilon^2)$ from each other. In the remaining (non-resonant) clusters, the spectral points are separated from each other by a distance of $O(\varepsilon)$. In Section 3.6 we present the main terms of the asymptotic behavior of the eigenvalues and eigenfunctions in both resonant and non-resonant clusters. Resonant clusters are controlled by algebras with polynomial permutation relations (28), which also arise in the problem of the Dirac monopole in a uniform magnetic field. But the parameters l and r of the structure functions of algebra (28) are determined here from the resonance condition

$$\frac{|H-3nE|}{|H+3nE|}=\frac{l}{r}$$

between electric and magnetic fields E and H.

In this work, we constructed irreducible representations of algebra (28) (corresponding to the possible values of the Casimir operators in a given physical model) by differential operators acting on one variable, as well7 as its hypergeometric coherent states. Using these states, the eigenfunctions of the original problem are expressed through solutions of the model ordinary differential equation.

In Chapter 4 (articles [3a], [4a], [9a], [10a]) we study the symmetry algebra of a quantum resonant harmonic oscillator (for brevity it is called “resonant”) in the case of three or more frequencies . This algebra plays an essential role in the study of the dynamics and spectrum of multidimensional physical systems, when states localized near a stable equilibrium position are studied. Resonance algebra is described using a finite number of generators and polynomial relations. For the classical version of this algebra, symplectic leaves and complex structures on them are studied, and for the quantum version, irreducible representations, coherent states, reproducing kernels and reproducing measures are constructed.

In sections 4.1 – 4.7 (articles [3a], [4a]) we study the case of elliptic "irreducible" resonance, when the oscillator frequencies are pairwise relatively prime natural numbers. **In section 4.8** (paper [9a]) we describe the reduction of the case of "reducible" elliptical resonance, where frequencies are not necessarily coprime, to the case of irreducible resonance. And in the last **section 4.9** (article [10a]) we study three-frequency hyperbolic resonance, when two frequencies are positive and the third is negative.

In Section 4.1 (papers [3a], [4a]) in the space $L^2(\mathbb{R}^n)$ we consider the Hamiltonian

$$\hat{H}[f] = \frac{1}{2}\sum_{j=1}^{n}\Big(-\hbar^2\frac{\partial^2}{\partial q_j^2} + f_j^2 q_j^2 - \hbar f_j\Big) \tag{29}$$

of quantum harmonic oscillator with natural pairwise mutually prime frequencies $f_1, \ldots, f_n$. Its symmetries are given by the formulas

$$\hat{S}_j \overset{\text{def}}{=} \hat{z}_j^* \hat{z}_j \quad (j = 1, \ldots, n), \qquad \hat{A}_\rho \overset{\text{def}}{=} \hat{z}^{*\rho_+}\hat{z}^{\rho_-} \quad (\rho \in \mathcal{R}). \tag{30}$$

Here $\hat{z}_j = (\hbar\partial/\partial q_j + f_j q_j)/\sqrt{2f_j}$ are annihilation operators; $\mathcal{R} = \{\rho \in \mathbb{Z}^n \mid \langle f, \rho\rangle = 0\}$ is the set of "resonant" vectors; and for each vector $\rho \in \mathbb{Z}^n$ the following two operations are defined:

$$(\rho_+)_j \overset{\text{def}}{=} \begin{cases} \rho_j, & \rho_j \geq 0, \\ 0, & \rho_j \leq 0, \end{cases}, \qquad (\rho_-)_j \overset{\text{def}}{=} \begin{cases} 0, & \rho_j \geq 0, \\ -\rho_j, & \rho_j \leq 0 \end{cases} \quad (j = 1, \ldots, n). \tag{31}$$

Since the operators $\hat{A}_\rho$ are not independent, the problem arises of selecting a minimal basis from an infinite set of such generators; this problem reduces to the problem of describing the set $\mathcal{M} \subset \mathbb{Z}^n$ of vectors ρ numbering the generators $\hat{A}_\rho$. In the work [97] it was shown that the set $\mathcal{M}$ is finite and consists of "minimal" resonant vectors, which cannot be obtained by adding two non-zero vectors from the intersection of the resonant lattice $mathcal R$ with one of the Cartesian quadrants.

Note that any resonant vector can be decomposed into a sum of minimal ones with non-negative integer coefficients:

$$\sigma = \sum_{\varkappa \in \mathcal{M}_\sigma} n_\varkappa^\sigma \cdot \varkappa, \qquad n_\varkappa^\sigma \in \mathbb{Z}_+, \quad \mathcal{M}_\sigma \subset \mathcal{M}. \tag{32}$$

Here $\mathcal{M}_\sigma$ is the set of minimal vectors in the Cartesian quadrant to which the vector σ belongs.

In the two-frequency case $n = 2$ the description of the set $\mathcal{M}$ is trivial. In the three-frequency case $n = 3$, the problem of explicitly describing the set $\mathcal{M}$ of minimal resonant vectors was solved in [3a].

Theorem 3. *Let $n = 3$, and let frequencies f_1, f_2, f_3 be pairwise relatively prime natural* *numbers. Let a pair of integers μ and ν be a solution to the Diophantine equation $\mu f_1 + \nu f_2 + f_3 = 0$ with the condition $0 \le \nu \le f_1 - 1$.*

Then the set $\mathcal{M}$ of minimal vectors in the resonant lattice $\mathcal{R}$ is the union of the following subsets: $\mathcal{M} = \mathcal{M}^{23} \cup \mathcal{M}^{31} \cup \mathcal{M}^{12} \cup \mathcal{M}^1 \cup \mathcal{M}^2 \cup \mathcal{M}^3$.

If $f_1 = 1$, then $\mathcal{M}^{23}$ consists of two vectors: $(-f_3, 0, f_1)$ and $(-f_2, f_1, 0)$.

If $f_1 \ge 2$, then in addition to these two vectors the subset $\mathcal{M}^{23}$ contains a sequence of vectors $\big(-(lf_3+\nu^{(l)}f_2)/f_1,\ l\nu \,(\bmod f_1),\ l\big)$, $l = 1, \dots, f_1-1$. In this case, the vector with number l is stored in this sequence only on condition that $\nu^{(l)} < \nu^{(j)}$ for all $j = 1, \dots, l-1$.

The minimal vectors of the subsets $\mathcal{M}^{31}$ and $\mathcal{M}^{12}$ are obtained from the previous descriptions of vectors in $\mathcal{M}^{23}$ by cyclic permutation of indices $1, 2, 3$.

The minimal vectors in the subset $\mathcal{M}^j$ have the form $(-\sigma)$, where σ is the minimal vector in the subset $\mathcal{M}^{kl}$, k and l are numbers that complement the number j to the triple of indices $1, 2, 3$.

In section 4.2 (article [4a]) we described the Poisson algebra of symmetries of a resonant oscillator with real generators S_k ($k = 1, \dots, n$) and complex generators A_ρ ($\rho \in \mathcal{M}$), subject to connections of Hermitian, commutative and noncommutative types, as well as polynomial relations with respect to the Poisson bracket.

In Section 4.3 (paper [4a]) we describe the symplectic leaves Ω of this algebra.

Here the concept of a resonant basis is introduced. This is a set of linearly independent minimal resonant vectors $\rho^{(1)}, \dots, \rho^{(n-1)} \in \mathcal{M}$ such that for any resonant vector $\sigma \in \mathcal{R}$ coefficients of its expansion into vectors $\rho^{(k)}$ are integers:

$$\sigma = \sum_{k=1}^{n-1} N_\sigma^{(k)} \rho^{(k)}, \qquad N_\sigma^{(k)} \in \mathbb{Z}. \tag{33}$$

An atlas of maps covering symplectic leaves is described; in each map the corresponding coordinate functions $w_1, \dots, w_{n-1}$ have no singularities; when moving from one map to another, i.e. from a resonant basis $\{\rho^{(j)}\}$ to another resonant basis $\{\tilde{\rho}^{(j)}\}$, these coordinates change $w \to \tilde{w}$ according to the power law. In addition, Darboux coordinates were introduced and the Kähler potential was calculated.

In section 4.4 (article [4a]) for each number $M \in \mathbb{Z}_+$ the Diophantine skeleton is defined $\mathbf{\Delta}[M] \stackrel{\text{def}}{=} \{k \in \mathbb{Z}_+^n \mid \langle f, k\rangle = M\}$, the question of the number of $d[M]$ points in it is considered (i.e., the question of the multiplicity of the eigenvalue $\hbar M$ of the Hamiltonian of the oscillator (29)), and the concept of a vertex $r \in \mathbf{\Delta}[M]$ of a Diophantine skeleton was introduced.

The point $r \in \mathbf{\Delta}[M]$ is called the vertex of the Diophantine skeleton $\mathbf{\Delta}[M]$ if there exists a resonant basis such that for all $l \in \mathbf{\Delta}[M]$ all coefficients of the expansion of the vector $l - r$ in basis vectors are non-negative. The vertex r and the corresponding resonant basis $\{\rho\}$ constitute the frame $R = (r, \{\rho\})$ of the Diophantine skeleton $\mathbf{\Delta}[M]$; rappers R number the maps in the atlas.

In Section 4.5 (article [4a]) for each number $M \in \mathbb{Z}_+$ such that $d(M) \neq 0$, we constructed the Hilbert space $\mathfrak{L}(\Omega)$ antiholomorphic polynomials with inner product

$$(\varphi, \psi) = \frac{1}{(2\pi\hbar)^{n-1}} \int_{\mathbb{C}^{n-1}} \varphi \, \overline{\psi} \, \mathcal{J}_R \, dw \, d\overline{w}. \tag{34}$$

Here w are local complex coordinates in the map with number R (where R is the frame of the Diophantine skeleton $\mathbf{\Delta}[M]$); $M = \langle f, r\rangle$; the density of the $\mathcal{J}_R$ measure is given by the formula

$$\mathcal{J}_R = \frac{S^{r-\Sigma\rho}}{r!\hbar^{|r|}} Q^{[M]}(S), \qquad Q^{[M]}(s) \stackrel{\text{def}}{=} \frac{s_1 \dots s_n}{\hbar} \int_0^\infty y^{M+|f|-1} \exp\Big\{ -\frac{1}{\hbar} \sum_{j=1}^{n-1} s_j y^{f_j} \Big\} \, dy,$$

in which $\sum \rho = \sum_{j=1}^{n-1} \rho^{(j)}$.

In this paper, the reproducing kernel $\mathcal{K}$ of this space is calculated. In local complex coordinates w in the map with number R it is a polynomial

$$\mathcal{K}_R = \sum_{\sigma \in \mathcal{R}_r} \frac{r!}{\hbar^{|\sigma_+|-|\sigma_-|}(r+\sigma)!} \overline{w}^{N_\sigma} w^{N_\sigma}. \tag{35}$$

Here the subset $\mathcal{R}_r \subset \mathcal{R}$ is given by the condition $\sigma \in \mathcal{R}_r \iff (r+\sigma) \in \mathbf{\Delta}[M]$, and the vectors $N_\sigma \in \mathbb{Z}_+^{n-1}$ are determined by expansion (33) of the vectors σ in a resonant basis from the frame R with vertex r.

In section 4.6 (article [4a]) the quantum resonance algebra $\mathcal{A}$ is described. For this purpose, generalized Pochhammer symbols are predefined $(s)_\rho \stackrel{\text{def}}{=} (s_1)_{\rho_1} \dots (s_n)_{\rho_n}$, where for any $a \in \mathbb{R}$, $m \in \mathbb{Z}$:

$$(a)_m \stackrel{\text{def}}{=} \begin{cases} (a+\hbar)\dots(a+m\hbar) & \text{for} \quad m \geq 1, \\ 1 & \text{for} \quad m = 0, \\ a(a-\hbar)\dots(a-\hbar(|m|-1)) & \text{for} \quad m \leq -1. \end{cases}$$

In addition, the following auxiliary operations are introduced on the lattice $\mathbb{Z}^n$ (index j runs through all values $1, \dots, n$):

$$\begin{aligned} &\alpha, \beta \to \alpha \cdot \beta, && (\alpha \cdot \beta)_j \stackrel{\text{def}}{=} \alpha_j \beta_j, \\ &\alpha, \beta \to [\alpha|\beta], && [\alpha|\beta]_j \stackrel{\text{def}}{=} \min\{(\alpha_-)_j, (\beta_+)_j\} - \min\{(\beta_-)_j, (\alpha_+)_j\}, \\ &\alpha, \beta \to [\alpha, \beta], && [\alpha, \beta] \stackrel{\text{def}}{=} \alpha_+ \cdot \beta_- - \alpha_- \cdot \beta_+, \end{aligned}$$

where the operations $\alpha \to \alpha_\pm$ are given in (31). Lattice vectors α, β are considered to commute if their commutator $[\alpha, \beta]$ is equal to zero.

The resonance algebra $\mathcal{A}$ is defined as an algebra with involution, generated by the generators $\mathbf{A}_\sigma$ ($\sigma \in \mathcal{M}$), $\mathbf{S}_j$ ($j = 1, \ldots, n$) and the following constraints and commutation relations.

– Quantum constraints of Hermitian type: $\mathbf{S}_j^* = \mathbf{S}_j$ $(j = 1, \ldots, n)$, $\mathbf{A}_\sigma^* = \mathbf{A}_{-\sigma}$ $(\sigma \in \mathcal{M})$.

– Quantum constraints of commutative type: $\prod_\rho (\mathbf{A}_\rho)^{k_\rho} = \prod_\sigma (\mathbf{A}_\sigma)^{m_\sigma}$ for any families of commuting vectors $\rho, \sigma \in \mathcal{M}$ and numbers $k_\rho, m_\sigma \in \mathbb{N}$ such that $\sum_\rho k_\rho \rho = \sum_\sigma m_\sigma \sigma$.

– Quantum constraints of noncommutative type: if the minimal vectors ρ and σ do not commute and $\rho \neq -\sigma$, then the relation $\mathbf{A}_\rho \mathbf{A}_\sigma = g_{\rho,\sigma}(\mathbf{S}) \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\mathbf{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}$ holds; here $g_{\rho,\sigma}(s) \overset{\text{def}}{=} (s - \hbar\rho)_{[\sigma|\rho]}$, $s \in \mathbb{R}^n$; $n_\varkappa^{\rho+\sigma}$ are the coefficients of decomposition (32) of the vector $\rho + \sigma$ in minimal vectors.

– Commutation relations:

$$[\mathbf{S}_j, \mathbf{S}_k] = 0, \quad [\mathbf{S}_j, \mathbf{A}_\rho] = \hbar \rho_j \mathbf{A}_\rho, \quad [\mathbf{A}_{-\rho}, \mathbf{A}_\rho] = \hbar F_{-\rho,\rho}(\mathbf{S}) \quad (j, k = 1, \ldots, n, \;\; \rho \in \mathcal{M}),$$

where the polynomials $F_{\rho,\sigma}$ are given by the formula $F_{\rho,\sigma} = (g_{\rho,\sigma} - g_{\sigma,\rho})/\hbar$.

In Section 4.7 (article [4a]) irreducible representations and coherent states of the resonance algebra $\mathcal{A}$ are constructed.

In the abstract representation of resonance algebra by the operators $\mathbf{S}_j$ $(j = 1, \ldots, n)$, $\mathbf{A}_\sigma$ $(\sigma \in \mathcal{M})$ in some Hilbert space $\mathcal{H}$ each non-minimal resonant vector σ is associated with the operator

$$\mathbf{A}_\sigma = \begin{cases} \mathbf{I}, & \text{if} \quad \sigma = 0, \\ \prod_{\varkappa \in \mathcal{M}_\sigma} \mathbf{A}_\varkappa^{n_\varkappa^\sigma}, & \text{if} \quad \sigma \neq 0, \end{cases}$$

where the subsets are $\mathcal{M}_\sigma$ and the numbers $n_\varkappa^\sigma$ defined according to (32).

For a given number $M \in \mathbb{Z}_+$ such that $d[M] \neq 0$ it is assumed that for at least one frame $R = (r, \{\rho\})$ in the space $\mathcal{H}$ there exists a normalized "vacuum" vector $\mathfrak{p}_R$ such that

$$\mathbf{A}_\rho \mathfrak{p}_R = 0 \quad (\rho \in \mathcal{R}_R^-), \qquad \mathbf{S}_j \mathfrak{p}_R = \hbar r_j \mathfrak{p}_R \quad (j = 1, \ldots, n), \tag{36}$$

where the subset $\mathcal{R}_R^- \subset \mathcal{R}$ is given by the condition $\rho \in \mathcal{R}_R^- \;\Leftrightarrow\; (r + \sigma) \notin \mathbf{\Delta}[M]$.

Theorem 4. (a) *In a local map with number $R = (r, \{\rho\})$, the coherent states of the algebra $\mathcal{A}$ are given by the formula*

$$\mathfrak{P}_R(w) = \sum_{t \in \mathbf{\Delta}[M]} \sqrt{\frac{\hbar^{|r|} r!}{\hbar^{|t|} t!}} \prod_{k=1}^{n-1} w^{N_{t-r}^{(k)}} \mathfrak{p}^t, \qquad \textit{where } \mathfrak{p}^t \overset{\text{def}}{=} (\hbar r)_{t-r}^{-1/2} \mathbf{A}_{t-r} \mathfrak{p}_R.$$

Here the non-negative exponents $N_{t-r}^{(k)}$ are determined by the decomposition (33) *of the resonance vector $t - r$ with respect to the basis $\{\rho\}$.*

Vectors $\{\mathfrak{p}^t\ t\in\mathbf{\Delta}[M]\}$ form an orthonormal basis in the space $\mathcal{H}_M$ of an irreducible representation of the algebra $\mathcal{A}$, where the Casimir operator $f_1\mathbf{S}_1+\cdots+f_n\mathbf{S}_n$ takes the value $\hbar M$. The operators of representation of the algebra $\mathcal{A}$ in this basis have the form $\mathbf{S}_j\mathfrak{p}^t=\hbar t_j\mathfrak{p}^t$ $(j=1,\dots,n)$, $\mathbf{A}_\rho\mathfrak{p}^t=(\hbar t)^{1/2}_\rho\mathfrak{p}^{\rho+t}$ $,(\rho\in\mathcal{R})$.

(b) *Inner product of coherent states $(\mathfrak{P},\mathfrak{P})_{\mathcal{H}}$ coincides with the reproducing kernel* (35) *of the space $\mathcal{L}(\Omega)$.*

(c) *Differential operators*

$$\overset{\circ}{S}_j\overset{\text{def}}{=}\hbar r+\hbar\sum_{k=1}^{n-1}\rho^{(k)}\,\overline{w}_k\frac{\partial}{\partial\overline{w}_k}\ \ (j=1,\dots,n),\qquad \overset{\circ}{A}_\sigma\overset{\text{def}}{=}(\overset{\circ}{S})_{\sigma_-}\prod_{k=1}^{n-1}(\overline{w}_k)^{N_k^\sigma}\ \ (\sigma\in\mathcal{M}),\quad(37)$$

given in local maps, are consistent on the intersections of maps and define an irreducible representation quantum resonance algebra $\mathcal{A}$ in Hilbert space $\mathcal{L}(\Omega)$ antiholomorphic polynomials with scalar product (34)*. The corresponding vacuum vector is the identity function.*

(d) *Using the coherent transform $\mathcal{P}:\mathcal{L}(\Omega)\to\mathcal{H}$ given by*

$$\mathcal{P}[\psi]\overset{\text{def}}{=}\frac{1}{(2\pi\hbar)^{n-1}}\int_{\mathbb{C}^{n-1}}\psi(\overline{w})\,\mathfrak{P}(w)\,\mathcal{J}_R\,dw\,d\overline{w},$$

abstract representation of the resonant algebra $\mathcal{A}$ in Hilbert space $\mathcal{H}$ with vacuum vector (36) *is intertwined with the irreducible representation* (37): $\mathbf{A}_\sigma\mathcal{P}[\psi]=\mathcal{P}[\overset{\circ}{A}_\sigma\psi]$, $\mathbf{S}_j\mathcal{P}[\psi]=\mathcal{P}[\overset{\circ}{S}_j\psi]$.

If $\mathcal{H}=L^2(\mathbb{R}^n)$, and the representation of the algebra $\mathcal{A}$ is given by formulas (30), for the basis vectors $\mathfrak{p}^t$ and coherent states $\mathfrak{P}_R(w)$, explicit formulas are obtained through Hermite polynomials.

coprime

Sections 4.1 – 4.7 are devoted to the study of “irreducible” resonance, when all frequencies f_j of the oscillator $\hat{H}[f]$ (29) are coprime natural numbers.

In section 4.8 (paper [9a]) we study the “reducible” elliptic case of three-frequency resonance, when the frequencies g_1, g_2, g_3 of the oscillator $\hat{H}[g]$ are arbitrary natural numbers (of course, it is assumed that $\mathrm{GCD}\{g_1,g_2,g_3\}=1$).

In this work, the reducible frequency vector (g_1,g_2,g_3) is associated with an irreducible frequency vector (f_1,f_2,f_3) according to the formula $f_j=g_j/(m_k\,m_l)$, where $m_j=\mathrm{GCD}\{g_k,g_l\}$,
(j,k,l) is a cyclic permutation of the numbers $(1,2,3)$, and it is shown that the study of the symmetry algebra of the oscillator $\hat{H}[g]$ reduces to the study of the symmetry algebra of the oscillator $hatH[f]$. As a result, irreducible representations in Hilbert spaces of antiholomorphic polynomials, coherent states, reproducing kernels and reproducing measures were obtained for reducible resonance.

In section 4.9 (paper [10a]) we studied three-frequency hyperbolic resonance, i.e. the case when two frequencies of the oscillator $\hat{H}[g]$ (29) are positive, and the third is negative.

The study of the hyperbolic case of resonance is in many ways similar to the study of the reducible case elliptical case. Moreover, the relations defining the algebra of hyperbolic resonance, formally coincide with the relations for the elliptic case. But the signs of the structural constants of resonant algebra in the elliptic and hyperbolic cases are, of course, different. Therefore the situation in the elliptic and hyperbolic cases are fundamentally different. Thus, in the elliptic case the symplectic leaves of Poisson algebra have compact closure, and irreducible representations of quantum resonant algebras are finite-dimensional. And in the hyperbolic case, symplectic leaves (of maximal dimension) have noncompact closure, and irreducible representations are infinite-dimensional.

Chapter 5, consisting of five sections, contains a series of problems on traps of charged particles [86, 87, 88].

In all problems of this cycle, the leading term of the Hamiltonian describes the so-called ideal Penning trap and represents the Hamiltonian of a hyperbolic harmonic oscillator with three frequencies. If these frequencies are in resonance, then the spectrum of the Hamiltonian has infinite degeneracy. In this case, standard perturbation theory is not applicable to the total Hamiltonian of the system. But it is possible to apply quantum averaging followed by reduction into the symmetry algebra of an ideal trap. This is resonant algebra, described in Chapter 4. In this work, we consider the question of at what trap parameters a frequency resonance can occur, and what resonant proportions arise. We studied the lower partial resonance, when only two of the three frequencies are commensurate, and the lower full resonance, when all three frequencies are commensurate.

The traps studied here differ not only in their resonant proportion, but also in the configuration of the electric and magnetic fields. The electric field is created by electrodes of various shapes; These are plates that form a cube, flat round or rectangular electrodes, and a cylindrical electrode. The perturbing magnetic field is either a uniform or inhomogeneous Ioffe field.

For each such system, the averaged Hamiltonian is explicitly calculated and written as a function of the generators of the resonance algebra of an ideal trap. In some problems of this cycle, the spectrum degeneracy is not removed in the subprincipal term of the averaged Hamiltonian. Then we examine its “secondary” symmetry algebra, which again turns out to be resonant, and the averaging procedure in the next term of perturbation theory is performed again. The twice averaged Hamiltonian is written as a function of the generators of the secondary resonant algebra.

Next, the technique of irreducible representations and coherent states is used. More precisely, using the integral representation of eigenfunctions through coherent states, the spectral problem is rewritten in an irreducible representation. As a result, firstly, the dimension of the space in which solutions are sought is reduced, and, secondly, the original partial differential equation is reduced to an ordinary differential equation.

In Section 5.1 (papers [5a], [27c]) we study the cubic Penning-Ioffe trap with resonance $3:(-1)$ and algebra with cubic commutation relations

$$\begin{gathered}
[\widehat{A}_1,\widehat{A}_2]=0,\quad [\widehat{A}_1,\widehat{A}_4]=-2i\hbar\widehat{A}_5,\quad [\widehat{A}_1,\widehat{A}_5]=2i\hbar\widehat{A}_4,\\
[\widehat{A}_2,\widehat{A}_4]=-6i\hbar\widehat{A}_5,\quad [\widehat{A}_2,\widehat{A}_5]=6i\hbar\widehat{A}_4,\quad [\widehat{A}_3,\widehat{A}_k]=0\quad (k=1,2,4,5),\\
[\widehat{A}_4,\widehat{A}_5]=i\hbar(15\hbar^2\widehat{A}_1+23\hbar^2\widehat{A}_2+9\widehat{A}_1\widehat{A}_2^2+\widehat{A}_2^3).
\end{gathered}\tag{38}$$

For algebra (38), we constructed irreducible representations by second-order differential operators acting in Hilbert spaces of antiholomorphic functions of one complex variable, found reproducing measures with the help of which the inner product is defined in representation spaces, and also obtained hypergeometric coherent states.

The averaged Hamiltonian is expressed as a function of the generators of this algebra and rewritten in an irreducible representation as an ordinary differential operator of Heun type [96].

For asymptotic eigenfunctions of the Penning trap Hamiltonian, we constructed an integral representation in terms of the hypergeometric coherent states of the cubic resonant algebra and in terms of solutions of the spectral problem for Heun's ordinary differential equation.

In Section 5.2 (papers [6a], [20b], [21b]) we study the double resonance in the Penning-Ioffe trap. The main resonance $2:(-1):2$ arises in the leading term of the Hamiltonian between the frequencies of the ideal Penning trap. It corresponds to an algebra with quadratic commutation relations:

$$\begin{gathered}
[\widehat{S}_+,\widehat{A}_\rho]=\hbar\widehat{A}_\rho,\quad [\widehat{S}_0,\widehat{A}_\rho]=-\hbar\widehat{A}_\rho,\quad [\widehat{S}_+,\widehat{A}_\sigma]=\hbar\widehat{A}_\sigma,\quad [\widehat{S}_-,\widehat{A}_\sigma]=2\hbar\widehat{A}_\sigma,\\
[\widehat{S}_-,\widehat{A}_\theta]=2\hbar\widehat{A}_\theta,\quad [\widehat{S}_0,\widehat{A}_\theta]=\hbar\widehat{A}_\theta,\quad [\widehat{A}_\rho,\widehat{A}_\sigma^*]=-\hbar\widehat{A}_\theta^*,\quad [\widehat{A}_\rho,\widehat{A}_\theta]=\hbar\widehat{A}_\sigma,\\
[\widehat{A}_\sigma,\widehat{A}_\theta^*]=-4\hbar\Big(\widehat{S}_-+\frac{\hbar}{2}\Big)\widehat{A}_\rho,\quad [\widehat{A}_\sigma^*,\widehat{A}_\sigma]=\hbar(4\widehat{S}_+\widehat{S}_-+\widehat{S}_-^2+2\hbar\widehat{S}_++3\hbar\widehat{S}_-+2\hbar^2),\\
[\widehat{A}_\rho^*,\widehat{A}_\rho]=\hbar(\widehat{S}_0-\widehat{S}_+),\quad [\widehat{A}_\theta^*,\widehat{A}_\theta]=\hbar(\widehat{S}_-^2+4\widehat{S}_-\widehat{S}_0+3\hbar\widehat{S}_-+2\hbar\widehat{S}_0+2\hbar^2)\\
\text{(other commutators are either conjugates to those listed or equal to zero)}
\end{gathered}\tag{39}$$

and constraints

$$\begin{gathered}
\widehat{A}_\rho\widehat{A}_\rho^*-\widehat{S}_+(\widehat{S}_0+\hbar)=0,\quad \widehat{A}_\sigma\widehat{A}_\sigma^*-\widehat{S}_+\widehat{S}_-(\widehat{S}_--\hbar)=0,\quad \widehat{A}_\theta\widehat{A}_\theta^*-\widehat{S}_0\widehat{S}_-(\widehat{S}_--\hbar)=0,\\
\widehat{A}_\rho\widehat{A}_\sigma^*-\widehat{S}_+\widehat{A}_\theta^*=0,\quad \widehat{A}_\sigma\widehat{A}_\theta^*-\widehat{S}_-(\widehat{S}_--\hbar)\widehat{A}_\rho=0,\quad \widehat{A}_\rho\widehat{A}_\theta-(\widehat{S}_0+\hbar)\widehat{A}_\sigma=0,\\
\widehat{S}_0^*=\widehat{S}_0,\quad \widehat{S}_-^*=\widehat{S}_-,\quad \widehat{S}_+^*=\widehat{S}_+.
\end{gathered}\tag{40}$$

The secondary resonance $k:l$ arises in the subprincipal term of the Hamiltonian. This resonance is described by algebra with polynomial commutation relations. In the case of lower resonance $1:0$, this algebra is quadratic:

$$\begin{gathered}
[\widehat{A}_0, \widehat{B}] = 2\hbar\widehat{B}, \qquad [\widehat{A}_-, \widehat{B}] = 2\hbar\widehat{B}, \qquad [\widehat{A}_+, \widehat{B}] = 0, \\
[\widehat{B}^*, \widehat{B}] = 2\hbar(\widehat{A}_0^2 + 2\widehat{A}_0\widehat{A}_- + 3\hbar\widehat{A}_0 + \hbar\widehat{A}_- + 2\hbar^2), \\
\widehat{A}_0^* = \widehat{A}_0, \quad \widehat{A}_-^* = \widehat{A}_-, \quad \widehat{A}_+^* = \widehat{A}_+.
\end{gathered} \tag{41}$$

For algebra (41), the following results were obtained in the work: irreducible representations of antiholomorphic functions by ordinary differential operators of second order in Hilbert spaces; reproducing kernels defined by the hypergeometric series; families of coherent states, which are the result of applying the Bessel function from the creation operator $\widehat{B}^*$ to the vacuum vector; reproducing measures expressed through the Tricomi function.

Using a coherent transform, the doubly averaged spectral problem for the trap Hamiltonian is rewritten in the irreducible representation of algebra (41) in the form of a second-order ordinary differential equation.

For the eigenstates of the Hamiltonian under study, we constructed an integral representation in terms of the solution of this differential equation and in terms of coherent states of algebra (41), to which two unitary averaging operators (arising in two averaging procedures) are applied.

In section 5.3 (articles [7a], [22b], [28c], [29c]) for a planar trap with round electrodes we studied the case of main resonance $2:(-1):2$ and secondary resonance $6:(-1)$. The main resonance is described by algebra (39), (40), and the secondary resonance is described by its subalgebra with quadratic commutation relations. In this problem, in addition to the quantum reduction of the trap Hamiltonian, its classical version is studied; here the emphasis is on the study of the doubly averaged Hamiltonian, written as a function of the generators of the Poisson $6:(-1)$-resonant algebra

$$\{A, B\} = 2iB, \qquad \{\overline{B}, B\} = i(6A^2 + 4dA), \qquad \overline{A} = A.$$

We investigated the dependence of the picture of its equilibrium points on the trap parameters.

In Section 5.4 (papers [8a], [23b], [30c]) we studied the planar Penning trap with rectangular electrodes. We have obtained a relationship between the control (geometric and physical) parameters of the trap, leading to a $3:(-1)$ resonance between the oscillator frequencies. This resonance is described by algebra (38). In addition to the quantum version of this algebra, we also considered the classical one - the Poisson algebra of symmetries of a $3:(-1)$-resonant oscillator. On the symplectic leaves of this algebra, the level lines of the averaged Hamiltonian are studied.

In Section 5.5 ([11a], [12a]), the last problem of this cycle - about a cylindrical Penning trap with resonance $2 : (-1) : 2$ - is studied using a new approach to calculating the coefficients of the averaged Hamiltonian. Namely, with the help of a twisted product, the procedure for averaging the Hamiltonian of a harmonic oscillator (including a hyperbolic one) perturbed by a differential operator with polynomial coefficients is transferred to the space of a graded symbol algebra. The averaging procedure performed in symbol space immediately gives an expression for the averaged Hamiltonian as a function of the generators $\widehat{S}_j$, $\widehat{A}_\rho$ (30) of the resonance algebra described in Section 4.6.

The developed approach is based on the fact that an arbitrary differential operator $\widehat{H}$ with polynomial coefficients on $\mathbb{R}^n$ can be uniquely represented as a finite linear combination $\widehat{H} = \sum_{\rho\in\mathbb{Z}^n} p_\rho(\widehat{S})\widehat{A}_\rho$ with polynomial coefficients p_ρ of the "actions" $\widehat{S}_j$, and it can be uniquely associated with its symbol, called the vs-polynomial in the paper. On the space of vs-polynomials, we defined the $*$-product operation corresponding to the usual associative product of differential operators with polynomial coefficients. Next, the quantum averaging procedure is rewritten in symbol space. In this form, the cumbersome routine calculations associated with averaging are easily transferred to a computer and implemented, for example, in the Wolfram Mathematica symbolic computing package.

In Chapter 6 (paper [13a]), consisting of three sections, instead of the family of ordinary coherent states of the Heisenberg algebra in $L^2(\mathbb{R})$, it is proposed to use a family of distributions in a rigged Hilbert space – Gelfand's triple. It is formed from two families of functionals on the Schwarz space. Each of these families has the key properties of ordinary coherent states: it intertwines representations of a given algebra, has the property of completeness, and minimizes the product of uncertainties in the Heisenberg relation. But, unlike ordinary coherent states, which are eigenstates of the annihilation operator, the constructed functionals belong to the continuous spectrum of Hermitian generators of the Heisenberg algebra. The inner product of functionals from different families has the main properties of the overlap function of ordinary coherent states: it is continuous in parameters, satisfies the reproducing identity, and has the corresponding geometric meaning. The listed properties give grounds to call the union of the constructed two families of functionals a family of coherent Schwarz distributions.

In Section 6.1 in a rigged Hilbert space (Gelfand triple) $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$, where $\mathcal{S}$ is a Schwartz space, and $\mathcal{S}'$ is the conjugate space of tempered distributions, we defined coherent distributions of the Heisenberg algebra and described their properties.

The family of coherent Schwartz distributions is defined as the union $\{X^+_{x^+},\, X^-_{x^-} \mid x^+, x^- \in \mathbb{R}\}$ of the following two families of functionals:

$$X^\pm = \Big\{ X^\pm_x(q) = \exp\Big\{ \pm \frac{i}{h} x\, q \mp \frac{i}{4h} x^2 \Big\}\, X^\pm_0(q) \mid x \in \mathbb{R} \Big\}, \tag{42}$$

where
$$X_0^{\pm}(q) = \frac{1}{\sqrt{2\pi h}} \exp\Big\{ \mp \frac{i}{2h} q^2 \pm \frac{i\pi}{8} \Big\}.$$

Theorem 5. (a) *The functionals $X_x^{\pm}(q)$ (42) are generalized eigenfunctions of the continuous spectrum of Hermitian generators $\hat{A}_{\pm} = \hat{q} \mp \hat{p}$ (where $\hat{q} \equiv q$, $\hat{p} = -ih\,\partial/\partial q$) of Heisenberg algebra $[\hat{A}_-, \hat{A}_+] = -2ih\hat{I}$. They satisfy the equations*

$$\hat{A}_{\mp}\, X_x^{\pm}(q) = x\, X_x^{\pm}(q), \qquad \hat{A}_{\pm}\, X_x^{\pm}(q) = \mp 2ih \frac{\partial}{\partial x} X_x^{\pm}(q).$$

(b) *Every family $X_x^{\pm}$ is complete in $\mathcal{S}$. Any vector $\psi \in \mathcal{S}$ can be expanded into functionals $\{X_x^{\pm} \mid x \in \mathbb{R}\}$:*
$$\psi = \int_{\mathbb{R}} \langle \psi,\, X_x^{\pm} \rangle\, X_x^{\pm}\, dx.$$

(c) *The inner product of two functionals from the same family has the form $\langle X_{x'}^{\pm},\, X_{x''}^{\pm} \rangle = \delta(x' - x'')$. The scalar product of two functionals from different families is given by the function*

$$K(x^+, x^-) \stackrel{def}{=} \langle X_{x^+}^{+},\, X_{x^-}^{-} \rangle = \frac{1}{2\sqrt{\pi h}} \exp\Big\{ \frac{i}{2h} x^+\, x^- \Big\} \tag{43}$$

and satisfies the reproducing property

$$\int_{\mathbb{R}} dy_+ \int_{\mathbb{R}} dy_-\, K(x^+, y^-)\, K(y^+, x^-)\, M(y^+, y^-) = K(x^+, x^-)$$

with (complex) density measure

$$M(x^+, x^-) = \frac{1}{2\sqrt{\pi h}} \exp\Big\{ -\frac{i}{2h} x^+\, x^- \Big\}. \tag{44}$$

(d) *Coherent transform $\mathcal{K}^{[\pm]} : L^2(\mathbb{R}_q) \to L^2(\mathbb{R}_{x^{\pm}})$, given by the formula*

$$\mathcal{K}^{[\pm]}[\psi(q)] = \langle \psi(q), X_{x^{\pm}}^{\pm}(q) \rangle,$$

is unitary. The inverse transform is given by the formula

$$(\mathcal{K}^{[\pm]})^{-1}[\varphi(x^{\pm})] = \int_{\mathbb{R}} \varphi(x^{\pm})\, X_{x^{\pm}}^{\pm}(q)\, dx^{\pm}.$$

The transform $\mathcal{K}^{[\pm]}$ intertwines an irreducible Hermitian representation of the Heisenberg algebra by operators $\hat{A}_- = q - ih\partial/\partial q$, $\hat{A}_+ = q + ih\partial/\partial q$ in the space $L^2(\mathbb{R}_q)$ with an equivalent irreducible Hermitian representation of this algebra by operators $\overset{[\pm]}{A}_{\mp} = x^{\pm}$, $\overset{[\pm]}{A}_{\pm} = \pm 2ih\, \partial/\partial x^{\pm}$ in the space $L^2(\mathbb{R}_{x^{\pm}})$:

$$\mathcal{K}^{[\pm]} \circ \hat{A}_- = \overset{[\pm]}{A}_- \circ \mathcal{K}^{[\pm]}, \qquad \mathcal{K}^{[\pm]} \circ \hat{A}_+ = \overset{[\pm]}{A}_+ \circ \mathcal{K}^{[\pm]}.$$

In Section 6.1 we also discuss the geometric meaning of the generalized reproducing kernel $K(x^+, x^-)$ (43) and density $M(x^+, x^-)$ (44) of the reproducing measure.

In section 6.2 (paper [13a]) we showed that the sequence of wave packets

$$\{\psi^{\pm}_{x,n}(q) \stackrel{def}{=} \int_{\mathbb{R}} \delta_n(x-y)\, X^{\pm}_y(q)\, dy \mid n \in \mathbb{N}\}, \qquad \text{where} \quad \delta_n(x) \stackrel{def}{=} ne^{-n^2x^2}/\sqrt{\pi},$$

approximates the coherent distribution (42), and the normalized wave packets

$$\widetilde{\psi}^{\pm}_{x,n}(q) \stackrel{def}{=} \psi^{\pm}_{x,n}(q)/\|\psi^{\pm}_{x,n}(q)\|_{L^2(\mathbb{R}_q)}$$

of this sequence provide a minimum to the product of uncertainties in the Heisenberg relation

$$(\Delta_\psi \hat{A}_+) \cdot (\Delta_\psi \hat{A}_-) \geq h$$

for operators $\hat{A}_-$ and $\hat{A}_+$. But for the operators of coordinate $\hat{q}$ and momentum $\hat{p}$ the product of uncertainties on the sequence of normalized wave packets $\widetilde{\psi}^{\pm}_{x,n}$ for $n\ to\infty$ tends to infinity.

In section 6.3 (article [13a]), using a coherent transform, the integral kernel of which is coherent distributions, we solved the spectral problem of the continuous spectrum of the Hamiltonian of an inverted oscillator

$$\hat{H} = \frac{1}{2}\Big(- h^2 \frac{\partial^2}{\partial q^2} - q^2 \Big). \tag{45}$$

This shows that coherent distributions can be used to solve problems with a continuous spectrum.

In addition, in this section it is proven that the family of coherent distributions (42) satisfies not only the first two (general) Gazeau-Clauder axioms (the axiom of continuity of the overlap function with respect to parameters and the property of completeness), but also two other (special) axioms of coherent states - the axiom of temporal stability and the so-called identity of action. These axioms hold for a normalized family of coherent distributions $\widetilde{X}^{\pm}_x = \sqrt{|x|}\, X^{\pm}_x$ with respect to the Hamiltonian (45) of the inverted oscillator. According to these axioms, the evolution of a normalized coherent distribution over time always remains a coherent distribution; in this case, the evolution of the parameters of coherent distributions corresponds to the classical evolution of coordinate functions (with which the parameters of coherent distributions are associated) in the phase space of the inverted oscillator.

At the conclusion of the dissertation, the results obtained are briefly listed.

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