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International Laboratory of Dynamical Systems and Applications

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Talanova Elena Anatolyevna

MORSE-SMALE DIFFEOMORPHISMS WITH NONWANDERING POINTS OF PAIRWISE DIFFERENT MORSE INDICES ON 3-MANIFOLDS

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professor
Pochinka Olga Vitalievna

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Introduction

An important class of structurally stable dynamic systems consists of Morse-Smale systems, an essential feature of which is the presence of a close relationship between the dynamic properties of the systems and the topology of the supporting manifolds. We present studies that in one way or another contributed to the identification of this class of systems.

In 1937 A.A. Andronov and L.S. Pontryagin [1] introduced the concept of a rough system in a limited part of the plane and established a criterion for the roughness of such a system. It turned out that these systems have a hyperbolic non-wandering set and do not have connections (trajectories going from saddle to saddle); moreover, they are dense in the space of all flows on the plane. This result was generalized by M. Peixoto [48], [49] to arbitrary closed surfaces with the concept of roughness replaced by the concept of structural stability (the equivalence of these concepts for flows on a plane was established by him in the same work). In the early 60s of the last century, S. Smale [65], like A.A. Andronov and L.S. Pontryagin, introduced into consideration the class of dynamical systems with a finite non-wandering hyperbolic set, whose invariant manifolds intersect transversally, and proved that the numbers of non-wandering orbits of different indices satisfy relations similar to Morse inequalities, after which such systems were called Morse-Smale systems, just like their discrete analogues. Later, S. Smale and J. Palis [46], [47] proved the structural stability of Morse-Smale dynamic systems (flows and cascades).

Despite the triviality of the non-wandering set, the topological classification of such systems is still very far from its completing. Morse-Smale flows are exhaustively classified up to topological equivalence on surfaces (in the works of E.A. Leontovich, A.G. Mayer [40], [41] M. Peixoto [50], A.A. Oshemkov and V.V. Sharko [45]). Let us list the known results related to the topological classification of various classes of Morse-Smale flows on manifolds of dimension three and higher. J. Fleitas [19] obtained a topological classification of polar flows (Morse-Smale flows, the non-wandering set of which contains exactly two node points and an arbitrary number of saddle periodic points) on three-dimensional manifolds. Ya.L. Umansky [66] obtained a topological classification of Morse-Smale flows with a finite number of heteroclinic trajectories on three-dimensional manifolds. A.O. Prishlyak [64] obtained a complete classification of three-dimensional gradient-like flows (Morse-Smale flows without periodic trajectories). S.Yu. Pilyugin [51] obtained a topological classification of Morse-Smale flows without heteroclinic intersections on a sphere of dimension greater than or equal to three. Classification results for some classes of multidimensional gradient-like flows were obtained in the papers of V.Z. Grines, E.Ya. Gurevich, E.V. Zhuzhoma, O.V. Pochinka [23], [24], [22], [21], [38]. Classification of three-dimensional non-singular flows with a small number of periodic orbits [67], [55], [56] was obtained in the papers of O.V. Pochinka and D.D. Shubin.

Morse-Smale diffeomorphisms on surfaces, unlike flows, allow trajectories going from saddle to saddle - heteroclinic trajectories (discovered by A. Poincare). Such movements lead to complex asymptotic behavior of invariant manifolds of saddle periodic orbits, which significantly increases the complexity of solving the topological classification problem. The classification of Morse-Smale diffeomorphisms on surfaces was obtained in 1998 by C. Bonatti and R. Langevin [18], as part of the classification of structurally stable diffeomorphisms with zero-dimensional basis sets (Smale diffeomorphisms). They proved that each Smale diffeomorphism corresponds to a finite combinatorial object, which is a set of geometric types of Markov partitions. However, Morse-Smale diffeomorphisms were not singled out for separate consideration, and therefore such the application of these invariants turned out to be unreasonably laborious for them. In the absence of heteroclinic points, a Morse-Smale diffeomorphism is called gradient-like, and for such diffeomorphisms various complete topological invariants were found in the papers of A.N. Bezdenezhnykh, V.Z. Grines, S.Kh. Kapkaeva, O.V. Pochinka [5], [6], [7], [25]. In the papers of V.Z. Grines, T.M. Mitryakova, A.I. Morozov, O.V. Pochinka [20], [43], [44] a complete topological classification of Morse-Smale surface diffeomorphisms with a finite number of heteroclinic orbits was obtained.

Difficulties in the transition from two-dimensional manifolds to manifolds of higher dimension are associated not only with the presence of heteroclinic orbits , but also with the possible wild embedding of separatrices of saddle periodic points (i.e. the closure of a separatrix is not a submanifold of a manifold). The first example of such a diffeomorphism on a three-dimensional sphere was constructed by D. Pixton [52] in 1977. V.Z. Grines and C. Bonatti [9] in 2000 proved that the topological conjugacy class of a Pixton diffeomorphism is described by a node in the space of orbits of the action of the diffeomorphism on some of its wandering sets. In the paper of C. Bonatti, V.Z. Grines, O.V. Pochinka [17] a complete topological classification of Morse-Smale diffeomorphisms on arbitrary closed 3-manifolds was obtained, which was preceded by a large series of papers by C. Bonatti, V.Z. Grines, E.Ya. Gurevich, E.V. Zhuzhoma, F. Laudenbach, V.S. Medvedev, E. Peku, O.V. Pochinka, that bring the solution to this problem closer [9], [10],[12],[14], [13], [15], [34], [53] [16], [17], [32], [35].

When the phase space of a diffeomorphism is three-dimensional, heteroclinic trajectories can form a one-dimensional set that breaks up into disjoint curves called heteroclinic. When studying deterministic processes described by Morse-Smale systems, a special role is played by non-compact heteroclinic curves, which in the case of a flow are trajectories, and in the case of diffeomorphism - curves invariant for some of its degree. From the end of the twentieth century to the present, in a series of works by E. Priest and T. Forbes [62], [63] much attention was paid to the problem of describing the topology of the magnetic field in the solar corona, in which the so-called separators play an important role. The mathematical model of separators is precisely heteroclinic trajectories and curves, and the question of their existence is one of the fundamental problems of magnetohydrodynamics. C. Bonatti, V.Z. Grines and V.S. Medvedev [12] in 2002 obtained a result, the consequence of which is a criterion for the existence of heteroclinic trajectories and curves. V.Z. Grines , together with E.V. Zhuzhoma, T.V. Medvedev and O.V. Pochinka [31] , was able to be used to install separators in the magnetic field of the solar corona.

From the results obtained in [11] it follows that a 3-manifold admits a Morse-Smale diffeomorphism without heteroclinic curves if and only if it is homeomorphic to $\mathbb{S}^3$ or a connected sum of a finite number of copies $\mathbb{S}^2 \times \mathbb{S}^1$, explicitly expressed through the number of saddle and nodal periodic orbits of the diffeomorphism. V.Z. Grines, E.V. Zhuzhoma and V.S. Medvedev [33] proved that in the case of a manual (not wild) embedding of one-dimensional separatrices of saddle points, the supporting manifold of a gradient-like 3-diffeomorphism admits a Heegaard decomposition, the genus of which is uniquely expressed in terms of the number of saddle points and nodal periodic orbits of diffeomorphism. Whether a similar connection exists in the case of wild embeddings of separatrices is an open question today.

Research goals and objectives

In this paper, we consider the class G of gradient-like diffeomorphisms f defined on orientable closed connected manifolds M^3 and having non-wandering points of pairwise different Morse indices (dimensions of unstable manifolds). From the definition of the class it follows that the non-wandering set of the diffeomorphism $f \in G$ consists of exactly four points $\omega_f, \sigma_f^1, \sigma_f^2, \alpha_f$ with Morse indices $0, 1, 2, 3$, respectively. The first examples of such diffeomorphisms with wild separatrices were constructed in the work of E.V. Kruglov and E.A. Talanova [39]. Any diffeomorphism $f \in G$ has exactly two saddle points σ_f^1, σ_f^2 of Morse indices 1 and 2, respectively, the intersection of two-dimensional manifolds of which forms a heteroclinic set (see Fig. 1)

$$H_f = W^s_{\sigma_f^1} \cap W^u_{\sigma_f^2}.$$

From the results of V.Z. Grines, E.V. Zhuzhoma and V.S. Medvedev, obtained in [33], it follows that in the case of manual embedding of one-dimensional separatrices, the supporting manifold of the diffeomorphism $f \in G$ is homeomorphic to the lens space $L_{p,q}$. Moreover, the set H_f contains at least p non-compact heteroclinic curves. The converse statement is also true: on any lens space there exists a diffeomorphism from the class G with tamely embedded one-dimensional separatrices. The main goal of this work is to prove the existence of diffeomorphisms with wild separatrices in the class under consideration and to describe the topology of the supporting manifold for such diffeomorphisms. The research objectives are to find topological invariants and construct quasi-energy functions of some subclasses of diffeomorphisms of the set G.

Scientific novelty of the results

All results are new. Exactly:

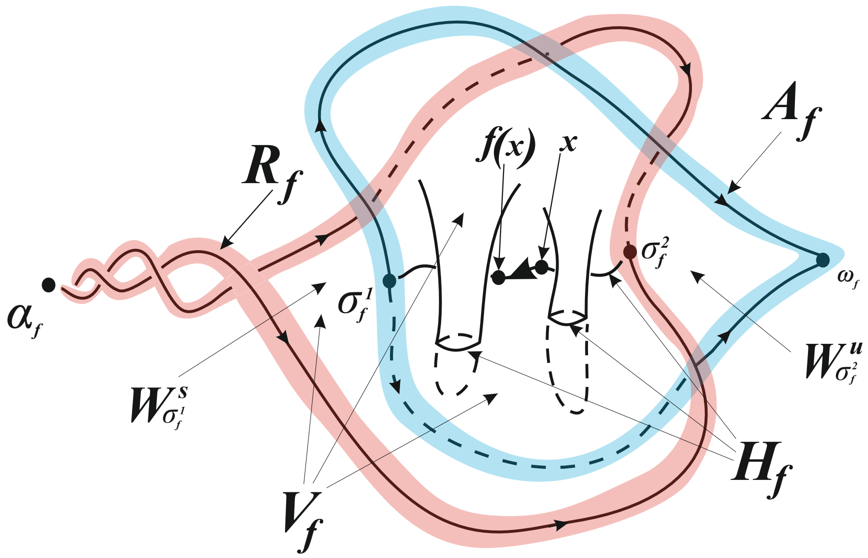

Figure 1: Phase portrait of a diffeomorphism $f \in G$ with a set H_f consisting of compact and non-compact heteroclinic curves

1. For Morse-Smale diffeomorphisms with four non-wandering points of pairwise distinct Morse indices on orientable closed connected 3-manifolds, a scenario for the transition from an arbitrary diffeomorphism to a diffeomorphism with the smallest number of heteroclinic curves is described.

2. It is proved that the ambient manifold for the diffeomorphisms under consideration is lens space.

3. A topological classification of diffeomorphisms from the class under consideration with a single heteroclinic curve is obtained; it has been proven that a complete invariant is the class (with respect to the ambient homeomorphism) of a Hopf node on the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

4. Quasi-energy functions was constructed for diffeomorphisms generated by an elementary Hopf knot.

5. An exact estimate for the number of critical points of the quasi-energy function for diffeomorphisms from the class under consideration was obtained.

Theoretical and practical significance of the conducted research

The research carried out relates to classical fundamental areas. These results contribute to the development of fundamental mathematics, while the direction of dynamical systems on 3-manifolds and in particular Morse-Smale diffeomorphisms have applications in mathematical models of most natural and social sciences. One example would be the modeling of the thinking process based on the theory of dynamic systems, which goes back to the model of J. Hopfield and M. Cohen, S. Grossberg [8], [36]. With fairly natural restrictions on the right-hand sides of an autonomous system of Cohen–Grossberg differential equations, in the phase space of the system there is a limited region that includes all trajectories of the system, which makes it possible to apply the results of this work (in particular, according to the classification of Morse-Smale diffeomorphisms from the class under consideration into 3 -manifolds) to the study of neural network models. Another example is the approach to modeling artificial neural networks (ANN), described in the works of V. Afraimovich, M. Rabinovich, P. Varona and others [4], [3], which is based on the principle of interaction of sections network under conditions of competition and leading to a sequential change of metastable (unstable) states of the model dynamic system. The dynamic image of a metastable state in the phase space of the model is a saddle equilibrium state, and the transition from one metastable state to another is described by a heteroclinic trajectory connecting two saddle equilibrium states. Today, the heteroclinic channel is the only known dynamic design with the help of which the fundamental contradiction between sensitivity (to information signals due to the information choice of metastable states — information reorganization of the heteroclinic channel) and reliability (stability of the channel) is resolved. And in this case it is also possible to apply the results obtained in this work for Morse-Smale diffeomorphisms with a single heteroclinic curve.

Methodology and research methods

The study used an original method, which consists in introducing the concept of the heteroclinic index p of the diffeomorphism f. It was found that the topological structure of the supporting manifold of diffeomorphisms from the class under consideration depends on it.

For a qualitative study of the systems considered in the work, classical methods of the theory of dynamical systems and topology are used: finding a suitable characteristic space, studying its topological properties and embedding into it traces of invariant manifolds of saddle equilibrium states. Homology theory and knot theory are also used. Morse theory and Morse permutations are used to construct energy functions.

Statements of defense

1. It is proved that for any Morse-Smale diffeomorphism with four non-wandering points of pairwise different Morse indices on an orientable closed connected 3-manifold with a heteroclinic intersection index equal to p, there exists a diffeomorphism isotopic to it, the index of which is equal to p, and the heteroclinic set is orientable (Theorem 1).

2. It is proved that if a diffeomorphism from the class under consideration has exactly one heteroclinic curve, then it is isotope to a source-sink diffeomorphism (Theorem 2).

3. It is proved that the class of topological conjugacy of a diffeomorphism with a single heteroclinic curve is completely determined by the class of a Hopf node, which is a projection of a one-dimensional separatrix into the space of orbits of the drainage basin. Moreover, any Hopf knot is realized by such a diffeomorphism (Theorems 3, 4).

4. Quasi-energy functions are constructed for diffeomorphisms generated by an elementary Hopf knot, and an exact estimate for the number of critical points of the quasi-energy function for diffeomorphisms from the class under consideration is obtained (Theorem 5).

5. It is proved that, regardless of the embedding of separatrices, the supporting manifold of any diffeomorphism with the heteroclinic intersection index equal to p is homeomorphic to the lens space $L_{p,q}$ (Theorem 6).

6. It is proved that every lens space $L_{p,q}$ admits a diffeomorphism with index equal to p with wildly nested one-dimensional separatrices (Theorem 7).

Structure and scope of paper

The dissertation consists of an introduction, six chapters, a conclusion and a list of references. The total volume of the work is 84 pages, including 47 drawings. The bibliography contains 86 titles.

Author's personal contribution

All results presented in the dissertation were obtained by the author independently. Scientific supervisor O.V. Pochinka is responsible for setting tasks and general management of the dissertation candidate's research activities in order to prepare for the defense of the dissertation. E.V. Kruglov and V.I. Shmukler were consultants on topological questions.

1 The content of the paper

The first chapter contains a list of articles and reports with the main results of the research presented at conferences. **The second chapter** contains the information and facts necessary for the study. **The third chapter** includes a description of the dynamics of diffeomorphisms of the class under consideration. **In the fourth chapter** for any diffeomorphism $f \in G$ it was introduce the concept of heteroclinic index I_f as follows. If the set H_f does not contain non-compact curves then we set $I_f = 0$. Otherwise denote by $\tilde{H}_f$ subset which consist of the non-compact curves. Since any curve $\gamma \subset \tilde{H}_f$ contains, together with any point $x \in \gamma$, the point $f(x)$, then we will assume that the curve γ is oriented in the direction from x to $f(x)$. We also fix the orientation on the manifolds $W^s_{\sigma_1}$ and $W^u_{\sigma_2}$. For a non-compact heteroclinic curve γ we denote by

$$v_\gamma = (\vec{v}^1_\gamma, \vec{v}^2_\gamma, \vec{v}^3_\gamma)$$

the triple of vectors with origin at point $x \in \gamma$ such that $\vec{v}^1_\gamma$ is the normal vector to $W^s_{\sigma_1}$, $\vec{v}^2_\gamma$ is the normal vector to $W^u_{\sigma_2}$ and $\vec{v}^3_\gamma$ are the tangent vector to the oriented curve γ. Let's call v_γ the frame of a non-compact heteroclinic curve γ (see Fig. 2).

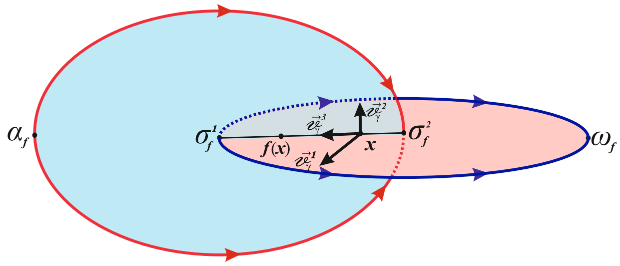

Figure 2: Frame of the non-compact heteroclinic curve

Obviously, the orientation (right or left) of the frame v_γ does not depend on the choice of point x on γ. Let us set $I_\gamma = +1\,(I_\gamma = -1)$ in the case of right (left) orientation. The number

$$I_f = \left| \sum_{\gamma \subset \tilde{H}_f} I_\gamma \right|$$

will be called *heteroclinic diffeomorphism index* f. For an integer $p \geqslant 0$, let $G_p \subset G$ denote the subset of diffeomorphisms $f \in G$ such that $I_f = p$.

The frame of a compact heteroclinic curve γ bounding the disk $d_\gamma \subset W^s_{\sigma^1_f}$ containing the saddle σ^1_f is defined in a similar way. In this case, the curve γ is oriented so that when moving along it, the disk d_γ remains on the left.

For a diffeomorphism $f \in G_p$, $p > 0$, we call the set H_f *orientable* if it consists only of non-compact curves and the frames of all curves in H_f have the same orientation (see Fig. refneor). .

For a diffeomorphism $f \in G_0$, we call a set H_f *orientable* if it is either empty or consists

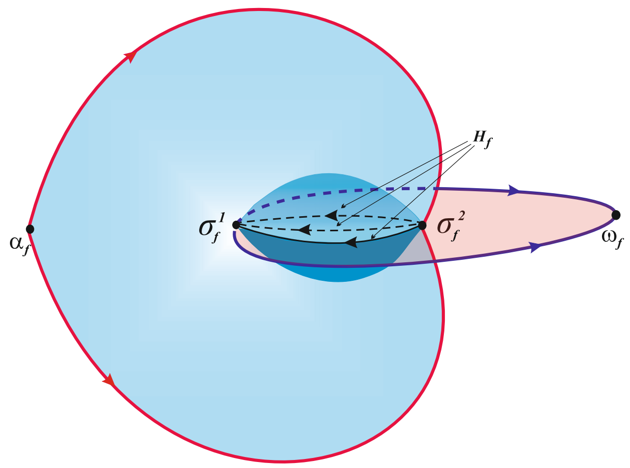

Figure 3: Diffeomorphism $f \in G_1$ with a non-orientable set H_f consisting of three non-compact curves

only of compact curves bounding disks on $W^s_{\sigma_f^1}$ containing the saddle σ_f^1, and the frames of all curves in H_f have the same orientation (see Fig. 4).

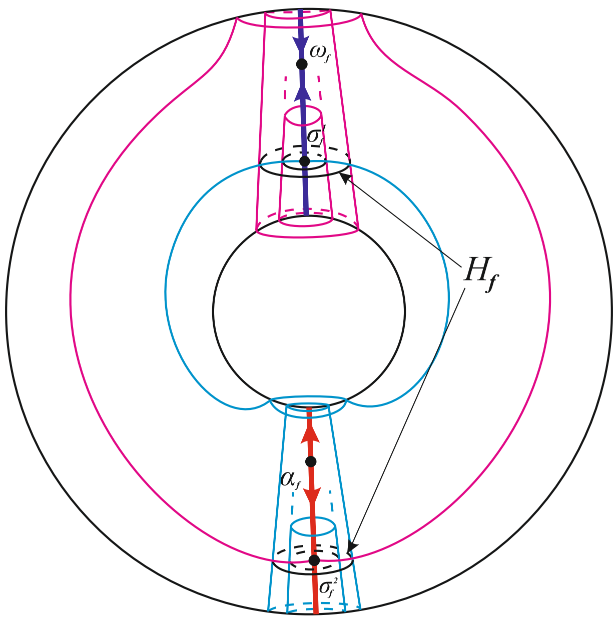

Figure 4: Diffeomorphism $f \in G_0^+$ with an orientable set H_f consisting of an infinite set of compact curves

Denote by $G_p^+ \subset G_p$, $p \geqslant 0$ the subset of diffeomorphisms $f \in G_p$ with an orientable set H_f.

The following fact proved in this chapter is the key to describing the topology of manifolds admitting diffeomorphisms of class G.

Theorem 1. **([57]*, Theorem 1)** *For any diffeomorphism $f : M^3 \to M^3$ from the class G_p, $p \geqslant 0$ in the set $Diff(M^3)$ there is an arc connecting the diffeomorphism f with some diffeomorphism $f_+ \in G_p^+$.*

This and the following results allow us to establish that the supporting manifold of any diffeomorphism $f \in G_1$ is homeomorphic to the sphere $\mathbb{S}^3$.

Theorem 2. ([60]*, Theorem 2) *For any diffeomorphism $f: M^3 \to M^3$ from the class G_1^+ in the set $Diff(M^3)$ there is an arc connecting the diffeomorphism f with the source-sink diffeomorphism.*

The complete topological invariant obtained in [17] for Morse-Smale 3-diffeomorphisms consists of a closed connected orientable simple 3-manifold and two transversally intersecting laminations embedded in it, consisting of tori and Klein bottles. In the paper [16] all admissible invariants are identified, and for each of them a Morse-Smale diffeomorphism is realizated. However, the lack of classification of simple 3-manifolds and laminations embedded in them does not always allow the realization of diffeomorphisms with given properties. In some special cases, invariants are often found in a more natural way, without considering them as part of the generality there are other more natural invariants that can be found without considering them as special case of the general one. Thus, for diffeomorphisms that have exactly one saddle point (Pixton diffeomorphisms), it was established in the paper [9] that their topological conjugacy is completely determined by the equivalence of the Hopf knot (knot $L \subset \mathbb{S}^2 \times \mathbb{S}^1$, belonging to the homotopy class of the standard knot $L_0 = \{s\} \times \mathbb{S}^1$), which is a projection of the one-dimensional unstable saddle separatrix into the orbital space of the sink basin. Among Hopf knots, a distinction is made between those that are equivalent to the standard knot and those that are nonequivalent. From the results of P.M. Akhmetyev and O.V. It follows from [2] that there are a countable number of pairwise non-equivalent Hopfian nodes to the standard one. By [52], [9], any Hopf knot can be realized by a Pixton diffeomorphism on the 3-sphere. **In the fifth chapter** of this work a similar result was obtained for diffeomorphisms of the class G_1^+.

Let $f \in G_1^+$. We denote by ℓ_f^1, ℓ_f^2 the unstable separatrices of the point σ_f^1 and set $L_f^i = p_{\omega_f}(\ell_f^i)$, $i = 1.2$ (see Fig. 5).

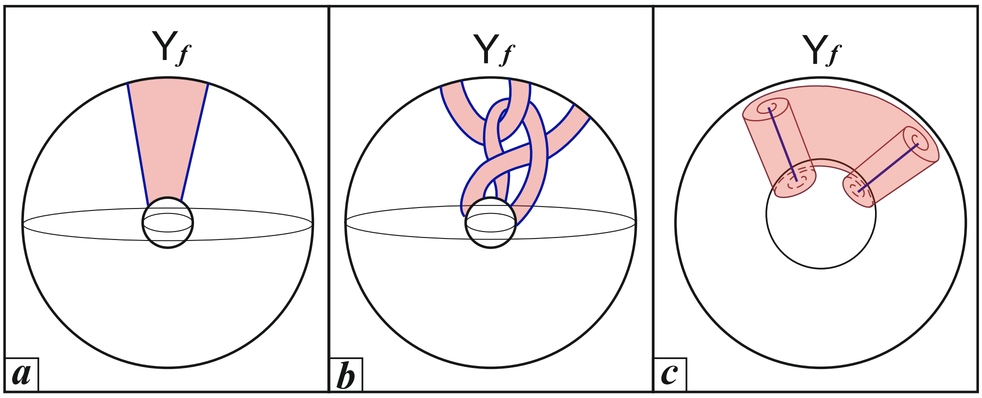

Figure 5: Possible versions for the projection Y_f of a two-dimensional unstable saddle manifold

Lemma 5.1 ([61]*, Lemma 1.1) *For any diffeomorphism $f \in G_1^+$, the sets L_f^1, L_f^2 are isotopic Hopf knotes.*

Let us denote by $\mathcal{L}_f = [L_f^1] = [L_f^2]$ the equivalence class of knotes L_f^1, L_f^2.

Theorem 3. ([61]*, Theorem 1.1) *Diffeomorphisms $f, f' \in G_1^+$ are topologically conjugate if and only if $\mathcal{L}_f = \mathcal{L}_{f'}$.*

Theorem 4. **([61]*, Theorem 1.2)** *For any equivalence class $\mathcal{L}$ of Hopf knots in $\mathbb{S}^2 \times \mathbb{S}^1$ there is a diffeomorphism $f \in G_1^+$ such that $\mathcal{L}_f = \mathcal{L}$.*

The steps of realization of the diffeomorphism $f_L \in G$ along the Mazur knot L are depicted in figures 6, 7, 8.

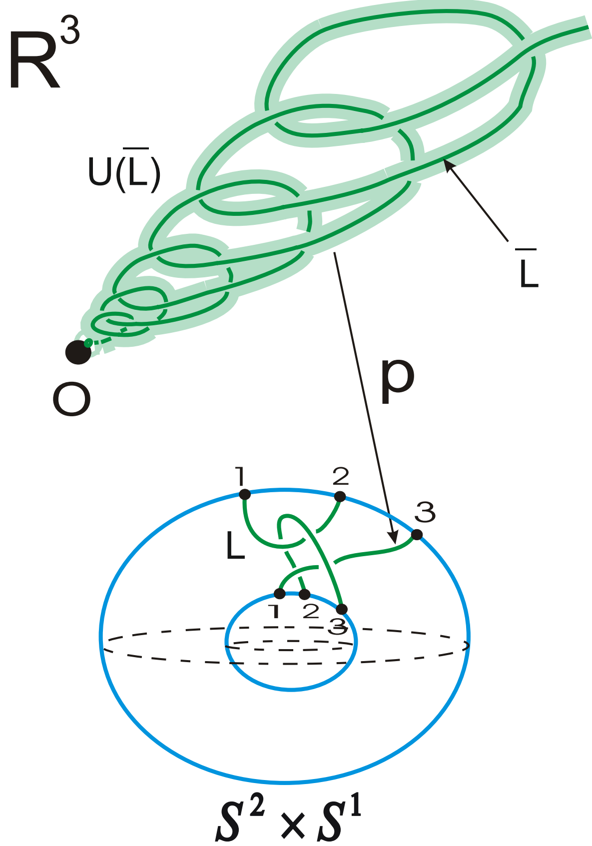

Figure 6: Lifting the Hopf knot L

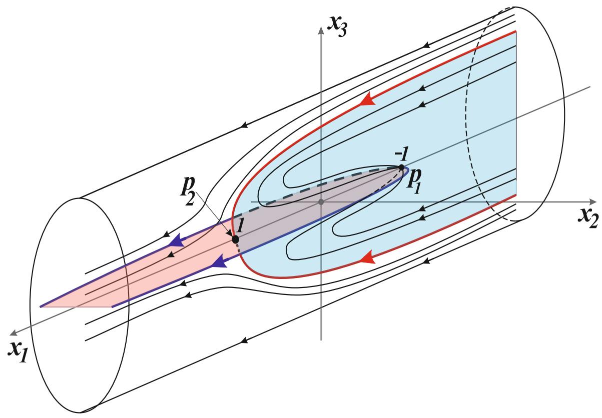

Figure 7: Trajectories of flow ϕ^t

From the results of V.Z. Grines, F. Laudenbach, O.V. Pochinka [26], [27], [29], [30] it is known that for any diffeomorphism $f : M^3 \to M^3$ from the class G there is a Morse-Lyapunov function. It's the Lyapunov function $\varphi : M^3 \to \mathbb{R}$, which is a continuous Morse

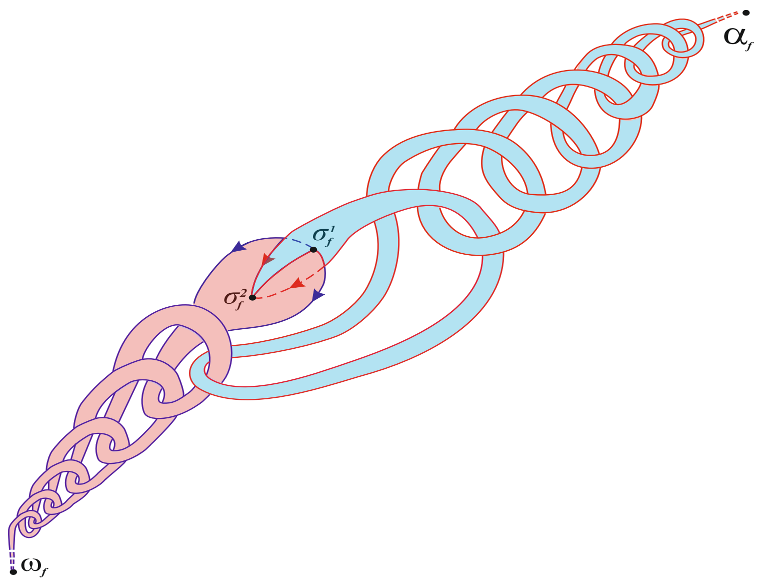

Figure 8: Diffeomorphism $f = f_L$

function. If the function φ has no critical points outside the non-wandering set of the diffeomorphism f, then, following [52], we call it the energy function for the diffeomorphism f. According to the paper [30], wild embedding of saddle separatrices is an obstacle to the existence of an energy function for the diffeomorphism $f \in G$. Because in the paper [28] the concept of a quasi-energy function was introduced for the diffeomorphism f (a Morse-Lyapunov function with a minimum number of critical points). Note that the number of critical points of the quasi-energy function is a topological invariant of the diffeomorphism f; let us denote it ρ_f.

Among Hopf knots, a distinction is made between those that are equivalent to the standard knot and those that are nonequivalent. According to [30], a diffeomorphism f_L has a Morse energy function if and only if the knot L is equivalent to the standard knot. According to the paper [30], a diffeomorphism f_L has a Morse energy function if and only if the knot L is equivalent to the standard knot.

Any Hopf knot is smoothly homotopic to a standard Hopf knot L_0 (see, for example, [37]), but is not isotopic or equivalent to it in the general case. B. Mazur [42] constructed a Hopf knot L_M, nonequivalent and nonisotopic to the knot L_0 (see Fig. 9). In the paper [2] a countable family of Hopf knots L_n was constructed (see Fig. 10), for which it was also proved there that they are pairwise nonequivalent.

In the fifth chapter of this work it was obtained the formula of the number of critical points of the Morse-Lyapunov function for the diffeomorphism f_{L_n}.

Theorem 5. ([58]*, Theorem 1) *For a diffeomorphism $f \in G_1^+$ constructed from a generalized Mazur knot L_n, $n \in \mathbb{N}$, the number ρ_f of critical points of the quasi-energy function of the diffeomorphism f is calculated using the formula*

$$\rho_f = 4 + 2n.$$

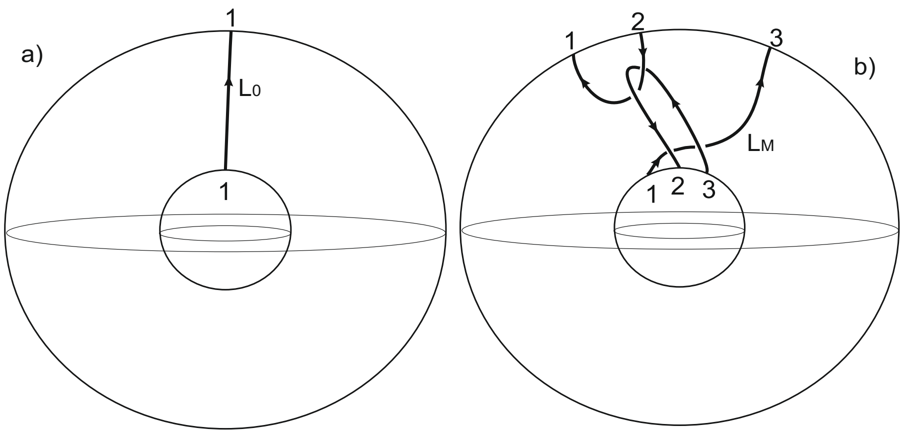

Figure 9: Non-isotopic and non-equivalent Hopf knots L_0 and L_M: a) standard Hopf knot L_0; b) Mazur node L_M

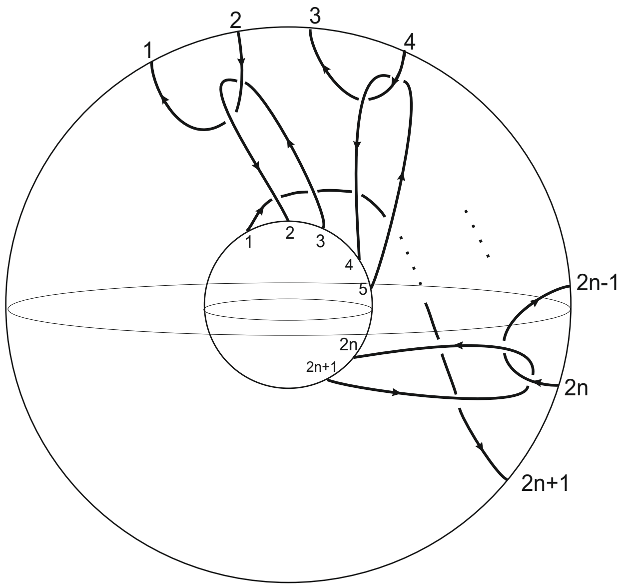

Figure 10: Generalized Mazur knot L_n

In the sixth chapter one of the main results of the work is the proof of the following fact.

Theorem 6. ([59]*, Theorem 1) *The supporting manifold of any diffeomorphism* $f \in G_p$, $p \geqslant 0$ *is homeomorphic to the lens space* $L_{p,q}$.

Also in this chapter it provides a constructive proof of the following statement.

Theorem 7. ([59]*, Theorem 2]) *On any lens space* $L_{p,q}$ *there is a diffeomorphism* $f \in G$ *with wildly embedded one-dimensional saddle separatrices.*

Note that all previously known examples of diffeomorphisms of the class under consideration with wildly embedded one-dimensional separatrices were constructed only on a three-dimensional sphere.

2 Publications based on research results

This dissertation was written on the basis of 4 articles published in journals included in international bibliographic databases.

- O. V. Pochinka, E. A. Talanova, *Quasi-energy function for 3-Morse-Smale diffeomorphisms with fixed points of pairwise different indices*, Mathematical Notes, 115:4, (2024), 1-13.
- O. V. Pochinka, E. A. Talanova, *Morse–Smale diffeomorphisms with nonwandering points of pairwise distinct Morse indices on 3-manifolds*, Russian Mathematical Surveys, 79:1, (2024), 135-184.
- O. Pochinka, V. Shmukler, E. Talanova, *Bifurcation of a disappearance of a non-compact heteroclinic curve*, Selecta Mathematica, New Series, 29:60, (2023), 1-14.
- E.V. Kruglov, E.A. Talanova, *On the realization of Morse–Smale diffeomorphisms with heteroclinic curves on a three-dimensional sphere*, Proceedings of Steklov Mathematical Institute, 236 (2002), 212-217.

Conclusion

In this paper, we consider the class of G gradient-like diffeomorphisms f defined on orientable closed connected manifolds M^3 and having non-wandering points of pairwise distinct indices: $\omega_f, \sigma_f^1, \sigma_f^2, \alpha_f$ with Morse indices $0, 1, 2, 3$, respectively. For any diffeomorphism $f \in G$, the concept of heteroclinic index I_f is introduced, as well as the orientability of the heteroclinic intersection. For an integer $p \geqslant 0$, $G_p \subset G$ denotes the subset of diffeomorphisms $f \in G$ such that $I_f = p$, and $G_p^+ \subset G_p$, $p \geqslant 0$ is a subset of diffeomorphisms $f \in G_p$ with orientable heteroclinic intersection.

The main results of the dissertation submitted for defense are the following facts proven in the work.

1. For any diffeomorphism $f : M^3 \to M^3$ from the class G_p, $p \geqslant 0$ in the set $Diff(M^3)$ there is an arc connecting the diffeomorphism f with some diffeomorphism $f_+ \in G_p^+$ (Theorem 1).

2. For any diffeomorphism $f : M^3 \to M^3$ from the class G_1^+ in the set $Diff(M^3)$ there is an arc connecting the diffeomorphism f with the source-sink diffeomorphism (Theorem 2).

3. The topological conjugacy of diffeomorphisms of class G_1^+ is completely determined by the equivalence of Hopf nodes, which are the projection of a one-dimensional saddle separatrix into the orbit space of the drainage basin. Moreover, any Hopf knot L can be realized by a diffeomorphism f_L from the class G_1^+, for which the equivalence class of the node L is a complete topological invariant (Theorems 3, 4).

4. For a diffeomorphism f_{L_n}, $n \in \mathbb{N}$, realizated by a generalized Mazur knot L_n, the number of critical points of its quasi-energy function is calculated by the formula $\rho_f = 4 + 2n$ (Theorem 5).

5. The supporting manifold of any diffeomorphism $f \in G_p$, $p \geqslant 0$ is homeomorphic to the lens space $L_{p,q}$ (Theorem 6).

6. Every lens space $L_{p,q}$ admits a diffeomorphism f from the class G_p with wildly embedded one-dimensional separatrices (Theorem 7).

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