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Structure of admissible subcategories in derived categories of algebraic varieties

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Introduction

In this thesis we study derived categories of coherent sheaves on algebraic varieties from several different points of view. Let X be an algebraic variety over a field $\Bbbk$. It has an associated abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X. We can take the derived category $D(\mathrm{Coh}(X))$ of that abelian category. The key object for this object is the following:

0.1. Definition. The *bounded derived category of coherent sheaves* $D^b_{\mathrm{coh}}(X)$ is the full subcategory of $D(\mathrm{Coh}(X))$ consisting of complexes of sheaves with only finitely many nonzero cohomology sheaves.

Remark. For technical reasons it is better to use another definition of $D^b_{\mathrm{coh}}(X)$: as a full subcategory in the derived category of the abelian category of quasi-coherent sheaves, consisting of complexes with only finitely many nonzero cohomology sheaves, for which all cohomology sheaves are coherent. Since for us X is always a Noetherian scheme, this definition is equivalent to the one above [Huy06, Prop. 3.5].

The category $D^b_{\mathrm{coh}}(X)$, which we will usually refer to as the *derived category of the variety* X, is a very large invariant of the variety X. Many more comprehensible invariants, such as algebraic K-theory or Hochschild (co)homology, can be recovered from the derived category of X. Despite the fact that the derived category of a variety is usually too large to be "computed" in a satisfactory sense, it can be productively studied, for example, by examining its connection to the derived categories of other varieties. Examples of such results and descriptions of the methods employed can be found in the survey [BO02].

The study of derived categories of algebraic varieties is a rapidly developing area of algebraic geometry. With regards to the history of this field we will mention only two classical articles that have had a significant impact on its further development. In the 1978 article [Bei78], A. Beilinson studied the derived category $D^b_{\mathrm{coh}}(\mathbb{P}^n)$ of projective space $\mathbb{P}^n$ and described it in terms of linear-algebraic objects. Using the terminology that didn't exist at that time, it can be said that Beilinson constructed an *exceptional collection* for $\mathbb{P}^n$. His article served as an important step in the future study of exceptional objects, exceptional collections, and semi-orthogonal decompositions. A bit later, in 1981, the article [Muk81] by S. Mukai was published, where he proved that for any abelian variety A there exists an equivalence of derived categories $D^b_{\mathrm{coh}}(A) \simeq D^b_{\mathrm{coh}}(A^\vee)$, where $A^\vee \cong \mathrm{Pic}^0(A)$ is the dual abelian variety of A. This equivalence identifies degree-zero line bundles on A with the skyscraper sheaves over the corresponding points of $\mathrm{Pic}^0(A)$. It should be noted that the varieties A and $A^\vee$ may not be isomorphic. This equivalence allowed Mukai to answer some questions related to Picard bundles and demonstrated that sometimes, extra symmetries arise between derived categories, and those symmetries can be highly non-trivial at the geometric level.

An important tool for studying derived categories is the concept of semi-orthogonal decomposition. It is a way to represent a category as a "gluing" of several smaller subcategories. We will need an auxiliary definition.

0.2. Definition. A strictly full triangulated subcategory $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ is called *admissible* if the embedding functor $\mathcal{A} \hookrightarrow D^b_{\mathrm{coh}}(X)$ has both left and right adjoint functors.

At first glance, the property of admissibility of a subcategory may not seem very restrictive, but its simplicity is deceptive: admissible subcategories are, in general, a rare occurrence. The existence of adjoint functors between "large" triangulated categories is often obtained automatically (see, for example, the article [Nee96]), but because we are working with the "small" category $D^b_{\mathrm{coh}}(X)$ rather than the unbounded derived category of quasi-coherent sheaves, checking the admissibility condition is much more challenging.

0.3. Definition. A set of admissible subcategories $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ in $D^b_{\mathrm{coh}}(X)$ is said to form a (strong) *semi-orthogonal decomposition* of the category $D^b_{\mathrm{coh}}(X)$ if the following conditions are satisfied:

- The subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ jointly generate $D^b_{\mathrm{coh}}(X)$ in the sense that the smallest triangulated subcategory of $D^b_{\mathrm{coh}}(X)$ that contains each $\mathcal{A}_i$ coincides with $D^b_{\mathrm{coh}}(X)$.
- Let $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ be two objects. If $j > i$, then $\mathrm{Hom}^\bullet_{D^b_{\mathrm{coh}}(X)}(A_j, A_i) = 0$ (*semi-orthogonality*).

Remark. In certain situations, it can be useful to weaken the concept of a semi-orthogonal decomposition by allowing the subcategories $\mathcal{A}_i$ not to be admissible. In many cases (for instance, if X is a smooth and proper variety), admissibility arises automatically. All the semiorthogonal decompositions appearing in this work have admissible components, so we omit the qualifying adjective "strong" and simply refer to them as semiorthogonal decompositions.

Admissible subcategories are closely related to semiorthogonal decompositions. For example, the following fact holds.

0.4. Lemma ([BK90]). *Let X be a smooth projective variety and let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. Consider the full subcategory ${}^\perp\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ defined as follows:*

$$ {}^\perp\mathcal{A} := \{F \in D^b_{\mathrm{coh}}(X) \mid \forall A \in \mathcal{A}\ \mathrm{Hom}^\bullet(F, A) = 0\}. $$

Then ${}^\perp\mathcal{A}$ is an admissible subcategory of $D^b_{\mathrm{coh}}(X)$ and the pair $\langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$ is a semi-orthogonal decomposition of $D^b_{\mathrm{coh}}(X)$. For an analogously defined subcategory $\mathcal{A}^\perp$ the pair $\langle \mathcal{A}^\perp, \mathcal{A} \rangle$ is also a semi-orthogonal decomposition of $D^b_{\mathrm{coh}}(X)$.

Let us note an important example of a semi-orthogonal decomposition, constructed by Orlov [Orl93]:

0.5. Theorem ([Orl93]). *Let X be a smooth variety, let $j \colon Z \hookrightarrow X$ be a smooth subvariety of codimension c, and let $\pi \colon Y \to X$ be the blow-up of X along Z. Denote by $j \colon E \hookrightarrow Y$ the inclusion of the exceptional divisor and by $p \colon E \to Z$ the restriction of the morphism π to E. Then there exists a semiorthogonal decomposition*

$$ D^b_{\mathrm{coh}}(Y) = \langle \pi^* D^b_{\mathrm{coh}}(X), \Phi_0(D^b_{\mathrm{coh}}(Z)), \ldots, \Phi_{c-2}(D^b_{\mathrm{coh}}(Z)) \rangle, $$

where the functors $\pi^*: D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ *and*

$$\Phi_i: F \in D^b_{\mathrm{coh}}(Z) \mapsto j_*(p^*(F) \otimes \mathcal{O}_\pi(i)) \in D^b_{\mathrm{coh}}(Y)$$

are inclusions of admissible subcategories.

Constructing semiorthogonal decompositions on a certain class of varieties often proves to be a challenging task. Currently, there are numerous examples of semiorthogonal decompositions, and there exist several methods for creating new decompositions. However, many open questions remain. Note that there are not so many known general properties of semiorthogonal decompositions that let us control their behavior. Some facts about their general behavior tend to have a negative character; for instance, semiorthogonal decompositions do not satisfy the Jordan–Holder property, meaning that two distinct decompositions of the same category do not admit a "common subdecomposition" in any reasonable sense. Explicit counterexamples to this property have been constructed in [Kuz13] and [BGS14]. On a positive note, the theorem of Kawatani and Okawa states that admissible subcategories are closed with respect to small deformations of objects [KO15].

The main part of this dissertation is divided into three sections, each of which heavily relies on the concepts of semiorthogonal decompositions and admissible subcategories.

The results of this thesis are published as three articles:

(a) Pirozhkov D. Semiorthogonal Decompositions on Total Spaces of Tautological Bundles // *International Mathematics Research Notices.* 2022. №3. P. 2250–2273.

(b) Pirozhkov D. Rouquier dimension of some blow-ups // *European Journal of Mathematics.* 2023. Vol. 9, art. 45.

(c) Pirozhkov D. Stably semiorthogonally indecomposable varieties // *Épijournal de Géométrie Algébrique.* 2023. Vol. 7.

In the Appendix A of the thesis, which is described in Section 1 of this summary, a semiorthogonal decomposition is constructed for a variety X obtained as the total space of a certain vector bundle over the Grassmannian. This decomposition happens to be similar to the exceptional collection in the derived category of the Grassmannian itself constructed by Kapranov. A global version of this result is also proven, which can be regarded as a generalization of Theorem 0.5.

In the Appendix B of the thesis, which is described in Section 2 of this summary, we confirm for some class of varieties that a certain invariant of triangulated categories, known as *Rouquier dimension*, of the derived category of coherent sheaves equals the usual geometric dimension of the variety. Conjecturally this is true for all varieties, although it is known to be the case for only a very small list. Semiorthogonal decompositions lead to an upper bound on the Rouquier dimension in terms of the Rouquier dimensions of each of its components, but typically, this estimate is highly inefficient. Using some very specific semiorthogonal decompositions we confirm the conjecture for, among other examples, the blow-up of nine distinct points on a projective plane or for the blow-up of three points in $\mathbb{P}^n$ for any n.

In the Appendix C of the thesis, which is described in Section 3 of this summary, the concept of *stable indecomposability* is introduced for the derived category of a variety. A simple consequence of stable indecomposability is that the derived category of the variety is indecomposable, meaning it does not admit non-trivial semiorthogonal decompositions, and furthermore, the same holds true for all subvarieties. Stable indecomposability is proved for abelian varieties, and this is used for some results regarding phantom subcategories.

1 Semiorthogonal decompositions and tautological bundles

Let V be a vector space of dimension n. Denote by $X = \mathrm{Gr}(k, V)$ the Grassmannian variety of k-dimensional linear subspaces in V. Let U and Q be, respectively, the tautological subbundle and quotient bundle on X, fitting into a short exact sequence:

$$0 \to U \to V \otimes \mathcal{O}_X \to Q \to 0.$$

Consider the variety $Y := \mathrm{Tot}_{\mathrm{Gr}(k,V)}(U)$, which is the total space of the tautological subbundle on X, and denote by $\pi\colon Y \to X$ the projection morphism. The points on Y are pairs $(v \in V, W \subset V)$, where W is a k-dimensional subspace of V containing the vector v. Let $p\colon Y \to \mathbb{A}(V)$ be a forgetful morphism that forgets the choice of a subspace. Here $\mathbb{A}(V)$ is the vector space V considered as an algebraic variety, i.e., $\mathrm{Spec}\,(\mathrm{Sym}^\bullet(V^\vee))$.

The fiber of the morphism p over a point $v \in \mathbb{A}(V)$ is a set of all k-dimensional subspaces of V containing the vector v. In other words, the fiber over any nonzero vector is isomorphic to a Grassmannian $\mathrm{Gr}(k-1, V/\langle v \rangle) \simeq \mathrm{Gr}(k-1, n-1)$, embedded into $\mathrm{Gr}(k, V)$. The fiber over the zero vector is the whole $\mathrm{Gr}(k, V) \simeq \mathrm{Gr}(k, n)$.

Consider what happens in the case when $k = 1$. In this case $X \simeq \mathbb{P}^{n-1}$, and the morphism from Y to $\mathbb{A}(V) \simeq \mathbb{A}^n$ is, as one can easily verify, the blow up of the affine space at the origin. For the projective space, there exists a Beilinson exceptional collection:

$$D^b_{\mathrm{coh}}(\mathbb{P}^{n-1}) = \langle \mathcal{O}_{\mathbb{P}^{n-1}}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^{n-1}}(-1), \mathcal{O}_{\mathbb{P}^{n-1}} \rangle, \tag{1.1}$$

And for the blow-up of the affine space at the origin, Theorem 0.5 implies the existence of a semiorthogonal decomposition:

$$D^b_{\mathrm{coh}}(Y) = \langle j_* \mathcal{O}_{\mathbb{P}^{n-1}}(-n+1), \ldots, j_* \mathcal{O}_{\mathbb{P}^{n-1}}(-1), D^b_{\mathrm{coh}}(\mathbb{A}^n) \rangle, \tag{1.2}$$

where $j\colon \mathbb{P}^{n-1} \hookrightarrow Y$ is the inclusion of the exceptional divisor of the blow-up, and the subcategory equivalent to $D^b_{\mathrm{coh}}(\mathbb{A}^n)$ is generated by the structure sheaf $\mathcal{O}_Y$. Observe that the decompositions (1.1) and (1.2) have a very similar structure. Furthermore, the case $k = n-1$, where the generic fiber of the morphism $p\colon Y \to \mathbb{A}^n$ is isomorphic to $\mathbb{P}^{n-2}$ and the central fiber is isomorphic to $\mathbb{P}^{n-1}$, was studied by Orlov, who in [Orl06, Prop. 2.10] constructed a semiorthogonal decomposition for $D^b_{\mathrm{coh}}(Y)$ similar to Beilinson's exceptional collection for projective space.

The main result of this section of the thesis, based on the paper [Pir22], is the existence of an analogous semiorthogonal decomposition of $D^b_{\mathrm{coh}}(Y)$ for other values of k:

1.1. Theorem ([Pir22, Th. 3.5]). *There exists a semiorthogonal decomposition*

$$D^b_{\mathrm{coh}}(Y) = \left\langle \binom{n-1}{k} \text{ copies of } D^b_{\mathrm{coh}}(\mathrm{Vect}), \binom{n-1}{k-1} \text{ copies of } D^b_{\mathrm{coh}}(\mathbb{A}(V)) \right\rangle.$$

Note that in the published version the variety Y is referred to as $\mathrm{Tot}(U)$.

In the case $k = 1$ this decomposition agrees with the Orlov's formula (1.2) for a blow-up of a point in an affine space. Since any blow-up of a smooth subvariety in a smooth variety locally looks like a product of a point blow-up in an affine space times another affine space, the decomposition (1.2) for a point blow-up can be used to deduce the general blow-up formula for derived categories (Theorem 0.5). Similarly, our Theorem 1.1 can be transformed into a global statement:

1.2. Theorem ([Pir22, Th. 4.5]). *Let X be a Cohen–Macaulay variety, let E be a vector bundle on X, and let $s \in \Gamma(X, E)$ be a regular global section of E. Denote by Z the zero locus of s, and by $Y \subset \mathrm{Gr}_X(k, E)$ the subvariety in the relative Grassmannian of k-dimensional subspaces in the fibers of E consisting only of those subspaces which over each point $x \in X$ contain the vector $s(x) \in E_x$. Then there exists a semiorthogonal decomposition*

$$D^b_{\mathrm{coh}}(Y) = \left\langle \binom{n-1}{k} \text{ copies of } D^b_{\mathrm{coh}}(Z), \binom{n-1}{k-1} \text{ copies of } D^b_{\mathrm{coh}}(X) \right\rangle.$$

Note that in the published version the variety Y is referred to as $\mathrm{Gr}_s(k, E)$.

Note that the variety we get in the basic special case where $X = \mathbb{A}^n$, the vector bundle E is a trivial rank-n bundle, and s is the tautological section, is exactly the variety we called Y in Theorem 1.1. This special case together with the theory of relative semiorthogonal decompositions imply the statement in general.

The decompositions in Theorems 1.1 and 1.2 are constructed explicitly. We have noted above the similarity between the decomposition (1.1) for the projective space and the decomposition (1.2) for a point blow-up, which is the $k = 1$ case of Theorem 1.1. In general, an important ingredient of the proof of Theorem 1.1 is a specific semiorthogonal decomposition (more precisely, an exceptional collection) of the derived category of the Grassmannian $\mathrm{Gr}(k, V)$. Note though, that unlike the cases $k = 1$ and $k = n-1$, the standard exceptional collection on the Grassmannian, constructed by Kapranov in [Kap84], is not suitable for constructing the desired semiorthogonal decomposition of $D^b_{\mathrm{coh}}(Y)$. Instead, we use a particular mutation of the Kapranov's collection.

2 Rouquier dimension of some blow-ups

Let T be a triangulated category. In the paper [Rou08] Rouquier defined an invariant of T which later became known as the *Rouquier dimension*. In order to define it, we need a piece of notation introduced in the paper [BV03]: for an object $E \in T$ and an integer $k \in \mathbb{Z}_{\geq 0}$ we denote by $\langle E \rangle_k$ the following inductively defined subset of objects in T:

- $\langle E \rangle_0$ is the set of finite direct sums of shifts of the copies of the object E, as well as direct summands of all objects like that. In other words, this is the set of objects that may obtained from E using only three kinds of operations: direct sums, shifts in the triangulated category, and passing to a direct summand.
- $\langle E \rangle_k$ is defined as the set of all objects $F \in T$ that fit into a distinguished triangle $F_0 \to F \to F_{k-1}$, where $F_0 \in \langle E \rangle_0$ and $F_{k-1} \in \langle E \rangle_{k-1}$, as well as all direct summands of such objects F.

An object E such that for some $k \in \mathbb{Z}_{\geq 0}$ we have $\langle E \rangle_k = T$ is called a *strong generator* of the category T. We are mostly interested in the geometric setting, and for the derived categories of coherent sheaves on smooth varieties strong generators always exist [BV03].

2.1. Definition. The *Rouquier dimension* of a triangulated category T is the smallest integer $k \in \mathbb{Z}_{\geq 0}$ such that there exists a strong generator $E \in T$ with $\langle E \rangle_k = T$. By convention the dimension is ∞ if there are no strong generators in T. The Rouquier dimension is denoted by $\mathrm{rdim}(T)$.

Determining the Rouquier dimension of a category is a difficult task since we need to consider all possible strong generators. In a geometric situation, i.e., for the triangulated category $D^b_{\mathrm{coh}}(X)$ where X is a smooth variety, Rouquier proved the following inequality

$$\dim(X) \leq \mathrm{rdim}(X) \leq 2\dim(X).$$

Here and below we use the shorthand notation $\mathrm{rdim}(X)$ instead of $\mathrm{rdim}(D^b_{\mathrm{coh}}(X))$ when X is a smooth variety. It turns out that for all varieties X where $\mathrm{rdim}(X)$ could be computed exactly the Rouquier dimension is always equal to the usual geometric notion of dimension. Orlov conjectured that this should always be the case:

2.2. Conjecture ([Orl09]). *For any smooth projective variety X the inequality* $\mathrm{rdim}(X) = \dim(X)$ *holds.*

There are not so many varieties for which Rouquier dimension can be computed. In the paper [Pir23a], on which this section of the thesis is based, we add some new varieties to the list of known cases of Conjecture 2.2. The varieties we consider are some blow-ups of projective spaces. The proof involves a construction of a semiorthogonal decomposition with special properties, which we use to bound the Rouquier dimension from above by the geometric dimension. The main result is the following theorem:

2.3. Theorem ([Pir23a, Th. 4.1]). *Let $\{Z_b\}_{b\in B}$ be a set of at most three pairwise disjoint linear subspaces in $\mathbb{P}^n$, where each Z_b is either a point or a linear subspace of codimension 2. Let X be the blow-up of $\mathbb{P}^n$ in the union of those subspaces. Then X satisfies Orlov's conjecture, i.e.,* $\operatorname{rdim}(X) = \dim(X) = n$.

Note that in the published article the notation for the blow-up is Y rather than X.

If the dimension n of the projective space is equal to 2 or 3, the construction of a semiorthogonal decomposition with special properties on the blow-up can be applied multiple times, which leads to stronger and more interesting statements:

2.4. Theorem ([Pir23a, Prop. 4.2]). *Consider a tower of blow-ups*

$$X_3 \to X_2 \to X_1 \to X_o = \mathbb{P}^2,$$

where each map $X_i \to X_{i-1}$ is a blow-up in at most three distinct points. Then X_3 satisfies Orlov's conjecture, i.e., $\operatorname{rdim}(X_3) = 2$.

Remark. Note that we can use a tower of this form to get as the resulting surface X_3 a blow-up of any nine distinct points on $\mathbb{P}^2$, as well as any del Pezzo surface. For del Pezzo surfaces Orlov's conjecture was proved in [BF12] using a completely different method.

2.5. Theorem ([Pir23a, Prop. 4.4]). *Consider a tower of blow-ups*

$$X_2 \to X_1 \to X_0 = \mathbb{P}^3,$$

where each map $X_i \to X_{i-1}$ is a blow-up of a disjoint union of some points and some lines, no more than three connected components per blow-up. Here by a line we mean a strict preimage of one-dimensional linear subspace in $\mathbb{P}^3 = X_0$. Then X_2 satisfies Orlov's conjecture, i.e., $\operatorname{rdim}(X_2) = 3$.

3 Stable indecomposability of triangulated categories

As noted above, constructing semiorthogonal decompositions in derived categories of coherent sheaves on varieties can be very helpful for understanding those categories. However, not every derived category admits a nontrivial semiorthogonal decomposition. We say that the derived category is *indecomposable* in that case. Which varieties have indecomposable derived categories is an interesting and important question.

Some examples of varieties with indecomposable derived categories are smooth proper varieties with trivial canonical bundle [Bri99], curves of positive genus [Oka11], and, more generally, varieties whose canonical linear system is globally generated [KO15].

The notion of indecomposability invites certain natural question. For example, assume that a variety X has an indecomposable derived category, and let Y be some other variety. Is it then true that any semiorthogonal decomposition of the category $D^b_{\mathrm{coh}}(X \times Y)$ is induced from a semiorthogonal decomposition of $D^b_{\mathrm{coh}}(Y)$? The induction here is the following process: if $\mathcal{A} \subset D^b_{\mathrm{coh}}(Y)$ is an admissible subcategory, consider the subcategory $D^b_{\mathrm{coh}}(X) \boxtimes \mathcal{A} \subset D^b_{\mathrm{coh}}(X \times Y)$ generated by the objects of the form $E \boxtimes F$, where $E \in D^b_{\mathrm{coh}}(X)$ is an arbitrary object and $F \in D^b_{\mathrm{coh}}(Y)$ is an object from the subcategory $\mathcal{A}$. Then this subcategory is admissible in $D^b_{\mathrm{coh}}(X \times Y)$ and one can check that a semiorthogonal decomposition of $D^b_{\mathrm{coh}}(Y)$ induces in this way a semiorthogonal decomposition of $D^b_{\mathrm{coh}}(X \times Y)$ (see, e.g., [Kuz11]).

The answer to this question is unknown in general. In the paper [Pir23b], on which this section of the thesis is based, we have introduced the notion of an *NSSI variety* (noncommutatively stably semiorthogonally indecomposable variety). If X is an NSSI variety, its derived category is indecomposable, and the same holds for any closed subvariety of X; moreover, this property also implies the positive answer to the question above, meaning that for any variety Y any semiorthogonal decomposition of $D^b_{\mathrm{coh}}(X \times Y)$ is, in fact, induced from $D^b_{\mathrm{coh}}(Y)$ provided that X is NSSI. In that paper we also establish the NSSI property for some class of varieties, including, for instance, abelian varieties (see Theorem 3.2 below).

The definition of the NSSI property is given in terms of "categories linear over $\mathrm{Perf}(X)$", where X is a scheme and $\mathrm{Perf}(X)$ is the triangulated category of perfect complexes on it. Roughly speaking, this is a triangulated category $\mathfrak{D}$ together with the action of the tensor-triangulated category $\mathrm{Perf}(X)$, i.e., for each object $E \in \mathrm{Perf}(X)$ we are given an endofunctor of $\mathfrak{D}$ which is "multiplication by the object E", and this set of endofunctors is compatible with the tensor product in $\mathrm{Perf}(X)$. In practice the triangulated structure happens to be too weak for notions like that to have nice properties and one needs to work with some enhancements. The rigorous definition of $\mathrm{Perf}(X)$-linear categories that we use and some of its properties are given in [Per18], with a brief reminder in [Pir23b]. As an illustration let us note that for any scheme Y the projection morphism $X \times Y \to X$ makes $\mathrm{Perf}(X \times Y)$ a $\mathrm{Perf}(X)$-linear category: the action functor corresponding to an object $E \in \mathrm{Perf}(X)$ is the endofunctor of $\mathrm{Perf}(X \times Y)$ given by the tensor product with the pullback of E to $X \times Y$.

3.1. Definition. The scheme X is said to be NSSI if for arbitrary choices of

1. $\mathfrak{D}$, a $\mathrm{Perf}(X)$-linear category which is proper over X and has a classical generator

2. $\mathcal{A} \subset \mathfrak{D}$ — an admissible subcategory of $\mathfrak{D}$;

the subcategory $\mathcal{A}$ is linear over $\mathrm{Perf}(X)$.

Despite the fact that this definition is highly abstract, for some schemes we can prove that the NSSI property holds. The two main results are below:

3.2. Theorem ([Pir23b, Th. 1.4]). *Let X be a scheme over a field $\Bbbk$ that admits an affine morphism into some abelian variety over $\Bbbk$. Then X is an NSSI scheme.*

Remark. If X is a smooth proper variety whose Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is a finite morphism, Theorem 3.2 implies that X is NSSI. In particular, all abelian varieties and all curves of positive genus are NSSI.

3.3. Theorem ([Pir23b, Th. 1.5]). *Let $\pi\colon X \to B$ be a flat proper morphism of quasi-compact separated schemes over the field $\Bbbk$. Assume that B is an NSSI scheme and that for any closed point $b \in B$ the fiber $X_b := \pi^{-1}(b)$ is an NSSI scheme. Then X is NSSI.*

Remark. In the published version of the paper in both theorems the notation for the scheme is Y rather than X.

Remark. If X is a bielliptic surface, then its Albanese morphism $\mathrm{alb}\colon X \to E$ is a fibration over an elliptic curve, and all fibers are elliptic curves. Thus by Theorem 3.2 all the assumptions of Theorem 3.3 are satisfied, and hence X is an NSSI variety.

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