Costs of stability

Jacques Drèze* Michel Le Breton† Alexei Savvateev‡
Shlomo Weber§

December 2005

Abstract

We consider a jurisdiction formation problem on the plane uniformly populated by a continuum of agents. This could be interpreted either as a real two-dimensional space where these agents live, or alternatively as a space of pairs of the two parameters of a public good on which the agents may have horizontally differentiated preferences. In the latter case, any agent is identified with the point of his best variety.

We study jurisdiction formation under transferable utility paradigm, and the main focus is on contribution schemes which lie in the core of a corresponding cooperative game. The proper core turns to be empty, and we consider the minimal ε-core. We show that, essentially, it contains only one allocation, namely the *egalitarian* (or *Rawlsian*) contribution scheme under which all agents are left with one and the same level of utility. It turns out that the minimal ε is extremely small — approximately, 0.0018.

Keywords: secession-proofness, optimal jurisdictions, Rawlsian allocation, hexagonal partition, (minimal) ε-core.

JEL Classification Numbers: D70, H20, H73.

*CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium.

†Université de Toulouse I, GREMAQ and IDEI, Toulouse, France.

‡Central Economics and Mathematics Institute, Moscow; Institute for Theoretical and Experimental Physics, Moscow; New Economic School, Moscow; CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium. Financial support through grants R98-0631 from the Economic Education and Research Consortium, # NSh-1939.2003.6 School Support, Russian Foundation for Basic Research No. 04-02-17227, and Russian Science Support Foundation is gratefully acknowledged.

§Southern Methodist University, Dallas, USA, and CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium.

1 Introduction

We consider a heterogeneous society whose population has to make selections from a two-dimensional space $\Re^2$ of horizontally[1] differentiated public projects of Mas-Colell (1983). We assume that members of a society have heterogeneous preferences over $\Re^2$, and that these preference relations are single-peaked, with respect to the standard Euclidean metric. This assumption could be interpreted as the existence of transportation costs. We identify each individual with the location of his best project on $\Re^2$, and assume *uniform distribution* of (best projects of) citizens over $\Re^2$.

For instance, population of a big city has to fix locations of several libraries, the overall number of which being also a matter of choice. Then, each citizen will be assigned to one and only one library. All the individuals assigned to one and the same library form a subset called "a jurisdiction"; as a result, we have a "jurisdiction structure" which is a partition of the society (or, equivalently, of the space $\Re^2$) into pairwise disjoint jurisdictions.

Following the choice of a jurisdiction structure, including selections of libraries in all jurisdictions, the contribution scheme towards costs of libraries must be chosen. We will assume budget balancing within every formed jurisdiction. In addition to these monetary costs, each citizen bears his personalized costs of being "far" from the library to which he is assigned. We assume that the monetary expression of a transportation cost is just the distance between the location of a citizen and the location of the library to which he is assigned.

Thus, the group choice of the locational problem described here consists of three items:

• - *jurisdiction structure*, which is a partition P of the space $\Re^2$ into subsets of individuals, *jurisdictions*, assigned to the same library;

• - *libraries locations* in each jurisdiction, and

• - *sharing rule*, that is a choice of a contribution scheme in order to cover the cost of a

[1]For the analysis of vertically differentiated projects, see Guesnerie (1995), Guesnerie and Oddou (1981,1988), Greenberg and Weber (1986), Jehiel and Scotchmer (2001), Weber and Zamir (1985), Westhoff (1977), Wooders (1978,1980).

library in every jurisdiction.

Our approach is that of *transferable utility (TU)* assumption.[2] First question of the analysis is that of a core, i.e. under which jurisdiction structures and subsequent contribution schemes the society will be stable, in a sense that no group of its members would wish to secede and form a new jurisdiction, in which total cost of every member is being decreased?

It turns out that the core is empty, i.e. there are no secession-proof partitions[3] and contribution schemes. Even if the most efficient partition of the society is chosen, which is the *hexagonal partition* (see below), there is no way to assign cost burden such that stability is assured.

Then, a natural question arises of how far we are from the core? Namely, if the government of the city decided to intervene and compensate a certain share of costs to citizens *in case* an efficient partition of $\Re^2$ is being implemented, what would be the minimal possible intervention necessary to reinforce secession-proofness?

We show that: (i) The minimal possible intervention is remarkably negligible, namely, it is sufficient to cover less than 0.002 per capita cost to reinsure stability, and (ii) Rawlsian (i.e. egalitarian) allocation, where everyone ends up with the same total cost (monetary plus transportation), plays a major part, being *the unique* allocation which is stable under the minimal necessary government intervention. Last observation provides an additional justification for egalitarism, together with (Bogomolnaia et al 2005a).

Results obtained here generalize (Le Breton et al 2004) to the simplest multi-dimensional case, which is the case of two dimensions. When the dimensionality is greater than 1, we encounter a problem of *inner geometry* of the space under consideration. Namely, we observe that the form of the optimal jurisdiction is not consistent with the space $\Re^2$ itself, in the sense that we cannot partition $\Re^2$ into most efficient jurisdictions (in fact, circles). This very fact is responsible for the emptiness of the core and, as a result, there is a case for exogenous

[2]For a treatment of NTU-case in the uni-dimensional setting, see (Bogomolnaia et al 2005).

[3]We will use terms *jurisdiction structure* and *partition* interchangeably, throughout the paper.

(government) intervention.

2 The Model

Let us specify a formal model. We assume that individuals are located uniformly over the whole (unbounded) space $\Re^2$. The volume of an arbitrary measurable[4] subset S will be denoted by $\lambda[S]$, or simply by $|S|$:

$$\lambda[S] = |S| = \int_S dt, \tag{1}$$

which is a real number from the closed segment $[0, +\infty]$.[5] Here, $t = (t_1, t_2)$ is a two-dimentional coordinate on $\Re^2$.

The cost of every library is given by a positive parameter g. The transportation cost incurred by individual[6] t, assigned to a library located at point p, is given by a cost function $d(t, p) = \sqrt{|t_1 - p_1|^2 + |t_2 - p_2|^2}$ which is the Euclidean distance on $\Re^2$.

Let us now introduce a concept of n-partition of an arbitrary measurable subset $S \subset \Re^2$ to an arbitrary positive or countable number n of its parts, *jurisdictions*:

Definition 1: An n-partition $P = (S_i)_{1 \le i \le n}$ is a jurisdiction structure that consists of n bounded measurable sets of a positive finite measure, pairwise disjoint up to a null-set, the union of which being equal to the entire subset S.[7] The set of all n-partitions P of

[4] An arbitrary subset is measurable if and only if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

[5] Throughout the paper, the following agreement will be made. Namely, when we calculate average values of functions over the whole space $\Re^2$, or over its Cartesian powers, or over an infinite-measured subset, we often write them as ratios of the two infinite integrals. This always makes sense in our story, since we impose sufficiently rigorous restrictions on all the functions used; namely, we require *periodicity* of allocations, sharing rules etc. so that we interpret these ratios as evaluated on the (finite-measured) periodicity generating set.

[6] We will not distinguish between individual t and an individual located at point $t \in \Re^2$.

[7] Restrictions on the measure and size of possible jurisdictions are imposed with the aim at having costs of all citizens uniformly bounded from above. If one allowed for null-set jurisdictions, then *all* its members would incur infinitely high costs; as for unbounded jurisdictions (including those of infinite measure), there is always the case that costs of members of such jurisdictions are unbounded from above.

a subset S will be denoted by $\mathcal{P}_n(S)$; the set of all partitions of S is denoted by $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \bigcup_{n=1}^{+\infty} \mathcal{P}_n(S) \bigcup \mathcal{P}_\infty(S). \tag{2}$$

Concerning partitions of $S = \Re^2$, we observe that they all contain infinite number of jurisdictions, according to Definition 1. Therefore, $\mathcal{P}(\Re^2) = \mathcal{P}_\infty(\Re^2)$, and we omit subscripts and superscripts, denoting the set of all partitions of $\Re^2$ by simply $\mathcal{P}$.

For the analysis of the location problem at hand, we construct a TU-game with the set of players coinciding with $\Re^2$, and the class of coalitions coinciding with all measurable sets, either finite- or infinite-measured. This will be done in two steps. As a first step, for each bounded measurable subset S of $\Re^2$ of a positive measure (which is a *possible* jurisdiction), denote by $D[S]$ the value of the following minimization problem:

$$D[S] := \min_{m \in \Re^2} \int_S d(t, m) dt. \tag{3}$$

This is called "MAT(S)" in Mathematical Programming, which is Minimal Aggregate Transportation of the set S. For obvious reasons, solution(s) to this problem exist (the integral in (3) is continuous in m, and for $m \to \infty$ the value of a program goes to $+\infty$). Any solution to (3) is called a *median location* of S, and we denote the set of all solutions to this program by $M(S)$ (by analogy with the uni-dimentional case where the set of solutions to the corresponding problem is

$$M(S) = \left\{ p \in \Re : \lambda\left(\{t \in S : t \leq p\}\right) = \lambda\left(\{t \in S : t \geq p\}\right) = \frac{1}{2}\lambda(S) \right\}, \tag{4}$$

i.e. coincides with the set of all medians of the uni-dimentional subset S). Once we have a possible jurisdiction S, its members would like to minimize transportation costs by placing their library to one of the points in $M(S)$. Next Lemma holds for the two-dimentional case under our consideration.

Lemma 1: For any possible jurisdiction $S \in \Re^2$, the set $M(S)$ is a singleton.

Due to Lemma 1, we will denote the unique median of a jurisdiction S by $m(S)$. As the second step, for an arbitrary possible jurisdiction $S \subset \Re^2$ we will define its *per capita characteristic function* (in terms of costs) as follows:

$$c[S] := \left(\frac{1}{|S|}\right)(g + D[S]). \tag{5}$$

So, we defined a characteristic function on the class of measurable subsets in $\Re^2$ with a positive finite measure. Alternatively, we could have allowed any group S to partition itself in an efficient way and not just to function as a unique jurisdiction; this is a concept of a *cover-game*. However, on this way we would not get any additional insights, in the sense that if a given allocation does not belong to the core of a cover-game, it will not be in the core of a simple game (see below).

At the same time, for the grand coalition, we of course partition it in the efficient way when consider stable contribution schemes. Efficiency here means minimizing the per capita total cost which we denote by $\bar{c}$. We have the following result, due to (Ballobas and Stern 1972; Haimovich and Magnanti 1988), see the discussion therein.

Theorem 1: Given the (uniformly populated) society $\Re^2$ with some g as fixed cost of a library, we have:

(i) $\bar{c} = c[H]$, where H is a hexagon of an optimal size;

(ii) Any partition $\bar{P} \in \mathcal{P}$ of $\Re^2$ into hexagons of the optimal size is efficient.

At the next step, we recall the notion of a core for this game, following a standard definition. Fix an arbitrary efficient partition $\bar{P} \in \mathcal{P}$ that consists of hexagons of the optimal size. Denote any of these hexagons by H. First, we introduce a concept of a *contribution scheme*, or equivalently of a *sharing rule* (these two terms are being used interchangibly, in what follows). Namely, a sharing rule, $x(t)$ describes monetary contribution of each individual t towards the cost of the library in his jurisdiction in the partition $\bar{P}$. We assume that each jurisdiction in

$\bar{P}$ balances its budget:[8]

$$\forall H \in \bar{P} \quad \int_H x(t)dt = g. \tag{6}$$

In addition, we assume for simplicity that any sharing rule is a periodic function, i.e. is a replication of one and the same sharing rule adopted within any jurisdiction.[9]

An *allocation* $c(\cdot)$ corresponding to a sharing rule, $x(\cdot)$, is the distribution of total costs in a population, after the rule $x(\cdot)$ is implemented:

$$c(t) = x(t) + d(t, m(H^t)), \tag{7}$$

where $H^t \in \bar{P}$ is the hexagonal jurisdiction containing t, and $m(H)$ is the center of the hexagon H (which is the location of a public good in a jurisdiction H). An allocation is uniquely defined up to a null-set where different jurisdictions can intersect. Now, we say that a given function $c(\cdot)$ is a *feasible cost allocation*, if there exists a sharing rule $x(\cdot)$ such that $c(\cdot)$ corresponds to $x(\cdot)$ via (7).

For any feasible allocation $c(\cdot)$ and an arbitrary measurable subset $S \subset \Re^2$ we define average costs of members of S, given the allocation $c(\cdot)$, by the following formula:

$$\bar{c}\{S, c(\cdot)\} = \left(\frac{1}{|S|}\right) \int_S c(t)dt; \tag{8}$$

when the allocation $c(\cdot)$ is fixed, we will abbrevate to write simply $\bar{c}_S := \bar{c}\{S, c(\cdot)\}$. For the grand coalition $S = \Re^2$, we of course have the following identity:

$$\bar{c}\left\{\Re^2, c(\cdot)\right\} \equiv \bar{c}, \tag{9}$$

for any feasible allocation, taking into account (6) and (7) — average total cost of all citizens under any feasible allocation following the efficient partition of the grand coalition $\Re^2$ equals the minimal per capita total cost $\bar{c}$.

[8]Since they all have the same form, there is no point in inter-jurisdictional transfers.

[9]This assumption simplifies calculus of the proof, while not being essential for the main result.

Now, we are ready to introduce a core of our cooperative game. The definition below simply says that if for some allocation $c(\cdot)$ there exists a coalition S such that $\bar{c}\{S, c(\cdot)\} > c[S]$, then this coalition S will secede and form a new jurisdiction, decreasing its overall cost.

Definition 2: Given the society $\Re^2$, we say that an allocation $c(\cdot)$ lies in the *core* (notation: $c(\cdot) \in C[\Re^2]$) if for any possible jurisdiction $S \subset \Re^2$ we observe that

$$\bar{c}\{S, c(\cdot)\} \leq c[S]. \tag{10}$$

Next definition introduces an allocation $r(\cdot)$ which plays a major part in what follows. This is the allocation for which the utility of the most disadvantaged individual is maximized (which allows us to name this allocation after Rawls). Under transferable utility, this implies equating costs, so that this allocation is also called *egalitarian*, and we use these two terms inter-changibly.

Definition 3: By the *Rawlsian*, or *egalitarian* allocation we mean the allocation $r(t) \equiv \bar{c}$.

3 The main result

Now we are ready to state and prove main findings of our study. First, we demonstrate that the core $C[\Re^2]$ is empty. This mere fact leaves us non-satisfied concerning questions "what then to do" and "what we expect to be the outcome of the game". When the core is empty, one is sometimes searching for a solution which is mostly close to being in the core. For instance, we may have assumed that there is a fixed per capita cost of a secession by any subgroup $S \neq \Re^2$; alternatively, one can consider the "government intervention" which compensates a certain fraction of a total cost to every citizen, in case they do not form seceding coalitions. Both approaches are essentially equivalent; using the latter approach, we come to a following definition of an ε-*core*:

Definition 4: Given the society $\Re^2$, we say that an allocation $c(\cdot)$ lies in the ε-*core* (notation: $C_\varepsilon[\Re^2]$), if for any measurable subset $S \subset \Re^2$ of a positive finite measure, we have

$$(1-\varepsilon)\bar{c}\{S, c(\cdot)\} \leq c[S]. \tag{11}$$

In other words, if people are following "the agreement" of the grand coalition, then the ε-part of costs is covered "from outside"; if, however, a certain coalition, S poses a threat for a secession, then their members have to bear costs on their own.

This definition relaxes the constraints which determine the core, and leaves us with a hope that, for certain values of ε non-emptiness of the core is being reinforced. Formally, we consider the value $\hat{\varepsilon}$ such that

- $C_{\hat{\varepsilon}}[\Re^2] \neq \emptyset$;
- For every $\varepsilon < \hat{\varepsilon}$, we observe that $C_\varepsilon[\Re^2] = \emptyset$.

We will demonstrate that this value exists, and even more, we give its full characterization. For the $\hat{\varepsilon}$-core, we have yet another name: we call it a *minimal* ε-*core*, and denote by $\mathcal{C}$.

In order to formulate the main result of the analysis, it is left to introduce the notion of an optimal jurisdiction form. Consider the following minimization problem over the class of all measurable subsets $S \subset \Re^2$ of a positive finite measure:

$$\min_{S \subset \Re^2} c[S]. \tag{12}$$

That is, we search for jurisdictions that minimize total per capita costs of its members. Denote any potential solution to (12) by $\hat{S}$. We have the following result.

Lemma 2: Denote $m(\hat{S})$ by $\alpha \in \Re^2$. Then, up to a null-set, $\hat{S}$ coincides with a circle B_α^l centered at the point α, and of an optimal radius l. (Therefore, all the solutions of (12) are parametrised by points in $\Re^2$, namely, centers of these circles: $\{B_\alpha\}_{\alpha \in \Re^2}$. When referring to optimal jurisdictions, we omit the superscript l, writing them as simply B_α.

Denote by $\hat{c}$ the value of the problem (12), that is, $\hat{c} = c[\hat{S}]$, for any $\hat{S}$ solving the problem (12). We are prepared to state and prove the main result of our analysis.

Theorem 2: Given the (uniformly populated) society $\Re^2$, we have:

(i) $\hat{\varepsilon} = 1 - \frac{\hat{c}}{\bar{c}} \neq 0$;

(ii) $\mathcal{C} = \mathcal{C}_{\hat{\varepsilon}} = \{r(\cdot)\}$ — a singleton, up to a null-set.

Summing this up, we state that the proper core is empty, and that the minimal ε-core is single-valued, containing ONLY the Rawlsian allocation.

4 References

Ahn, S., Cooper, C., Cornuéjols, G. and A. Frieze (1988) "Probabilistic Analysis of a Relaxation for the k-Median problem", *Mathematics of Operations Research*, 13, 1-31.

Alesina, A. and E. Spolaore (1997) "On the Number and Size of Nations", *Quarterly Journal of Economics,* 113, 1027-1056.

Aumann, R. and J. Drèze (1974) "Cooperative Games with Coalition Structure", *International Journal of Game Theory,* 3, 217-237.

Balinski, M.L. (1965) " Integer Programming : Methods, Uses, Computation", *Management Science*, 12, 253-313.

Billingsley, P. (1995) *Probability and Measure*, Third edition. Wiley, New York.

Bollobas, B. and N. Stern (1972) "The Optimal Structure of Market Areas", *Journal of Economic Theory,* 4, 174-179.

Christaller, W. (1933) *Die Zentralen Orte in Suddeutschland*, Jena: Fisher (English Translation, *Central places in Southern Germany*, Englewood Cliffs, Prentice Hall, 1966).

Chudak, F.A. (1998) "Improved Approximation Algorithms for Uncapacited Facility Location", in R.E. Bixby, E.A. Boyd and R.Z. Rios-Mercado, Eds, *Integer Programming and*

Combinatorial Optimization, Springer LNCS, Volume 1412 of *Notes in Computer Science*, Springer, 180-194.

Chudak, F.A. and D.B. Shmoys (1999) ”Improved Approximation Algorithms for the Uncapacited Facility Location Problem”, Mimeo, Cornell University.

Cornuéjols, G. Nemhauser, G.L. and L.A. Wolsey (1990) ”The Uncapacited Facility Location Problem” in P. Mirchandani and R. Francis, Eds, *Discrete Location Theory*, John Wiley and Sons, New York, 119-171.

Fejes Toth, L. (1953) *Lagerungen in der Ebene, auf der Kugel und im Raum,* Vol 65, Springer-Verlag, Berlin-Gottingen-Heidelberg

Fisher, M.L. and D.S. Hochbaum (1980) ”Probabilistic Analysis of the Planar k-Median Problem”, *Mathematics of Operations Research*, 5, 27-34

Goemans, M.X. and M Skutella (2004) ”Cooperative Facility Location Games”, *Journal of Algorithms*, 50, 192-214.

Guha, S. and S. Khuller (1998) ”Greedy Strikes Back : Improved Facility Location Algorithms”, in *Proceedings of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, 649-657.

Haimanko, O., Le Breton, M. and S. Weber (2004) “Voluntary Formation of Communities for the Provision of Public Projects”, *Journal of Economic Theory,* 115, 1-34.

Haimovich, M. and T.L. Magnanti (1988) ”Extremum Properties of Hexagonal Properties and the Uniform Distribution in Euclidean Location”, *Siam Journal of Discrete Mathematics*, 1, 50-64

Hales, T.C. (2001) ”The Honeycomb Conjecture”, *Discrete and Computational Geometry*, 25, 1-22.

Le Breton, M. and S. Weber (2003) “The Art of Making Everybody Happy: How to Prevent a Secession?” *IMF Staff Papers,* 50 (3), 403-435.

Le Breton, M., Weber, S. and J. Drèze (2005) “Rawlsian Pricing of Access to Public facilities : A Unidimensional Illustration", Mimeo, Université Catholique de Louvain.

Lösch, A. (1954) *The Economics of Location*, Yale University Press, New Haven.

Madhian, M., Ye, Y. and J. Zhang (2002) "Improved Approximation Algorithms for Metric Facility Location Problems" in *The 5th International Workshop on Approximation Algorithms for Combinatorial Optimization*, 229-242.

P. Mirchandani and R. Francis, Eds, *Discrete Location Theory*, John Wiley and Sons, New York, 1990.

Papadimitriou, C.H. (1981) " Worst-Case and Probabilistic Analysis of a Geometric Location problem", *Siam Journal of Computing*, 10, 542-557.

Shmoys, D.B., Tardos, E. and K. Aardal (1997) "Approximation Algorithms for Facility Location Problems", in *Proceedings of the 29th Symposium on Theory of Computing*, 265-274.

Stern, N. (1972) "The Optimal Size of Market Areas", *Journal of Economic Theory,* 4, 154-173.

Sviridenko, M. (2002) "The 1.582-Approximation Algorithm for the Metric Uncapacited Facility Location Problem" in *The 9th Conference on Integer Programming and Combinatorial Optimization*, 240-257.

Young, H.P. (1994) "Cost Allocation", Chapter 34 in R.J. Aumann and S. Hart, Eds, *Handbook of Game Theory*, Vol 2, North Holland, Amsterdam.