Abstract

This work extends the spatial voting model to include variable voter turnout. I consider two alternative assumptions. First, I look at voter indifference, when the probability of a voter turning out depends on the difference in utility from the election of her most preferred and second most preferred candidate. The second assumption is voter alienation, when the probability of turning out depends on the utility from the election of her most preferred candidate. For a deterministic model, I show that in an equilibrium, the positions of the candidates do not necessarily converge to the median voter. I then study how the positions of the candidates, their relative shares of winning, and turnout depend on the distribution on voter preferences and on nonspatial candidate characteristics. In a probabilistic voting model, indifference is shown to reduce the stability of the convergent equilibrium.

1 Introduction

A spatial model of elections involves candidates who propose policy platforms and voters who choose which candidate to support based on the proximity of the candidate's platform to the voter's most preferred policy. The early and best-known result (Downs, 1957) was that if there are two vote-maximizing candidates, the policy space is one-dimensional, and the voter preferences are single-peaked, then both candidates should choose the same policy platform, identical to the median most preferred policy of the voters. If the policy space was more than one-dimensional, there was no stable outcome (Plott, 1967, McKelvey, 1976).

This contradicts the empirical evidence, as the observed policy positions of candidates and parties are relatively stable, and they do not usually converge. This observed disparity was the motivation behind a large body of theoretical work, analyzing such concepts as probabilistic voting (Hinich, Ledyard, and Ordeshook, 1972a, Enelow and Hinich, 1982), office-motivated candidates (Wittman, 1977), or valence (Groseclose, 2001).

One possible explanation for policy divergence is that the political platforms of candidates or parties affect the decisions of individual voters whether or not to vote. Turnout consequences are almost certainly taken into account when political platforms are announced. Losing the support of the base voters is one the main reasons that keeps politicians from trying to "steal the political clothes" of their opponents and converge to the median voter[1].

Explaining positive turnout in the framework of the rational choice theory is a major theoretical challenge. There is a large and growing body of literature on the topic[2], but there is no consensus on what kind of behavior makes individuals participate in large elections.

The "paradox of the rational voter" is a consequence of the fact that each single vote is unlikely to be decisive if the overall number of voters is large (Riker and Ordeshook, 1968, Davis, Hinich, and Ordeshook, 1970, Chamberlain and Rotschild, 1982, Myerson, 2000). Thus, if there are positive costs of participating in the elections (like travel expenses, time, gathering information, etc.), then the voter is better off not voting.

This paradox was first remarked upon by Anthony Downs (1957) in his well-known work. He did not address the issue directly, attributing widespread voting to extra-theoretic (and irrational) factors.

The game-theoretic models with strategic voters did not produce conclusive results (Ledyard, 1984, Palfrey and Rosenthal, 1983, 1985). The basic argument of such models can be

[1] Turnout is one of several mechanisms that can result in policy divergence. For a review of literature on this topic see Zakharov (2006).

[2] Other literature reviews include, among others, Aldrich (1993), Lijphart (1997), and Feddersen (2004).

formulated as follows. If no one is voting, then the outcome of the elections will depend on the choice of any single voter who decides to vote. Other voters will become active as long as the benefit of voting exceeds the cost. Thus it is possible to have an equilibrium where all voters are rational and some level of voting activity is present. In such an equilibrium, every active voter's expected benefit of voting will be no less than the cost. Nevertheless, this argument is insufficient, since the participation rate in large electorates will be very small.

Different models of voter behavior were suggested. Ferejohn and Fiorina (1974) looked at voters as regret minimizers. A more recent strand of literature views abstention as a result of rational behavior of voters who are not perfectly informed about their benefit from the election of a particular candidate. It was argued that if voters are not perfectly informed about their preferences, some voters might abstain even if the cost of voting is zero. It was argued that the less informed voters may abstain in order to allow the more informed voters to decide the outcome (Feddersen and Pesendorfer, 1996, Feddersen, 2004).

The evidential decision model of Grafstein (1991) treats every voter as thinking that her action will influence the actions of all other voters. In this setting both the perceived probability of being pivotal and turnout are higher. In a model by Kanazawa (1998), backward-looking voters similarly associate their past voting behavior with the outcome of the previous elections.

Eldin, Gelman, and Kaplan (2005) considered voters who care not only for her own utility, but also for the utility of every other individual in the society. Similar approach is followed by Harsanyi (1980), Feddersen and Sandroni (2002), and Coate and Conlin (2005).

Other theoretical arguments involve a third type of agent — the group leader, who may reward individual voters for their participation and support of a particular candidate (Morton, 1991, Uhlaner, 1989).

There are several hypotheses relevant to the spatial theory that can be tested empirically. The best-studied prediction is that turnout depends on the closeness of the election, as in a closer election the probability of casting the decisive vote is higher (Geys, 2006, contains a review of relevant literature). Most tests support the hypothesis, although the evidence is sometimes contradictory, such as in Kirchgässner and Zu Himmern (1997) study of German General Elections for 1983–1994.

The so-called "mobilization hypothesis" provides an alternative explanation to the (possible) positive relationship between turnout and election closeness. It can be argued that if the elections are more closely contested, then the competing candidates are mobilized to procure additional turnout (see, for example, a study by Cox and Munger, 1989, linking campaign spending and election closeness).

2 Spatial voting models under indifference and alienation hypotheses.

The results of spatial models with variable turnout and strategic voters are mixed and inconclusive, especially when one is interested in the effect that turnout has on the positions of the candidates. One proposed solution is to simplify the model by assuming that the voters decide whether to vote or to abstain according to some fised rule.

The two well-known conjectures linking the likelihood of turnout and the policy positions of the candidates are known as *indifference* and *alienation* hypotheses. Under the indifference hypothesis, a voter casts her ballot if and only if there is sufficient difference between payoffs that the candidates offer to the voter. Hence, a voter who is indifferent between the candidates will abstain. Under the *alienation hypothesis*, a voter will abstain if she is sufficiently dissatisfied with the policies promised by either of the candidates.

Until recently, there were relatively few works investigating indifference and alienation hypothesis. The in the earliest work on the subject (Brody and Page, 1973) the authors analyzed survey data collected after 1968 U.S. Presidential elections. It was found that the likelihood of the respondent having voted was greater if her evaluation of her most preferred candidate was more favorable. A similar relationship supporting the indifference hypothesis was also found. Later works by Zipp (1985) and Plane and Gershtenson (2004) also found support for both indifference and alienation hypotheses using survey-level data.

In a recent paper, Adams, Dow, and Merrill (2006) used a conditional logit model to estimate the alienation and indifference components of abstention. The authors found that for the 1980-1992 Presidential elections both factors contributed to depressed turnout, with no substantial partisan differences in their effects.

A plausible way to introduce indifference and alienation in a spatial model is to assume that the probability of voting is a function of the positions of the candidates and the policy preference of the voter. Under the indifference hypothesis, one would assume that the probability of voting declines with the policy distance between the candidates; under the alienation hypothesis, the probability of voting would decline with the distance between the voter and the nearest candidate.

In the original work by Downs (1957) it was noted that voter alienation may result in the divergence of candidate platforms from the median voter If we assume that alienated voters abstain, then a candidate who decides to move her platform closer to the median voter' bliss point faces a tradeoff: on one hand, she gains some votes closer to the center of the political spectrum (the "moderate" voters), but on the other hand she may be bound to lose some votes on the far left (those of the "extreme" voters).

More formal analysis, starting with Davis, Hinich, and Ordeshook (1970), Hinich, Ledyard, and Ordeshook (1972a,b), and McKelvey (1975), suggested that voter indifference by itself should not be sufficient to dislodge a median voter equilibrium. Voter alienation is more likely to lead to the divergence of policy positions, as a candidate who decides to move her platform closer to the median voter' bliss point faces a tradeoff: on one hand, she gains some votes closer to the center of the political spectrum (the "moderate" voters), but on the other hand she may be bound to lose some votes on the far left (those of the "extreme" voters). If the distribution is unimodal, and the voter density at the peak of the distribution is sufficiently high, then both candidates converging to the mode is a local equilibrium. If the distribution is symmetric and the median and the mean coincide with the mode, then the equilibrium is a global one.[3]

The effect of voter indifference and alienation on candidate behavior in a Downsian framework was studied by Kirchgässner (2003). He assumed that the probability that a voter turning out is a function of voter's position, as well as of the positions of the candidates. It was assumed to be a decreasing function of the "relative distance" — the ratio between the difference and the sum of the distances from the voter's position to the positions of either candidate (thus, the relative distance is large if the voter is either in the middle between the candidates or far away from either candidate, and is zero if the voter's position coincides with the position of one of the candidates).

If the relationship between relative distance and the probability of turnout is linear, the degree to which candidate policy platforms diverge depends on whether the candidates maximize absolute or relative voteshare, with no convergence in the first case, and convergence to the median voter in the second. If the relationship is between relative distance and the probability of turnout is threshold, then there is no convergence.

[3]If the distribution of voter preferences is asymmetric, a local equilibrium in which candidates select different policy platforms may not be a global equilibrium (Comanor, 1976).

The model presented in this work is similar in spirit to that of Kirchgässer (2003). However, there are two important differences. First, the authors consider the voters with policy preferences distributed over a single-dimensional policy space according to a general-form continuous distribution (it was taken to be uniform in the latter work).

The second feature of the model studied in this work is that we use the concept of candidate valence (notion of candidate valence is attributed to Stokes, 1963). This term refers to candidate characteristics such as popularity, name recognition, experience[4], and other factors that contribute to a voter's satisfaction with the candidate regardless of that candidate's policy position.

Different candidate valence has several implications for spatial models of voting. First, there is no equilibrium in the standard voter[5]. Any position of the low valence candidate can be matched by her rival, who will obtain all the votes as a result. Second, changes in the political platforms of the candidates have asymmetric effects on the position of the indifferent voter if the voters are risk-averse. A change in the policy position of the candidate with the greater valence will have a greater impact on the indifferent voter's position than an equal change in the position of her rival.

3 Model assumptions.

There is a continuum of voters with policy ideal policies distributed on a convex compact set $X \subset \mathbf{R^n}$ with a continuous density $f(\cdot)$.

There are $K \geq 2$ candidates with policy positions $y_1, \cdots, y_K$.

If candidate j is elected, a voter with the ideal policy $v \in [0, 1]$ receives a utility of

$$u_j(v) = \epsilon_j - \phi(\|y_j - v\|). \tag{3.1}$$

Here, y_j is the policy position of Candidate j, e_j is the valence of Candidate j, and $\phi(\cdot)$ is a twice-differentiable function with $\phi'(\cdot) > 0$, $\phi'(\cdot) > 0$, $\phi'(0) = 0$, and $\phi(d) = \phi(-d)$ for all $d > 0$. This function reflects the voter's disutility from the difference between the realized policy and the voter's preferred policy v.

The voters are assumed to be sincere. The choice of a voter with the ideal policy v depends on the utilities $u_j(v)$ for $j = 1, \cdots, K$ and is described by the function $t : \mathbf{R}^K \rightarrow \Delta^{K+1}$, where t_{K+1} denotes the probability of abstention.

Under the regular sincere voting hypothesis, a voter supports a candidate who delivers the highest utility, or fairly randomizes if there are several such voters.

Sincere voting (SV)

$$t_j(u_1(v), \cdots, u_K(v)) = \begin{cases} \frac{1}{\#\{i|u_i(v)=\max_k u_k(v)\}}, & u_j(v) = \max_k u_k(v) \\ 0 & , u_j(v) \neq \max_k u_k(v). \end{cases} \tag{3.2}$$

For $K = 2$, the sincere voting hypothesis is consistent with the behavior of a rational voter whose voting costs are zero.

Under the indifference hypothesis, a voter supports a candidate only if the utility that the candidate delivers to the voter is significantly bigger than the next highest utility.

[4]The implications of endogenous valence were analyzed by Zakharov (2005).

[5]One can either analyze the mixed equilibrium, as Aragones and Palfrey (2002), assume that the candidates are policy-motivated, as was done by Groseclose (2001), or look at the conditions for the existence of an equilibrium in several dimensions, as in Ansolabehere and Snyder (2000).

Indifference (IH).

$$t_j(u_1(v), \cdots, u_K(v)) = \begin{cases} 1, & u_j(v) - c \geq \max_{k \neq j} u_k(v) \\ 0, & u_j(v) - c < \max_{k \neq j} u_k(v). \end{cases} \tag{3.3}$$

The indifference assumption is consistent with the behavior of a rational voter who has voting costs of C, believes that her vote will be pivotal with the exogenous probability $p = \frac{C}{c}$, and believes that when her vote is pivotal, the alternative is the election of the next-best candidate.

Alienation (AH).

$$t_j(u_1(v), \cdots, u_K(v)) = \begin{cases} 1, & u_j(v) \geq \max\{d, \max_{k \neq j} u_k(v)\} \\ 0, & u_j(v) < \max\{d, \max_{k \neq j} u_k(v)\}. \end{cases} \tag{3.4}$$

The alienation hypothesis is consistent with the behavior of a rational voter who has voting costs C, believes that her vote will be pivotal with the exogenous probability $p = \frac{C}{d}$, and believes that when her vote is pivotal, the alternative is the implementation of a status quo policy which delivers zero utility to the voter.

A more conventional interpretation of the alienation hypothesis is that a voter is simply reluctant to support a candidate who delivers a low level of utility, with the psychological benefit of abstaining exceeding the possible value of being pivotal against the candidates who are even less acceptable to the voter.

Next we must define the payoffs of the candidates. The voteshare of candidate j is equal to

$$V_j = \int t_j dF(v). \tag{3.5}$$

There are potentially two sources of candidate motivation. The classical Downsian view is that candidates are motivated solely by winning the elections. The probability of winning office is an increasing function of either the share of vote obtained by the candidate, or of the difference between the candidate's voteshare and the largest voteshare among the opposing candidates (the candidate's *plurality*). This distinction is irrelevant for a two-candidate model with perfect turnout, but may become important if the voteshares of the candidates do not add up to one[6].

Office-motivated candidates (OMC). The utility of Candidate j is

$$U_j = (1 - \lambda)V_j + \lambda(V_j - \max_{k \neq j} V_k). \tag{3.6}$$

The parameter $0 \leq \lambda \leq 1$ is the weight of plurality versus voteshare in determining the candidate's chances of winning.

[6]In a work by Hinich, Ledyard, and Ordeshook (1972), the probability $P_1(y_1, y_2, v)$ of a voter with policy preference v supporting candidate 1 was a continuously differentiable functions with the following properties: $P_1 = 0$ if $\phi(|v - y_1|) > \phi(|v - y_2|)$, P_1 is decreasing in $\phi(|v - y_1|) - \phi(|v - y_2|)$ and in $\phi(|v - y_1|)$ if $\phi(|v - y_1|) < \phi(|v - y_2|)$, and $P_1(y_1, y_2, v) = P_2(y_2, y_1, v)$. A one-shot game between two plurality-maximizing candidates produced policy convergence. Crucially, their result depended on the candidate objective functions being symmetric in y_1 and y_2. This is not the case if the candidates have different valence.

An alternative hypothesis is that a candidate is interested in having a certain policy objective realized after the elections.[7]

Policy-motivated candidates (PMC). The utility of Candidate i is

$$U_i = -\psi(|y - \bar{y}_i|). \tag{3.7}$$

The value $\bar{y}_i$ is the ideal policy of Candidate i, while $\psi(\cdot)$ is the disutility function of the candidate.[8] The value y is the policy that the candidate expects will be realized after the elections.

4 Indifference and alienation in the deterministic model.

In this section I look at a one-dimensional, two-candidate deterministic voting model. The advantage of using a deterministic model is the the possibility of doing a comparative statics analysis at the equilibrium. One can derive the conditions that describe the positions of the candidates in an equilibrium, and see how the positions change with the changes in the model parameters — cost of voting, valence of the candidates, and the distribution of voters.

The voter with policy preference $\tilde{y}$ is the *indifferent voter* if

$$\epsilon - \phi(y_1 - \tilde{y}) = -\phi(y_2 - \tilde{y}), \tag{4.8}$$

where $\epsilon = \epsilon_1 - \epsilon_2$. Provided that $y_1 < y_2$, all voters with $y < \tilde{y}$ receive higher utility under Candidate 1, while the rest of the voters receive higher utility under Candidate 2.

Under **IH**, the indifferent voter will abstain, as well as the voters in his neighborhood. Denote by $\bar{y}_1$, $\bar{y}_2$ the leftmost and rightmost abstaining voter. We have

$$\epsilon - \phi(\bar{y}_1 - y_1) - c + \phi(y_2 - \bar{y}_1) = 0 \tag{4.9}$$

and

$$\epsilon - \phi(\bar{y}_2 - y_1) + c + \phi(y_2 - \bar{y}_2) = 0. \tag{4.10}$$

The following result is straightforward.

Proposition 4.1 *Let $n = 1$, $K = 2$. Under* **IH** *and* **OMC**,

1. *No pure-strategy equilibria exists if $\epsilon_1 \neq \epsilon_2$.*
2. *If $\epsilon_1 = \epsilon_2$, an equilibrium exists, with $y_2 - y_1 = \phi^{-1}(c)$ and $f(\bar{y}_1) = f(\bar{y}_2)$.*

If one of the candidates shifts her policy position toward that of her opponent, her voteshare may change for two reasons. First, the position of the indifferent voter will change; second, the turnout will be affected. Since we assumed that voter disutility is concave in policy distance, the turnout will decrease by a greater amount if the positions of the two candidates are closer. At some point, the marginal voteshare effect of a change in a candidate's position will be zero.

[7] This idea was first exploited in the works of Donald Wittman (1983) and Randall Calvert (1985).

[8] A work of Timothy Groseclose (2001) explores a two-candidate game with the candidates maximizing a weighted average of (3) for $\lambda = 0$ and (3). His work did include candidates with different valence, but did not consider the possibility of voters abstaining. For tractability's sake we restrict our attention to one of the two cases.

If the two candidates have equal valence, then the changes in the positions of the candidates will have symmetric effects on their voteshare, so an equilibrium is possible. If the valence is asymmetric, so is the effect of a candidate's position on her voteshare. Thus if y_1 is candidate 1's best response y_2, then y_2 is not a best response to y_1.

Example. Let $\phi(x) = x^2$ and $\epsilon_1 - \epsilon_2 = \epsilon \geq 0$. Then from (6.73) and (6.74) we have

$$\bar{y}_1 = \frac{y_1 + y_2}{2} + \frac{\epsilon - c}{2(y_2 - y_1)} \tag{4.11}$$

and

$$\bar{y}_2 = \frac{y_1 + y_2}{2} + \frac{\epsilon + c}{2(y_2 - y_1)}. \tag{4.12}$$

The utilities of the candidates (6.73), (6.74) will be given by

$$U_1 = (1+\lambda)\frac{y_1 + y_2}{2} + \frac{1}{2(y_2 - y_1)}((1+\lambda)\epsilon - (1-\lambda)c) \tag{4.13}$$

and

$$U_2 = 1 - (1+\lambda)\frac{y_1 + y_2}{2} - \frac{1}{2(y_2 - y_1)}((1+\lambda)\epsilon + (1-\lambda)c). \tag{4.14}$$

The best responses of the two candidates are given by

$$y_1(y_2) = \begin{cases} y_2 + \sqrt{2\frac{(1-\lambda)c - (1+\lambda)\epsilon}{1+\lambda}}, & (1-\lambda)c > (1+\lambda)\epsilon \\ y_2, & (1-\lambda)c < (1+\lambda)\epsilon. \end{cases} \tag{4.15}$$

and

$$y_2(y_1) = y_1 + \sqrt{2\frac{(1-\lambda)c + (1+\lambda)\epsilon}{1+\lambda}}. \tag{4.16}$$

I now consider the effects of the alienation hypothesis. The alienation hypothesis claims that a voter will abstains if the positions of both candidates are sufficiently different from her own ideal position. Thus a candidate who decides to move her position closer to her opponent's faces a dilemma. On one hand, the candidate will capture additional votes from her opponent. On the other hand, the candidate will lose votes on the far end of the political spectrum. The effects of voter density and candidate valence on policy positions in such an equilibrium are nontrivial and demand investigation.

Under an alienation hypothesis, a voter votes only if her distance from the nearest candidate is sufficiently small. Denote by

$$d_i = \phi^{-1}\left(\epsilon_i - \frac{c'}{p}\right) \tag{4.17}$$

the distance between the the position of candidate i and the position of the voter who prefers candidate i to the other candidate and is on the verge of abstaining because of alienation.

We call the voters who are on the threshold of abstaining *ambivalent voters.*

There are potentially two cases (see Fig. 1). In the first case, with

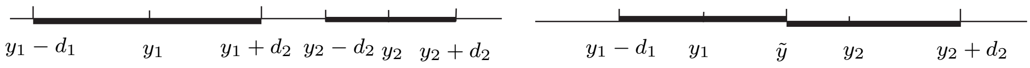
$$y_2 - y_2 > d_1 + d_2, \tag{4.18}$$

there are alienated voters with positions between y_1 and y_2. In the opposite case, with

$$y_2 - y_1 \leq d_1 + d, \tag{4.19}$$

all voters with positions between y_2 and y_1 participate.

The equilibrium conditions and comparative statics are different in for case. For the first case, the following result holds:



(a) Case 1: $y_2 - y_1 > d_1 + d_2$ (b) Case 2: $y_2 - y_1 \leq d_1 + d_2$

Figure 1: Voter choice depending on candidate location

Proposition 4.2 *Let y_1, y_2 be a LNE under* **OMC** *and* **AH***, such that $y_2 - y_1 \geq 2d$. Then we have:*

$$\begin{array}{ll} f(y_1 + d_1) = f(y_1 - d_1), & f'(y_1 + d_1) - f'(y_1 - d_1) < 0 \\ f(y_2 + d_2) = f(y_2 - d_2), & f'(y_2 + d_2) - f'(y_2 - d_2) < 0. \end{array} \tag{4.20}$$

If the candidates are so far apart that there are alienated voters with intermediate positions, then changes in the position of one candidate have no effect on the voteshare of the other candidate.

The comparative statics in this equilibrium are straightforward:

Corollary 4.1 *Let y_1, y_2 be a LNE under* **OMC** *and* **AH***, such that $y_2 - y_1 \geq 2d$. Then we have*

$$\frac{\partial y_1}{\partial d_1} \geq 0 \tag{4.21}$$

if and only if $f'(y_1 - d_1) + f'(y_1 + d_1) \geq 0$.

The effect of both a reduction of the voting cost and the increase in candidate valence is identical, as both the distance between the ambivalent voters $2d_1$ and the candidate's share of vote increases.

The second case is more interesting. The equilibrium is described as follows:

Proposition 4.3 *Let y_1, y_2 be a LNE under* **OMC** *and* **AH***, such that $y_2 - y_1 > d_1 + d_2$. Let $f'(\tilde{y}) \neq 0$ and $f'(y_1 - d_1) = f'(y_2 + d_2) = 0$. Then, the following holds:*

$$f(\tilde{y}) \frac{\phi'(\tilde{y} - y_1)}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})} = (1 - \lambda) f(y_1 - d_1), \tag{4.22}$$

$$f(\tilde{y}) \frac{\phi'(y_2 - \tilde{y})}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})} = (1 - \lambda) f(y_2 + d_2), \tag{4.23}$$

$$\epsilon - \phi(\tilde{y} - y_1) = -\phi(y_2 - \tilde{y}), \tag{4.24}$$

and

$$\frac{f'(\tilde{y})}{f(\tilde{y})} < \frac{1}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})} \min \left\{ \frac{\phi'(\tilde{y} - y_1)\phi''(y_2 - \tilde{y})}{\phi'(y_2 - \tilde{y})}, \frac{\phi'(y_2 - \tilde{y})\phi''(\tilde{y} - y_1)}{\phi'(\tilde{y} - y_1)} \right\}. \tag{4.25}$$

In an equilibrium, each candidate loses and gains an equal amount of votes by moving her position. Since the candidate with the valence advantage gains more votes than her opponent if she moves her position toward the indifferent voter, it follows that the density of voters is greater in the neighborhood of the voter who is ambivalent between voting for the advantaged candidate and abstaining, than in the neighborhood of the other ambivalent voter.

The first corollary is an immediate consequence of the fact that the voters are risk-averse:

Corollary 4.2 *Let $\epsilon_1 > \epsilon_2$. Then $f(y_1 - d_1) > f(y_2 + d_2)$.*

Since a voter with a higher valence gains more voteshare if she moves toward the indifferent voter, in an equilibrium she must also be bound to lose more voter due to alienation.

We want to know how the equilibrium positions of the candidates and the position of the indifferent voter are affected by changes in the valence advantage of the first candidate, by changes in voter densities in the neighborhood of the indifferent voter and the ambivalent voters, and by cost of voting c'.

We have to make several assumptions. First, we assume that the third derivative of the disutility function $\phi(\cdot)$ is negative. This assumption has the following interpretation. Suppose that a voter with the ideal policy v has to choose between two options: policy $y > v$ and a lottery where policies $y - a$ and $y + a$ are realized with probability $\frac{1}{2}$ each. If the third derivative of the disutility function is negative, then the difference in utility from these two potions declines with the policy distance $y - v$.

The third derivative assumption is inherently consistent with the alienation hypothesis. As the distance between the candidates and the voter increases, the voter is willing to pay less in order to insure herself against a lottery on the candidate's positions, and thus is less likely to vote.

The second assumption that we make is that the voter density is constant in the neighborhood of the ambivalent voters.

Corollary 4.3 *Let $y_1 < y_2$ be equilibrium positions of the candidates, and let $\tilde{y}$ be the position of the indifferent voter. Let $f'(y_1 - d_1) = f'(y_2 + d_2) = 0$ and $f'(\tilde{y}) \neq 0$. Then, the following holds:*

$$\frac{\partial y_1}{\partial d} = 0, \ \frac{\partial y_2}{\partial d} = 0, \ \frac{\partial \tilde{y}}{\partial d} = 0, \ \frac{\partial (y_2 - y_1)}{\partial f(\tilde{y})} = 0, \ \frac{\partial (y_2 - y_1)}{\partial \lambda} = 0, \ \frac{\partial \tilde{y}}{\partial \epsilon} = 0. \tag{4.26}$$

Each of the following holds if and only if $f'(\tilde{y}) > 0$:

$$\frac{\partial (y_2 - y_1)}{\partial f(y_1 - d_1)} < 0^{9}, \ \frac{\partial (y_2 - y_1)}{\partial f(y_2 + d_2)} < 0, \ \frac{\partial \tilde{y}}{\partial f(y_1 - d_1)} > 0, \ \frac{\partial \tilde{y}}{\partial f(y_2 + d_2)} < 0, \tag{4.27}$$

$$\frac{\partial y_1}{\partial f(\tilde{y})} < 0, \ \frac{\partial y_2}{\partial f(\tilde{y})} < 0, \ \frac{\partial \tilde{y}}{\partial f(\tilde{y})} < 0, \ \frac{\partial y_1}{\partial \lambda} < 0, \ \frac{\partial y_2}{\partial \lambda} < 0, \ \frac{\partial \tilde{y}}{\partial \lambda} < 0.$$

Suppose that, in addition, we have $\phi'''(\cdot) < 0$. Then each of the following holds if and only if $\epsilon_1 > \epsilon_2$:

$$\frac{\partial y_1}{\partial \epsilon} < 0, \ \frac{\partial y_2}{\partial \epsilon} > 0, \ \frac{\partial (y_2 - y_1)}{\partial \epsilon} > 0. \tag{4.28}$$

Let $\phi'''(\cdot) < 0$ and $\epsilon_1 > \epsilon_2$. Then the following is true if $f'(\tilde{y}) > 0$:

$$\frac{\partial y_1}{\partial f(y_1 - d_1)} < 0, \ \frac{\partial y_2}{\partial f(y_2 + d_2)} > 0. \tag{4.29}$$

Let $\phi'''(\cdot) < 0$ and $\epsilon_1 > \epsilon_2$. Then the following is true if $f'(\tilde{y}) < 0$:

$$\frac{\partial y_1}{\partial f(y_2 + d_2)} < 0, \ \frac{\partial y_2}{\partial f(y_1 - d_1)} > 0. \tag{4.30}$$

We find that the cost of voting C and the perceived probability of being decisive p do not affect the equilibrium positions of the candidates and of the indifferent voter. This is because

[9] We assume that $f(y)$ uniformly increases in some neighborhood of y.

of our assumption of uniform voter density in the neighborhood of the ambivalent voters. An increase in $d = \frac{C}{p}$ will reduce the voteshare of both candidates, but the marginal effect of a change in a candidate's position on the voteshare of the candidate will remain unaffected. For the same reason, an equal change in the valence of both candidates will not affect their policy positions, although it will change their absolute and, likely, their relative voteshares.

An increase in the valence advantage of one of the candidates will lead to a divergence of candidate positions, with the positions of both candidates moving away from the indifferent voter. This, in turn, should lead to a higher turnout.

The effect of an increase in the voter density in the neighborhood of the indifferent voter depends on an additional factor. If $f'(\tilde{y}) > 0$, that is, the indifferent voter lies to the left of a local maximum in the density of voters, then an increase in $f(\tilde{y})$ will lead to a leftward shift in the positions of both candidates and of the indifferent voter. In the new equilibrium, $f(\tilde{y})$, $\tilde{y} - y_1$ and $y_2 - \tilde{y}$ will remain the same, as we have assumed constant density around the ambivalent voters. An increase of the role of plurality in a candidate's objective function has the same effect as an increase in the voter density near the indifferent voter. When the weight of plurality increases, so does the value of capturing the indifferent voter, as the candidate not only gains votes, but also decreases the voteshare of her opponent.

Note that the policy distance $y_2 - y_1$ is unaffected by both the changes in $f(\tilde{y})$ and λ. Thus increases of these parameters should result in an increase in turnout if and only if the left candidate has higher valence.

Finally, I consider the case when the voter of each type votes with a certain probability (perhaps due to different costs of voting). The probability of voting is taken to depend on the policy difference between the candidates[10].

Uniform effect of policy distance on turnout (UEH). A voter votes with probability $\psi(|y_2 - y_1|)$, where $\psi(\cdot)$ is a twice differentiable function.

The payoffs to the candidates are

$$U_1 = \psi(y_2 - y_1)F(\tilde{y}) \tag{4.31}$$

and

$$U_2 = \psi(y_2 - y_1)(1 - F(\tilde{y})). \tag{4.32}$$

The following equilibrium result has been obtained:

Proposition 4.4 *Under UEH, y_1, y_2 are a LNE if*

$$f(\tilde{y})\psi(y_2 - y_1) - \psi'(y_2 - y_1) = 0, \tag{4.33}$$

[10]This approach is similar to that of Kirchgässer (2003), where the probability of voting function is defined directly and does not follow from any rational behavior. The function proposed there is consistent with both alienation and indifference hypotheses:

$$P(v, y_1, y_2) = \begin{cases} \frac{y_2 - y_1}{y_1 + y_2 - 2v}, & v < y_1 \\ \frac{y_1 + y_2 - 2v}{y_2 - y_1}, & y_1 \le v \le \frac{y_1 + y_2}{2} \\ \frac{2v - y_1 - y_2}{y_2 - y_1}, & \frac{y_1 + y_2}{2} \le v \le y_2 \\ \frac{y_2 - y_1}{2v - y_1 + y_2}, & y_2 \le v. \end{cases}$$

Here participation is a continuous function of v with $\lim_{v \to -\infty} P = 0$, $\lim_{v \to \infty} P = 0$, $P(\frac{y_1 + y_2}{2}, y_1, y_2) = 0$, and $P(y_1, y_1, y_2) = P(y_2, y_1, y_2) = 1$. However, this function is not used here for two reasons. First, it is appropriate only if both candidates have identical valence. Second, in order to calculate candidate voteshares one has to integrate voting probabilities over all voters to the left and to the right of the indifferent voter $\frac{y_1 + y_2}{2}$. Hence, voter preferences must be distributed uniformly in order for the results to be tractable.

$$F(\tilde{y}) = \frac{\phi'(\tilde{y} - y_1)}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})}, \tag{4.34}$$

and

$$\epsilon - \phi(y_1 - \tilde{y}) = -\phi(y_2 - \tilde{y}).$$

First, her voteshare relative to her opponent will increase. Second, the overall voteshare may decrease since turnout may increase with smaller policy distance. In an equilibrium, the turnout increases with the policy distance. Otherwise, each candidate (or at least the candidate with the valence advantage) will benefit from moving in the direction of her opponent. Thus the conditions of the indifference hypothesis are satisfied in automatically.

One of our goals is to examine the comparative statics of the model. We want to know how will the positions and the voteshares of the candidates shift if the valence of one of the candidates or the voter density in the neighborhood of the indifferent voter changes. There is the following result:

Corollary 4.4 *Let y_1, y_2 be a local Nash equilibrium in the election game with payoffs (4.31), (4.32), and let $\psi''(\tilde{y}) > 0$ and $f'(\tilde{y}) > 0$. Then, we have*

$$\frac{\partial \tilde{y}}{\partial \epsilon} > 0, \tag{4.35}$$

$$\frac{\partial (y_2 - y_1)}{\partial \epsilon} < 0, \tag{4.36}$$

and

$$\frac{\partial (y_2 - \tilde{y})}{\partial \epsilon} < 0. \tag{4.37}$$

An increase in the valence advantage the first candidate has several competing effects on the position of the indifferent voter and on the policy distance. The candidate who has the valence advantage can now obtain greater voteshare by moving her position toward that of her opponent, and will be better off given the position of her opponent. At the same time, the opponent will be better off moving away from the candidate. Thus the overall effect on the positions of the candidates is not clear. However, the effects on the position of the indifferent voter and on the policy distance are more certain.

5 Indifference and alienation in the probabilistic model.

In the second part of the work, I study the implications that the indifference and alienation assumptions will have on the probabilistic voting model.

The principal assumption of a spatial probabilistic voting model is that the candidates are not fully aware of the effect of their policies on the utility of a voter. Thus, from a candidate's perspective, a voter's action is a random variable conditional on the ideal policy of the voter, the platforms of all candidates, and other observable factors.

This uncertainty can arise for several reasons. Voters with identical attitudes toward policy may have different perception of candidates' personal qualities, such as her competence or honesty[11]. Uncertainty can also be a result of idiosyncratic random events affecting an individual's voting decision.

The equilibrium in a probabilistic voting model is common, as the expected voteshares of the candidates depend continuously on their policy positions. However, virtually all works investigate the existence of an equilibrium where all candidates select identical policy platforms.

Hinich, Ledyard, and Ordeshook (1972) proved that an equilibrium in a two-candidate positioning game exists as long as the probability that a voter supports a candidate is concave in the voter's utility from the election of that candidate, and convex in the utility the voter receives if the opponent is elected. The equilibrium is a convergent one if the probability of a voter supporting a candidate is a function of the difference in utilities that the voter derives from the election of each candidate. The well-known result is that both candidates choose the mean voter's ideal policy if the voter utility is the negative squared Eucledian distance between the policies of the candidates. Lin, Enelow, and Dorussen (1999) obtained the conditions for a convergent equilibrium for a multi-candidate game for some other distance metrics.

A sufficient condition for the existence of an equilibrium is the concavity of probabilities of voting in candidate locations. This assumption is a very strong one and has been criticized in several works, most recently by Kirchgässer (2000). If the domain of candidate positions is unrestricted, then the probability that a voter supports a candidate cannot be concave in the candidate's position. Thus the existence of the equilibrium cannot be guaranteed.

The analysis of a probabilistic voting model typically addresses this question in one of the three ways.

First, one may try to answer whether the concavity conditions for a convergent equilibrium are satisfied locally. The most recent work here is Schofield (2006), who derived the local equilibrium conditions for several candidates with different valence. The second question (and a more difficult one to answer) is whether the local convergent equilibrium is also a global one. This issue was addressed in many works, starting with Hinich (1978) and Enelow and Hinich (1982). The general result have been that the convergent equilibrium will unravel if voting is close to being deterministic, or if the variance of the voter ideal policies is large. Finally, one may try to find nonconvergent equilibria. This is the most difficult problem of all, and it has not been solved analytically. Numeric solutions were proposed in several works, such as Schofield, Sened, and Nixon (1998), Lin, Enelow, and Dorussen (1999), or Schofield (2006). The nonconvergent equilibria were found to be local. Moreover, the degree of in local Nash equilibria, simulated with the use of real survey data to estimate voter ideal points, was greater than the degree of convergence of estimated candidate positions in the same elections.

In this work, I will address the first two questions and obtain local and global conditions for a convergent equilibrium for voters with squared Eucledian disutility, under the assumption of voter indifference.

5.1 The probabilistic voting model

I consider N voters of equal mass with the ideal policies $v_i \in \mathbf{R}^n$, and K candidates.

The utility of voter i if candidate j wins is given by

$$u_{ij} = e_j - \beta \|v_i - y_j\|^2 + \epsilon_{ij}, \tag{5.38}$$

there v_i is the ideal policy of voter i, y_j and e_j are the policy position and valence of candidate j, and ϵ_{ij} is zero-mean random variable, IID with the distribution $F(\cdot)$. We take $\beta = 1$ for

[11] A candidate is said to have a higher valence if she has, on average, a higher perceived ability (Stokes, 1963). Deterministic models incorporating valence include Groseclose (2001) that assumes the candidates to be partially motivated by policy, and Aragones and Palfrey (2002) that looks at a mixed-strategy equilibrium.

a general functional form of $F(\cdot)$ and $e_1 \leq \cdots \leq e_K$. Without the loss of generality we let $\sum_i v_i = 0$.

This specification follows Hinich (1977) and the majority of other works. Under an alternative specification, such as in Coughlin and Nitzan (1981), the value ϵ_{ij} has a multiplicative effect on voter utility.

The indifference and alienation assumptions for a multi-candidate model probabilistic model can be formulated as follows:

Indifference (IH). Voter i votes for candidate j if and only if $u_{ij} - c \geq \max_{k \neq j} u_{ik}$ for some $c \geq 0$.

Alienation (AH). Voter i votes for candidate j if and only if $u_{ij} \geq \max_{k \neq j} u_{ik}$ $u_{ij} \geq d$ for some d.

If the random variable ϵ_{ij} is continuous with unrestricted domain, then the probability that $u_{ij} = u_{ik}$ is equal to zero for $k \neq j$.

Denote by P_{ij} the probability that voter i supports candidate j. Then the expected voteshare of candidate j is

$$V_j = \sum_{i=1}^{N} P_{ij}. \tag{5.39}$$

5.2 Local conditions for $K = 2$, $n = 1$, and the general form of $F(\cdot)$.

We first look at the probabilistic voting model under the indifference hypothesis. Without loss of generality, assume that $e_{i1} - e_{i2}$ are identically distributed with the distribution function $G(\cdot)$.

The probabilities that voter i abstains or votes for one of the candidates are given by

$$P_{i1} = 1 - G((v_i - y_1)^2 - (v_i - y_2)^2 + c), \tag{5.40}$$

$$P_{i2} = G((v_i - y_1)^2 - (v_i - y_2)^2 - c), \tag{5.41}$$

and

$$P(\text{Voter } i \text{ abstains}) = 1 - P_{i1} - P_{i2} = G((v_i - y_1)^2 - (v_i - y_2)^2 + c) - G((v_i - y_1)^2 - (v_i - y_2)^2 - c). \tag{5.42}$$

The expected voteshares of the candidates are

$$V_1 = \sum_{i=1}^{N} (1 - G((v_i - y_1)^2 - (v_i - y_2)^2 + c)) \tag{5.43}$$

$$V_2 = \sum_{i=1}^{N} G((v_i - y_1)^2 - (v_i - y_2)^2 - c). \tag{5.44}$$

I assume that the candidates maximize a weighted sum of plurality and voteshare:

$$U_i = \lambda V_i + (1 - \lambda)(V_i - V_{-i}) = V_i - \lambda V_{-i}. \tag{5.45}$$

The conditions for a convergent local Nash equilibrium in a game with are similar to those for the case of perfect turnout.

Proposition 5.1 *Let the utility of the candidates be given by (5.43), (5.44), and (5.45). Suppose that $G(\cdot)$ has a differentiable density $g(\cdot)$. Denote by $\bar{v}$ and σ_v^2 the mean and variance of v_i. Then, $y_1 = y_2 = \bar{v}$ is a local Nash equilibrium if and only if*

$$\begin{aligned} g(-c) + \lambda g(c) - 2\sigma_v^2(g'(-c) + \lambda g'(c)) &> 0, \\ g(c) + \lambda g(-c) + 2\sigma_v^2(g'(c) + \lambda g'(-c)) &> 0 \end{aligned} \tag{5.46}$$

The expected utility of the two candidates in the convergent equilibrium would be

$$U_1^* = 1 - G(c) - \lambda G(-c), \tag{5.47}$$

$$U_2^* = G(-c) - \lambda(1 - G(c)). \tag{5.48}$$

The intuition of behind this result is as follows. Since the voter's disutility from policy distance is concave, the marginal effect a candidate's position on the voter's utility is increasing in policy distance. But so is the effect of a change in a candidate's position on the voter's probability of supporting that candidate — so the policy choice of each candidate is weighted in favor of more distant voters. If the disutility is linear, then these weights are linear in policy distance.

It is worth comparing the second-order conditions for this model and the case with perfect turnout. For $c = 0$, the condition (5.46) becomes

$$\sigma_v^2 < \frac{f(0)}{2|f'(0)|}. \tag{5.49}$$

Thus the electoral mean is a local equilibrium if a change in a candidate's position has a significant impact on her probability of winning (high $f(0)$, low σ_v^2). The concavity condition is also satisfied if the density of ϵ is sufficiently close to constant. Conditions similar to (5.46) were obtained by Hinich (1978), Enelow (1989), Lin, Enelow, and Dorussen (1999), Schofield (2006), and by a number of other works.

If there is a possibility of voter abstention due to indifference, a slightly different condition is required.

Corollary 5.1 *Suppose that $g(\cdot)$ is symmetric around zero mean. Then the following is true.*

1. *If $g'(c) \geq 0$, then $y_1 = y_2 = \bar{v}$ is a local equilibrium.*
2. *If $g'(c) < 0$ and*

$$\lambda > \frac{g(c) - 2\sigma_v^2 g'(c)}{g(c) + 2\sigma_v^2 g'(c)}, \tag{5.50}$$

or

$$\sigma_v^2 < \frac{g(c)(1+\lambda)}{-2g'(c)(1-\lambda)}, \tag{5.51}$$

then $y_1 = y_2 = \bar{v}$ is a local equilibrium.

The equilibrium is more likely to exists if the density of ϵ is multimodal, with the voters with the realizations of ϵ at the modes of the distribution not abstaining. Thus a local equilibrium becomes less likely if c is large. Finally, the local convergent equilibrium exists if the variance of voter ideal policies is small or the candidates are plurality maximizers.

Example. Let ϵ_i be uniformly distributed on $[-e+a, e+a]$. The value a is the expected valence advantage of Candidate 1 over Candidate 2. The voter ideal policies are distributed with mean $\bar{v} = 0$ and variance σ_v^2. We further assume that $e - a > c$. This is a sufficient condition for every voter to have a positive probability of voting for every candidate if both candidates select identical policy platforms. First we investigate whether $y_1 = y_2 = \bar{v} = 0$ is a local Nash equilibrium.

The probability that voter i will support Candidate 1 is

$$P_{i1} = \begin{cases} 1, & (y_1 - v_i)^2 - (y_2 - v_i)^2 < a - e - c, \\ 1 - \frac{(y_1-v_i)^2-(y_2-v_i)^2+c-a+e}{2e}, & a - e - c \leq (y_1 - v_i)^2 - (y_2 - v_i)^2 < a + e - c, \\ 0, & (y_1 - v_i)^2 - (y_2 - v_i)^2 \geq a + e - c. \end{cases} \tag{5.52}$$

The probability that voter i will support Candidate 2 is

$$P_{i2} = \begin{cases} 0, & (y_1 - v_i)^2 - (y_2 - v_i)^2 < a - e + c, \\ \frac{(y_1-v_i)^2-(y_2-v_i)^2-c-a+e}{2e}, & a - e + c \leq (y_1 - v_i)^2 - (y_2 - v_i)^2 < a + e + c, \\ 1, & (y_1 - v_i)^2 - (y_2 - v_i)^2 \geq a + e + c. \end{cases} \tag{5.53}$$

Letting $y_2 = 0$, we then calculate the marginal probabilities with respect to y_1:

$$\frac{\partial P_{i1}}{\partial y_1} = \begin{cases} -\frac{1}{e}(y_1 - v_i), & y_1^2 - 2v_i y_1 \in [a - e - c, a + e - c], \\ 0, & y_1^2 - 2v_i y_1 \notin [a - e - c, a + e + c], \end{cases} \tag{5.54}$$

$$\frac{\partial P_{i2}}{\partial y_1} = \begin{cases} \frac{1}{e}(y_1 - v_i), & y_1^2 - 2v_i y_1 \in [a - e + c, a - e + c], \\ 0, & y_1^2 - 2v_i y_1 \notin [a - e + c, a + e + c], \end{cases} \tag{5.55}$$

It follows that if Candidate 1 maximizes a utility function (**??**), then $y_1 = \bar{v} = 0$ is Candidate 1's locally best response to $y_2 = 0$. Similarly, it can be shown that $y_2 = 0$ is a locally best response to $y_1 = \bar{v} = 0$. Thus, $y_1 = y_2 = \bar{v} = 0$ is a local Nash equilibrium.

The probabilities of voting for Candidates 1 and 2 in this equilibrium are

$$P_{i1}^* = \frac{1}{2} - \frac{c-a}{2e}, \; P_{i2}^* = \frac{1}{2} - \frac{c+a}{2e}. \tag{5.56}$$

The expected utilities for both candidates will be

$$U_1^* = N(P_{i1}^* - \lambda P_{i2}) = \frac{N}{2}\left(1 - \lambda - \frac{(1+\lambda)c - (1-\lambda)a}{e}\right), \tag{5.57}$$

and

$$U_2^* = N(P_{i2}^* - \lambda P_{i1}) = \frac{N}{2}\left(1 - \lambda - \frac{(1+\lambda)c + (1-\lambda)a}{e}\right) \tag{5.58}$$

5.3 Global conditions for $K = 2$, $n = 1$, and the general form of $F(\cdot)$.

The conditions for a global equilibrium are more difficult to obtain. Here I present an existence result for the case when there are 2 groups of voters.

Proposition 5.2 *Suppose that the following assumptions are satisfied:*

1. *There are 2 voters with weights w and $1 - w$ and ideal policies $v_1 = -\frac{w}{1-w}$, $v_2 = 1$. Let $\lambda = 0$ and $w \geq \frac{1}{2}$.*

2. *The value* ϵ_i *be distributed on* $[a-e, a+e]$, $e-a > c$, *according to a nonzero differentiable density* $f(\cdot)$.

3. *For* $i = 1, 2$, *we have*

$$2g'((v_i - y_1)^2 + c)(v_i - y_1)^2 > -g((v_i - y_1)^2 + c) \tag{5.59}$$

for all y_1 *such that* $|v_i - y_1| \leq \sqrt{a+e-c}$ *and*

$$2g'((v_i - y_2)^2 - c)(v_i - y_2)^2 > -g((v_i - y_2)^2 - c) \tag{5.60}$$

for all y_2 *such that* $|v_i - y_2| \leq \sqrt{a+e+c}$.

Then, $y_1 = 0$ *is Candidate 1's globally best response to* $y_2 = 0$ *if one of the following three conditions is satisfied:*

1. $c > a - e + \frac{w^2}{(1-w)^2}$ *and* $1 - G(c) \geq w(1 - G(c - \frac{w^2}{(1-w)^2}))$,

2. $c \in [a - e + 1, a - e + \frac{w^2}{(1-w)^2}]$, $G(c) \leq \min\{1 - w, 1 - (1 - w)(1 - G(c - 1))\}$,

3. $c < a - e + 1$, $G(c) \leq 1 - w$.

Similarly, $y_2 = 0$ *is a globally best response to* $y_1 = 0$ *if one of the following three conditions is satisfied:*

1. $c > \frac{w^2}{(1-w)^2} - a - e$ *and* $G(-c) \geq wG(\frac{w^2}{(1-w)^2} - c)$,

2. $c \in [1 - a - e, \frac{w^2}{(1-w)^2} - a - e]$ *and* $G(-c) \geq \max\{w, (1 - w)G(1 - c)\}$,

3. $c < 1 - e - a$ *and* $G(-c) \geq w$.

We assume $\lambda = 0$ for simplicity's sake. The condition $e - a > c$ is sufficient for both groups of voters to have a positive probability of voting for each candidate at the convergent equilibrium. Equation (5.59) is sufficient to ensure that if the maximum of V_1 for $y_2 = 0$ is attained at $y_1 \neq 0$, then it is reached at the maximum of either P_{11} or P_{12}. Equation (5.60) is the similar condition for the maximum of V_2.

The first inequality corresponds to the case when $\max_y P_{12}(y, 0) < 1$ and $\max_y P_{11}(y, 0) < 1$, second — when $\max_y P_{12}(y, 0) < 1$ and $\max_y P_{22}(y, 0) = 1$, third — when $\max_y P_{12}(y, 0) = \max_y P_{11}(y, 0) = 1$.

The first special case to consider is the one with a symmetric distribution of ϵ_1. In that case, $y_1 = 0$ is the best response for $y_2 = 0$ if and only if $y_2 = 0$ is the best response for $y_2 = 0$.

Suppose that the third inequality is satisfied. In this case, Candidate 1 can deviate from $y_1 = 0$ to ensure that Voter 1 supports him with probability 1. Since the utility of Candidate 1 at $y_1 = y_2 = 0$ is decreasing in c, the equilibrium is more likely to be a global one if c is smaller. Moreover, if $e < 1$, then for every w there exists a c small enough such that $y_1 = y_2 = 0$ is a global equilibrium.

Next consider conditions 2 and 3. It follows that for every c there exists w large enough so that the voter mean is not a global equilibrium.

Example. Suppose that ϵ_i are distributed as in the example above, and that there are three voters with positions $v_1 = -2b$, $v_2 = v_3 = b$ for some $b > 0$. Let $\lambda = 0$. Denote $\bar{P}_{12} = \max_{y>0} P_{12}$. Denote by $\tilde{y}_1$ the largest $y_1 > 0$ such that $P_{11}(y_1, 0) = 0$. Since the second-order condition is satisfied for $y \leq \tilde{y}_1$, $V_1(0, 0) \geq V_1(y, 0)$ for all $y > 0$ if and only if $2\bar{P}_{12} \leq V_1(0, 0)$. There are two cases.

1. $\bar{P}_{12} < 1$ or $b^2 < e - a + c$. Then the condition $2\bar{P}_{12} \leq V_1(0,0)$ is $b^2 < \frac{e-c+a}{2}$.
2. $\bar{P}_{12} = 1$ or $b^2 \geq e - a + c$. Then the condition is $V_1(0,0) \geq 2$ is $e + 3c - 3a < 0$.

Similar conditions for the second candidate are

1. $\bar{P}_{21} < 1$ or $b^2 < e + a + c$. Then the condition $2\bar{P}_{21} \leq V_2(0,0)$ is $b^2 < \frac{e-c+5a}{2}$.
2. $\bar{P}_{12} = 1$ or $b^2 \geq e + a + c$. Then the condition $V_2(0,0) \geq 2$ is $e + 3c + 3a \leq 0$ that is never satisfied.

For $a = 0$, the sole condition for the global equilibrium is $b^2 < \frac{e-c}{2}$.

5.4 Alienation

Under the alienation hypothesis the analysis becomes much less tractable, since for every i, the voting decision is affected by the realizations of ϵ_{i1}, ϵ_{i2}, and $\epsilon_{i1} - \epsilon_{i2}$. The probabilities of voting and abstaining are given by

$$\begin{aligned} & P(\text{Voter } i \text{ supports candidate 1}) = (1 - F((v_i - y_1)^2 + d))F((v_i - y_2)^2 + d) + \\ + & \int_{(v_i-y_1)^2+d}^{\infty} \int_{(v_i-y_2)^2+d}^{e_1} f(e_1)f(e_2)de_2de_1, \end{aligned} \tag{5.61}$$

$$\begin{aligned} & P(\text{Voter } i \text{ supports candidate 2}) = F((v_i - y_1)^2 + d)(1 - F((v_i - y_2)^2 + d)) + \\ + & \int_{(v_i-y_1)^2+d}^{\infty} \int_{e_1}^{\infty} f(e_1)f(e_2)de_2de_1, \end{aligned} \tag{5.62}$$

and

$$P(\text{Voter } i \text{ abstains}) = F((v_i - y_1)^2 + d)F((v_i - y_2)^2 + d), \tag{5.63}$$

where $d < 0$. The choice of the voter is illustrated on Figure 2.

Figure 2: Voter choice depending on the realization of ϵ_{i1} and ϵ_{i2}.

The first-order conditions for voteshare maximization are

$$\begin{aligned} \frac{\partial V_1}{\partial y_1} &= -2\sum_{i=1}^{N}(v_i - y_1)\Big(f((v_i - y_1)^2 + d)F((v_i - y_2)^2 + d) + \\ &+ \int_0^{\infty} f((v_i - y_1)^2 + d + h)f((v_i - y_2)^2 + d + h)dh\Big) = 0 \end{aligned} \tag{5.64}$$

and

$$\begin{aligned} \frac{\partial V_2}{\partial y_2} &= -2\sum_{i=1}^{N}(v_i - y_2)\Big(f((v_i - y_2)^2 + d)F((v_i - y_1)^2 + d) + \\ &+ \int_0^{\infty} f((v_i - y_1)^2 + d + h)f((v_i - y_2)^2 + d + h)dh\Big) = 0. \end{aligned} \tag{5.65}$$

If the density $f(\cdot)$ is constant, a variant of the electoral mean is a convergent equilibrium. Let ϵ_i be uniformly distributed on $[-a, a]$. The conditions (5.64), (5.65) become

$$\frac{\partial V_1}{\partial y_1} = -\frac{1}{a} \sum_{i:(v_i - y_1)^2 + d < a} (v_i - y_1) \tag{5.66}$$

and

$$\frac{\partial V_2}{\partial y_2} = -\frac{1}{a} \sum_{i:(v_i - y_2)^2 + d < a} (v_i - y_2). \tag{5.67}$$

It follows that in a convergent equilibrium, we have[12]

$$y^* = \sum_{|v_i - y^*| < \sqrt{a-d}} v_i. \tag{5.68}$$

At least one such y^* exists, but there can be multiple equilibria. At each such equilibrium, both candidates select the mean ideal policy of the voters who do not abstain with probability one.

Example. There are 6 voters with positions $v_1 = 0, v_2 = 1$, $v_3 = 2$, $v_4 = 8$, $v_5 = 9$, and $v_6 = 10$. If $5 \leq \sqrt{a-d} \leq 6$, there are 3 local equilibria satisfying (5.68): $y^* = 1$, $y^* = 5$, and $y^* = 9$.

5.5 Local conditions for $K \geq 2$, $n \geq 1$, and a specific form of $F(\cdot)$.

In order to obtain tractable second-order conditions for a model with more than two candidates and a multi-dimensional policy space, one must assume some specific functional form for $F(\cdot)$. Here I modify the model of Schofield (2006) to account for voter indifference. I take

$$F(x) = e^{-e^{-x}}. \tag{5.69}$$

Under this assumption, the probabilities of voting are given by

$$P_{ij} = \frac{\exp(e_i - c - \beta \|y_j - v_i\|^2)}{\sum_{k=1}^{K} \exp(e_k - \beta \|y_k - v_i\|^2)}. \tag{5.70}$$

For $y_1 = \cdots = y_K = 0$, we must have

$$P_{ij} = P_j = \frac{1}{1 + \sum_{k \neq j} \exp(e_k - e_j + c)}. \tag{5.71}$$

The candidates are assumed to maximize their expected voteshare. Denote ∇ by the $n \times n$ covariance matrix of v_i. The main existence result is identical to that of Schofield (2006):

Proposition 5.3 *In a game with candidate payoffs V_j, the joint origin $y_1 = \cdots y_K$ is a local Nash equilibrium only if every eigenvalue of the characteristic matrix*

$$C_i = 2\beta(1 - 2P_1)\nabla - I \tag{5.72}$$

is negative.

It follows that a convergent equilibrium is less likely given a higher c.

A more interesting issue is the effect that voter indifference and alienation may have on the existence and location of nonconvergent equilibria. Empirical literature suggests that there is an inconsistency between measured candidate positions in multi-party elections and simulated Nash equilibria for the same elections. The observed candidate positions are significantly less convergent than the predicted positions. It may be that voter indifference and alienation can account for at least some part of this difference.

[12]The second-order condition is always satisfied.

6 Conclusion

This work formalizes a two-candidate Downsian election game where voters may choose to abstain. There are two hypotheses regarding voter's behavior. First, a voter may abstain if the difference between the candidates (from that voter's point of view) is insignificant. This assumption is known as the indifference hypothesis. Under the second, alienation hypothesis a voter abstains if the utility from the election of either candidates is below a certain threshold value. The candidates are assumed to maximize a weighted sum of absolute voteshare and plurality.

The first observation is that the equilibrium fails to exist under the first assumption, as the candidates continue to converge to the median voter. Under the second assumption, an equilibrium is likely to exist. The key observation is that the equilibrium is unaffected by small changes in the threshold utility.

Separately, the author considers the case when the probability of voting for each voter is defined as a function of the policy distance between candidates. It was shown that in an equilibrium, the probability of voting declines with policy distance.

The hypotheses are analyzed under both deterministic and probabilistic voting. If the voting is probabilistic and the disutility of the voters is quadratic in policy distance, then the positions of the candidates can converge to the mean of the distribution of voter preferences only under the indifference assumption.

There are potentially several ways to expand this paper's analysis. It would be interesting to explore the implications of the candidates being motivated by policy instead of office.

Appendix

Proof of Proposition 4.1

We have

$$U_1 = (1-\lambda)F(\bar{y}_1) + \lambda(F(\bar{y}_1) - 1 + F(\bar{y}_2) = F(\bar{y}_1) - \lambda(1 - F(\bar{y}_2)) \tag{6.73}$$

and

$$U_2 = (1-\lambda)(1 - F(\bar{y}_2)) + \lambda(1 - F(\bar{y}_2) - F(\bar{y}_1)) = 1 - F(\bar{y}_2) - \lambda F(\bar{y}_1), \tag{6.74}$$

where $\bar{y}_i$ is the voter who supports Candidate i and is on the threshold of abstaining. Thus

$$\frac{\partial U_1}{\partial y_1} = f(\bar{y}_1)\frac{\phi'(\bar{y}_1 - y_1)}{\phi'(\bar{y}_1 - y_1) + \phi'(y_2 - \bar{y}_1)} \tag{6.75}$$

and

$$\frac{\partial U_2}{\partial y_2} = f(\bar{y}_2)\frac{\phi'(y_2 - \bar{y}_2)}{\phi'(y_2 - \bar{y}2) + \phi'(\bar{y}_2 - \bar{y}_1)}. \tag{6.76}$$

If $\epsilon_1 > \epsilon_2$, five cases are possible:

1. $y_1 > \bar{y}_1 > \bar{y}_2 > y_2$: $\frac{\partial U_1}{\partial y_1} > 0$, $\frac{\partial U_2}{\partial y_2} < 0$.
2. $y_1 > \bar{y}_1 > \bar{y}_2 = y_2$: $\frac{\partial U_1}{\partial y_1} > 0$, $\frac{\partial U_2}{\partial y_2} = 0$.
3. $y_1 > \bar{y}_1 > y_2 > \bar{y}_2$: $\frac{\partial U_1}{\partial y_1} > 0$, $\frac{\partial U_2}{\partial y_2} > 0$.
4. $y_1 = \bar{y}_1 > y_2 > \bar{y}_2$: $\frac{\partial U_1}{\partial y_1} = 0$, $\frac{\partial U_2}{\partial y_2} > 0$.
5. $\bar{y}_1 > y_1 > y_2 > \bar{y}_2$: $\frac{\partial U_1}{\partial y_1} < 0$, $\frac{\partial U_2}{\partial y_2} > 0$.

None of these five cases can be an equilibrium.

1. $y_1 > \bar{y}_1 > \bar{y}_2 > y_2$: $\frac{\partial U_1}{\partial y_1} > 0$, $\frac{\partial U_2}{\partial y_2} < 0$.
2. $y_1 = \bar{y}_1 > \bar{y}_2 = y_2$: $\frac{\partial U_1}{\partial y_1} = 0$, $\frac{\partial U_2}{\partial y_2} = 0$.
3. $\bar{y}_1 > y_1 > y_2 > \bar{y}_2$: $\frac{\partial U_1}{\partial y_1} < 0$, $\frac{\partial U_2}{\partial y_2} > 0$.

In the second case we have $y_2 - y_1 = \phi^{-1}(c)$. The second-order condition for Candidate 1 is

$$\frac{\partial^2 U_1}{\partial y_1^2} = -\frac{\phi''(0)\phi'(y_2 - y_1)}{(2\phi'(y_2 - y_1))^2} < 0. Likewise, the second-order condition is satisfied for Candidate 2. \tag{6.77}$$

Proof of Proposition 4.2

The utilities of the candidates are

$$U_1 = F(y_1 + d_1) - F(y_1 - d_1) + (1-\lambda)(F(y_2 + d_2) - F(y_2 - d_2)) \tag{6.78}$$

and

$$U_2 = F(y_2 + d_2) - F(y_2 - d_2) + (1-\lambda)(F(y_1 + d_1) - F(y_1 - d_1)). \tag{6.79}$$

The proposition's statement contains first- and second-order conditions for the maximization of (6.78) and (6.79).

Proof of Proposition 4.3

The utilities of the candidates for this case are

$$U_1 = F(\tilde{y}) - (1-\lambda)F(y_1 - d_1) - \lambda F(y_2 + d_2) \tag{6.80}$$

and

$$U_2 = -F(\tilde{y}) + (1-\lambda)F(y_2 + d_2) + \lambda F(y_1 - d_1) \tag{6.81}$$

Differentiating (6.80) and (6.81) with respect to y_1 and y_2 we obtain the proposition's statement.
We then check second-order conditions. Denote $x_1 = \tilde{y} - y_1$, $x_2 = y_2 - \tilde{y}$.

$$\frac{\partial^2 U_1}{\partial y_1} = \frac{\partial}{\partial y_1^2}(F(\tilde{y}) - (1-\lambda)F(y_1 - d_1)) = -f(\tilde{y})\frac{\phi_1^{''}\phi_2^{'}}{(\phi_1^{'} + \phi_1^{'})^2} + f^{'}(\tilde{y})\frac{\phi_1^{'}}{\phi_1^{'} + \phi_2^{'}} - (1-\lambda)f^{'}(y_1 - d_1) \tag{6.82}$$

and

$$\frac{\partial^2 U_2}{\partial y_2} = \frac{\partial}{\partial y_2^2}\left((1-\lambda)F(y_2+d_2) - F(\tilde{y})\right) = -f(\tilde{y})\frac{\phi_2^{''}\phi_1^{'}}{(\phi_1^{'}+\phi_2^{'})^2} + f^{'}(\tilde{y})\frac{\phi_2^{'}}{\phi_1^{'}+\phi_2^{'}} + (1-\lambda)f^{'}(y_2+d_2). \quad (6.83)$$

If the second set of equilibrium conditions are satisfied, then the second derivatives are negative.

Proof of Corollary 4.3

Let

$$G = \begin{pmatrix} f(\tilde{y})\frac{\phi'(\tilde{y}-y_1)}{\phi'(\tilde{y}-y_1)+\phi'(y_2-\tilde{y})} - f(y_1-d) \\ f(\tilde{y})\frac{\phi'(y_2-\tilde{y})}{\phi'(\tilde{y}-y_1)+\phi'(y_2-\tilde{y})} - f(y_2+d) \\ \epsilon - \phi(\tilde{y}-y_1) + \phi(y_2-\tilde{y}) \end{pmatrix}. \quad (6.84)$$

Denote $\tilde{f} = f(\tilde{y})$, $\tilde{f}' = f'(\tilde{y})$, $f_1 = f(y_1)$, $f_2 = f(y_2)$, $\phi_1 = \phi(\tilde{y}-y_1)$, $\phi_2 = \phi(y_2-\tilde{y})$,

$$V = \begin{pmatrix} y_1 & y_2 & \tilde{y} \end{pmatrix} \quad (6.85)$$

and

$$P = \begin{pmatrix} f_1 & f_2 & \tilde{f} & \epsilon & d & \lambda \end{pmatrix}. \quad (6.86)$$

If the equilibrium conditions (4.22), (4.23), and (4.24) are satisfied, then we have

$$G(V,P) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.87)$$

According to the implicit function theorem, we must have

$$\frac{\partial V}{\partial P} = -\left(\frac{\partial G}{\partial V}\right)^{-1}\frac{\partial G}{\partial P}. \quad (6.88)$$

We have:

$$\frac{\partial G}{\partial V} = \begin{pmatrix} -\frac{\tilde{f}\phi_1^{''}\phi_2^{'}}{(\phi_1^{'}+\phi_2^{'})^2} & -\frac{\tilde{f}\phi_1^{'}\phi_2^{''}}{(\phi_1^{'}+\phi_2^{'})^2} & \frac{\tilde{f}^{'}\phi_1^{'}}{\phi_1^{'}+\phi_2^{'}} + \frac{\tilde{f}(\phi_1^{'}\phi_2^{''}+\phi_2^{'}\phi_1^{''})}{(\phi_1^{'}+\phi_2^{'})^2} \\ \frac{\tilde{f}\phi_1^{''}\phi_2^{'}}{(\phi_1^{'}+\phi_2^{'})^2} & \frac{\tilde{f}\phi_1^{'}\phi_2^{''}}{(\phi_1^{'}+\phi_2^{'})^2} & \frac{\tilde{f}^{'}\phi_2^{'}}{\phi_1^{'}+\phi_2^{'}} - \frac{\tilde{f}(\phi_1^{'}\phi_2^{''}+\phi_2^{'}\phi_1^{''})}{(\phi_1^{'}+\phi_2^{'})^2} \\ \phi_1^{'} & \phi_2^{'} & -(\phi_1^{'}+\phi_2^{'}) \end{pmatrix}, \quad (6.89)$$

$$\frac{\partial G}{\partial P} = \begin{pmatrix} -(1-\lambda) & 0 & \frac{\phi_1^{'}}{\phi_1^{'}+\phi_2^{'}} & 0 & 0 & f_1 \\ 0 & -(1-\lambda) & \frac{\phi_2^{'}}{\phi_1^{'}+\phi_2^{'}} & 0 & 0 & f_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad (6.90)$$

$$\left(\frac{\partial G}{\partial V}\right)^{-1} = \begin{pmatrix} \frac{\tilde{f}L - \tilde{f}'\phi_2'^2(\phi_1'+\phi_2')}{\tilde{f}\tilde{f}'L} & \frac{\tilde{f}L - \tilde{f}'\phi_1'\phi_2'(\phi_1'+\phi_2')}{\tilde{f}\tilde{f}'L} & -\frac{\phi_1'\phi_2'}{L} \\ \frac{\tilde{f}L + \tilde{f}'\phi_1'\phi_2'(\phi_1'+\phi_2')}{\tilde{f}\tilde{f}'L} & \frac{\tilde{f}L - \tilde{f}'\phi_2'^2(\phi_1'+\phi_2')}{\tilde{f}\tilde{f}'L} & \frac{\phi_1'\phi_2'}{L} \\ \frac{1}{\tilde{f}'} & \frac{1}{\tilde{f}'} & 0 \end{pmatrix}, \quad (6.91)$$

where

$$L = \phi_1''\phi_2'^2 - \phi_2''\phi_1'^2. \quad (6.92)$$

According to the implicit function theorem, we must have

$$\frac{\partial V}{\partial P} = -\left(\frac{\partial G}{\partial V}\right)\frac{\partial G}{\partial P} \quad (6.93)$$

at V that is the solution to the equilibrium conditions (4.22), (4.23), (4.24). This gives us

$$\frac{\partial y_1}{\partial f_1} = -\frac{1-\lambda}{\tilde{f}'} + \frac{(1-\lambda)\phi_2'^2(\phi_1'+\phi_2')}{\tilde{f}L}, \quad (6.94)$$

$$\frac{\partial y_2}{\partial f_1} = -\frac{1-\lambda}{\tilde{f}'} - \frac{(1-\lambda)\phi_1'\phi_2'(\phi_1'+\phi_2')}{\tilde{f}L}, \quad (6.95)$$

$$\frac{\partial(y_2-y_1)}{\partial f_1} = \frac{-(1-\lambda)\phi_2'(\phi_1'+\phi_2')^2}{\tilde{f}L}, \quad (6.96)$$

$$\frac{\partial \tilde{y}}{\partial f_1} = -\frac{1-\lambda}{\tilde{f}'}, \tag{6.97}$$

$$\frac{\partial y_1}{\partial f_2} = \frac{1-\lambda}{\tilde{f}'} + \frac{(1-\lambda)\phi_1'\phi_2'(\phi_1'+\phi_2')}{\tilde{f}L}, \tag{6.98}$$

$$\frac{\partial y_2}{\partial f_2} = +\frac{1-\lambda}{\tilde{f}'} - \frac{(1-\lambda)\phi_1'^2(\phi_1'+\phi_2')}{\tilde{f}L}, \tag{6.99}$$

$$\frac{\partial (y_2-y_1)}{\partial f_2} = -\frac{(1-\lambda)\phi_1'(\phi_1'+\phi_2')^2}{\tilde{f}L}, \tag{6.100}$$

$$\frac{\partial \tilde{y}}{\partial f_2} = \frac{1-\lambda}{\tilde{f}'}, \tag{6.101}$$

$$\frac{\partial y_1}{\partial \tilde{f}} = -\frac{1}{\tilde{f}'}, \tag{6.102}$$

$$\frac{\partial y_2}{\partial \tilde{f}} = -\frac{1}{\tilde{f}'}, \tag{6.103}$$

$$\frac{\partial (y_2-y_1)}{\partial \tilde{f}} = 0, \tag{6.104}$$

$$\frac{\partial \tilde{y}}{\partial \tilde{f}} = -\frac{1}{\tilde{f}'}, \tag{6.105}$$

$$\frac{\partial y_1}{\partial \epsilon} = \frac{\phi_1'\phi_2'}{L}, \tag{6.106}$$

$$\frac{\partial y_2}{\partial \epsilon} = -\frac{\phi_1'\phi_2'}{L}, \tag{6.107}$$

$$\frac{\partial (y_2-y_1)}{\partial \epsilon} = -2\frac{\phi_1'\phi_2'}{L}, \tag{6.108}$$

$$\frac{\partial \tilde{y}}{\partial \epsilon} = 0, \tag{6.109}$$

$$\frac{\partial y_1}{\partial d} = 0, \tag{6.110}$$

$$\frac{\partial y_2}{\partial d} = 0, \tag{6.111}$$

$$\frac{\partial (y_2-y_1)}{\partial d} = 0, \tag{6.112}$$

$$\frac{\partial \tilde{y}}{\partial d} = 0, \tag{6.113}$$

$$\frac{\partial y_1}{\partial \lambda} = -\frac{\tilde{f}_1+\tilde{f}_2}{\tilde{f}'}, \tag{6.114}$$

$$\frac{\partial y_2}{\partial \lambda} = -\frac{\tilde{f}_1+\tilde{f}_2}{\tilde{f}'}, \tag{6.115}$$

$$\frac{\partial (y_2-y_1)}{\partial \lambda} = 0, \tag{6.116}$$

$$\frac{\partial \tilde{y}}{\partial \lambda} = -\frac{\tilde{f}_1+\tilde{f}_2}{\tilde{f}'}. \tag{6.117}$$

Proof of Proposition 4.4

If the local Nash equilibrium we must have

$$\frac{\partial U_1}{\partial y_1} = f(\tilde{y})\frac{\partial \tilde{y}}{\partial y_1}\psi(y_2-y_1) - F(\tilde{y})\psi'(y_2-y_1) = 0 \tag{6.118}$$

and

$$\frac{\partial U_2}{\partial y_2} = -f(\tilde{y})\frac{\partial \tilde{y}}{\partial y_2}\psi(y_2 - y_1) + (1 - F(\tilde{y}))\psi'(y_2 - y_1) = 0. \tag{6.119}$$

Subtracting the two equations we obtain

$$f(\tilde{y})\psi(y_2 - y_1)\left(\frac{\partial \tilde{y}}{\partial y_1} + \frac{\partial \tilde{y}}{\partial y_2}\right) - \psi'(y_2 - y_1) = 0. \tag{6.120}$$

We have

$$\frac{\partial \tilde{y}}{\partial y_1} = \frac{\phi'(\tilde{y} - y_1)}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})} \tag{6.121}$$

and

$$\frac{\partial \tilde{y}}{\partial y_2} = \frac{\phi'(y_2 - \tilde{y})}{\phi'(\tilde{y} - y_1) + \phi'(y_2 - \tilde{y})}, \tag{6.122}$$

so $\frac{\partial \tilde{y}}{\partial y_1} + \frac{\partial \tilde{y}}{\partial y_2} = 1$ and

$$f(\tilde{y})\psi(y_2 - y_1) - \psi'(y_2 - y_1) = 0. \tag{6.123}$$

Equation (4.33) is obtained by substituting (6.123) into (6.118). The proof is complete.

Proof of Corollary 4.4

Denote $x_1 = \tilde{y} - y_1$, $x_2 = y_2 - \tilde{y}$. Denote

$$G = \begin{pmatrix} f(\tilde{y})\psi(x_1 + x_2) - \psi'(x_1 + x_2) \\ F(\tilde{y}) - \frac{\phi'(x_1)}{\phi'(x_1) + \phi'(x_2)} \\ \epsilon + \phi(x_2) - \phi(x_1) \end{pmatrix} \tag{6.124}$$

The equilibrium conditions can be expressed as $G(\tilde{y}, x_1, x_2, \epsilon) = 0$. Put

$$H = (\tilde{y}, x_1, x_2). \tag{6.125}$$

The following holds:

$$\frac{\partial G}{\partial h} = \begin{pmatrix} f'(\tilde{y})\psi(x_1 + x_2) & f(\tilde{y})\psi'(x_1 + x_2) - \psi''(x_1 + x_2) & f(\tilde{y})\psi'(x_1 + x_2) - \psi''(x_1 + x_2) \\ f(\tilde{y}) & -\frac{\phi''(x_1)\phi'(x_2)}{(\phi'(x_1) + \phi'(x_2))^2} & \frac{\phi''(x_2)}{(\phi'(x_1) + \phi'(x_2))^2} \\ 0 & -\phi'(x_1) & \phi'(x_2) \end{pmatrix} \tag{6.126}$$

and

$$\det\left(\frac{\partial G}{\partial h}\right) = -f(\tilde{y})(f(\tilde{y})\psi'(x_1 + x_2) - \psi''(x_1 + x_2))(\phi'(x_1) + \phi'(x_2)) -$$

$$-\frac{f'(\tilde{y})\psi(x_1 + x_2)(\phi'^2(x_2)\phi''(x_1) + \phi'(x_1)\phi''(x_2))}{(\phi'(x_1) + \phi'(x_2))^2}. \tag{6.127}$$

Applying the implicit function theorem we obtain

$$\frac{\partial H}{\partial \epsilon} = -\left(\frac{\partial G}{\partial h}\right)^{-1}\frac{\partial G}{\partial \epsilon} = \tag{6.128}$$

$$= -\frac{1}{\det(\frac{\partial G}{\partial h})}\begin{pmatrix} \frac{(f(\tilde{y})\psi'(x_1+x_2) - \psi''(x_1+x_2))(\phi'(x_2)\phi''(x_1) + \phi''(x_2))}{(\phi'(x_1)+\phi'(x_2))^2} \\ f(\tilde{y})(f(\tilde{y})\psi'(x_1 + x_2) - \psi''(x_1 + x_2)) - \frac{f'(\tilde{y})\psi(x_1+x_2)\phi''(x_2)}{(\phi'(x_1)+\phi'(x_2))^2} \\ -f(\tilde{y})(f(\tilde{y})\psi'(x_1 + x_2) - \psi''(x_1 + x_2)) - \frac{f'(\tilde{y})\psi(x_1+x_2)\phi'(x_2)\phi''(x_1)}{(\phi'(x_1)+\phi'(x_2))^2} \end{pmatrix}.$$

If $\psi''(\tilde{y}) < 0$ and $f'(\tilde{y}) > 0$, then we obtain the corollary's statement.

Proof of Proposition 5.1

The first-order conditions for the maximization of the objective function are

$$\frac{\partial U_1}{\partial y_1} = -2\sum_{i=1}^{N}(y_1 - v_i)\left[g((v_i - y_1)^2 - (v_i - y_2)^2 + c) + \lambda g((v_i - y_1)^2 - (v_i - y_2)^2 - c)\right] \tag{6.129}$$

and

$$\frac{\partial U_2}{\partial y_2} = -2\sum_{i=1}^{N}(y_2 - v_i)\left[g((v_i - y_1)^2 - (v_i - y_2)^2 - c) + \lambda g((v_i - y_1)^2 - (v_i - y_2)^2 + c)\right]. \tag{6.130}$$

They are satisfied at $y_1 = y_2 = \bar{v}$. The second-order conditions at this point are

$$\frac{\partial^2 U_1}{\partial y_1^2} = -2\sum_{i=1}^{N}\left[g(c) + \lambda g(-c) + 2(y_1 - v_i)^2(g'(c) + \lambda g'(-c))\right] < 0 \tag{6.131}$$

and

$$\frac{\partial^2 U_2}{\partial y_2^2} = -2\sum_{i=1}^{N}\left[g(-c) + \lambda g(c) - 2(y_2 - v_i)^2(g'(-c) + \lambda g'(c))\right] < 0. \tag{6.132}$$

Condition (5.46) follows immediately.

Proof of Proposition ??

The proof is nearly identical to the proof of the main theorem in Schofield (2006). Putting

$$h_{kj} = \lambda_k + c - \lambda_j + \beta\|v_i - y_j\| - \beta\|v_i - y_k\| \tag{6.133}$$

and

$$u_{ij}^* = \lambda_j - \beta\|v_i - y_j\| \tag{6.134}$$

we obtain

$$P_{ij} = \frac{\exp(u_{ij}^* - c)}{\sum_{k\neq j}^{K}\exp(u_{ik}^*) + \exp(u_{ij}^* - c)} = \frac{1}{1 + \sum_{k\neq j}\exp(h_{kj})} \tag{6.135}$$

We have

$$\frac{\partial P_{ij}}{y_{jl}} = \tag{6.136}$$

where y_{jl} is the lth component of y_l.

References

[1] Adams, James, Jay Dow, and Samuel Merrill III. 2006. "The Political Consequences of Alienation-Based and Indifference-Based Voter Abstention: Applications to Presidential Elections." *Political Behavior* 28(1): 65–86

[2] Aldrich, John. 1993. "Rational Choice and Turnout." *American Journal of Political Science* 37(1): 246–278

[3] Ansolabehere, Stephen, and James M. Snyder, Jr. 2000. "Valence Politics and Equilibrium in Spatial Elections Model." *Public Choice* 103: 327–336

[4] Aragones, Enriqueta, and Thomas R. Palfrey. 2002. Mixed Equilibrium in a Downsian Model with a Favored Candidate. *Journal of Economic Theory*, 103(1): 131-161

[5] Brody, Richard A., and Benjamin J. Page. 1973. "Indifference, Alienation, and Rational Decisions: The Effects of Candidate Evaluations on Turnout and Vote." *Public Choice* 15: 1–17

[6] Calvert, R. (1985) Robustness of multidimensional voting model: Candidate motivations, uncertainty and convergence. *American Journal of Political Science* 29: 69–95

[7] Chamberlain, Gary, and Michael Rotschild. 1981. "A Note on the Probability of Casting a Decisive Vote." *Journal of Economic Theory* 25: 152–162

[8] Coate, Steven, and Michael Conlin. 2004. "A Group Rule-Utilitarian Approach to Voter Turnout: Theory and Evidence." *The American Economic Review* 94(5): 1476–1504

[9] Cox, Gary W., and Michael C. Munger. 1989. "Closeness, Expenditures, and Turnout in the 1982 U.S. House Elections." *American Political Science Review* 83(1): 217–231

[10] Downs, A. (1957) An economic theory of democracy. Hew York: Harper & Row

[11] Eldin, Aaron, Andrew Gelman, and Noah Kaplan. 2005. "Voting as a Rational Choice: The Effect of Preferences Regarding the Well-Being of Others." Unpublished manuscript.

[12] Enelow, J. and Hinich, M. 1982. "Nonspatial candidate characteristics and electoral competition." *Journal of Politics* 44: 115–130

[13] Enelow, J. and Hinich, M. 1984. "Probabilistic Voting and the Importance of Centrist Ideologies in Democratic Elections." *Journal of Politics* 459–478

[14] Feddersen, Timothy. 2004. "Rational Choice THeory and the Paradox of Not Voting." *Journal of Economic Perspectives* 18(1): 99–112

[15] Feddersen, Timothy. 1992. "A Voting Model Implying Duverger's Law and Positive Turnout." *American Journal of Political Science* 36(4): 938–962

[16] Feddersen, Timothy, and Wolfgang Pesendorfer. 1996. "The Swing Voter's Curse." *American Economic Review* 86(3): 408–424

[17] Feddersen, Timothy, and Wolfgang Pesendorfer. 1999. "Abstention in Elections with Asymmetric Information and Diverse Preferences. " *The American Political Science Review* 93(2): 381–398

[18] Feddersen, Timothy J. and Alvaro Sandroni. 2002. “A Theory of Participation in Elections.” Unpublished manuscript.

[19] Ferejohn, John A. and Fiorina, Morris P. 1974. “The Paradox of not Voting: A Decision Theoretic Analysis” *The American Political Science Review* 68: 525–536

[20] Gelman, A. and King, G. 1990 “Estimating incumbency advantage without bias.” *American Journal of Political Science* 34: 1142–1164

[21] Geys, Benny. 2006. “Explaining Voter Turnout: A Review of Aggregate-Level Research.” Unpublished manuscript.

[22] Grafstein, Rodert. 1990. “An Evidential Decision Theory of Turnout.” *Americn Journal of Political Science* 35(4): 989–1010

[23] Groceclose, T. (2001) A model of candidate location when one candidate has a valence advantage. *Americn Journal of Political Science* 45(5): 862–886

[24] Harsanui, John D. 1980. “Rule Utilitarianism, Rights, Obligations and the Theory of Rational Behavior.” *Theory and Decision* 12(1), 115–133

[25] Hinich, M., Ledyard, J., and Ordeshook, P. (1972a) Nonvoting and the existence of equilibrium under majority rule. *Journal of Economic Theory* 4: 144–153

[26] Kanasawa, Satoshi. 1998. “A Possible Solution to the Paradox of Voter Turnout.” *Journal of Politics* 60(4): 974–995

[27] Kirchgässner, Gebhard. 2000. “Probabilistic Voting and Equilibrium: An Impossibility Result.” *Public Choice* 103: 35–48

[28] Kirchgässner, Gebhard. 2003. Abstention Because of Indifference and Alienation, and Its Consequences for Party Competition: A Simple Psychological Model. University of St. Gallen Department of Economics working paper series 2003 2003-12, Department of Economics, University of St. Gallen.

[29] Kirchgässer, Gebhard, and Anne M. Zu Himmern. 1997. “Expected Closeness and Turnout: An Empirical Analysis for the Genman General Elections, 1983–1994. *Public Choice* 91: 3–25

[30] Lijphart, Arend. 1997. “Unequal Participation: Democracy’s Unresolved Dilemma.” *The American Political Science Review* 91(1), 1–14

[31] Ledyard, John. 1984. “The Pure Theory of Large Two-Candidate Elections.” *Public Choice* 44: 7–41

[32] Leighley, Jan. 1996. “Group Membership and the Mobilization of Political Participation.” *Jounal of Politics* 58: 447–463

[33] Morton, Rebecca B. 1991. ”Groups in Rational Turnout Models.” *Americn Journal of Political Science* 35(3): 758–776

[34] Mueller, Dennis. 2003. *Public Choice III.* Cambridge: Cambridge University Press.

[35] Myerson, Roger. 2000. “Large Poisson Games.” *Journal of Economic Theory* 94(1): 7–45

[36] Palfrey, Thomas R., and Howard Rosenthal. 1983. "A Strategic Calculus of Voting". *Public Choice* 41: 7–53

[37] Palfrey, Thomas R., and Howard Rosenthal. 1985. "Voter Participation and Strategic Uncertainty". *The American Political Science Review* 79: 62–78

[38] Plane, Dennis L., and Joseph Gershtenson. 2004. "Candidate's ideological locations, abstention, and turnout in US midterm Senate elections" *Political Behavior* 26: 69–93

[39] Riker, William H., and Peter C. Ordeshook. 1968. "A Theory of the Calculus of Voting". *American Political Schence Review* 62: 25–42

[40] Stokes, D. (1963) Spatial models of party competition. *American Political Science Review* 57: 368-77

[41] Thurder, Paul W., and Eymann, A. 2000. "Policy-Specific Alienation and Indifference in the Calculus of Voting: A Simultaneous Model of Party Choice and Abstention". *Public Choice* 102: 51–77

[42] Uhlaner, Carole J. 1989. "Rational Turnout: The Neglected Role of Groups." *Americn Journal of Political Science* 33(2): 390–422

[43] Zakharov, Alexei V. 2005. "Candidate Location and Endogenous Valence." EERC No. 05/17

[44] Zakharov, Alexei V. 2006. "Policy divergence in spatial voting models." Unpublished manuscript.

[45] Zipp, John F. 1985. "Perceived Representativeness and Voting: An Assessment of the Impact of "Choices" vs. "Echoes". *American Political Science Review* 79: 50–61