

Kähler metrics whose geodesics are circles

V. Timorin *

We classify all Kähler metrics in an open subset of \mathbb{C}^2 whose real geodesics are circles. All such metrics are equivalent (via complex projective transformations) to Fubini metrics (i.e. to Fubini-Study metric on $\mathbb{C}P^2$ restricted to an affine chart, to the complex hyperbolic metric in the unit ball model or to the Euclidean metric).

Introduction

All Riemannian metrics in an open subset of \mathbb{R}^2 or \mathbb{R}^3 whose geodesics are arcs of circles are classical, i.e., isometric to Euclidean, Riemann or Lobachevsky geometries. This was proved by A. Khovanskii [1] in dimension 2 and by F. Izadi [2] in dimension 3. But in dimension 4 this is wrong. There are remarkable Kähler metrics whose real geodesics are circles — Fubini metrics (see Appendices 1 and 2).

Our main result is as follows:

Theorem 1 *Consider a Kähler metric in an open subset of \mathbb{C}^2 such that all geodesics are parts of circles (or straight lines). Then this metric is (up to a complex projective transformation) some Fubini metric.*

In the next Section we will prove this result. Then we mention (without proof) a local geometric classification of complete families of circles that are point-wise rectifiable by means of complex projective transformations.

Acknowledgements. I am grateful to A.G. Khovanskii and R.W. Sharpe for useful discussions.

*Partially supported by RFBR 99-01-00245 and CRDF RM1-2086

Proof of the main result

For a definition of Kähler metrics see Appendix 2. Let g be a Kähler metric in an open region $\Omega \subset \mathbb{C}^2$ such that all geodesics with respect to g are parts of circles. First note that for any point $p \in \Omega$ the set of geodesics passing through p coincides with the image under the exponential map of the set of lines passing through p .

The following theorem [3] holds:

Theorem 2 *Fix some identification of \mathbb{R}^4 with the algebra \mathbb{H} of quaternions. Suppose that a local diffeomorphism $\Phi : (\mathbb{R}^4, 0) \rightarrow (\mathbb{R}^4, 0)$ takes sufficiently many lines (through 0) in general position to circles. If $d_0\Phi = id$, then the second derivative of Φ has the form $x \mapsto A(x)x$ or $x \mapsto xA(x)$ where A is some \mathbb{R} -linear map, and the multiplication is in the sense of quaternions.*

In particular, the bundle of geodesics at a point $p \in \Omega$ is given by $p + xt + \frac{1}{2}A(x)xt^2$ or $p + xt + \frac{1}{2}xA(x)t^2$ where t is a parameter, x is the velocity vector and A is some linear map. To fix the idea assume that the multiplication by $A(x)$ is from the left. Now recall the following well-known fact (for a proof, see Appendix 2)

Proposition 3 *Exponential maps with respect to a Kähler metric are holomorphic up to third order terms (i.e., their 2-jets are holomorphic).*

Therefore, the map $x \mapsto A(x)x$ must be holomorphic. Now we need

Lemma 4 *Suppose that the map $x \mapsto A(x)x$ (where A is some linear operator) is holomorphic. Then $A(x)$ is complex linear and takes complex values only. In other words, A is a complex linear functional.*

PROOF. In general, $A(x) = a(x) + b(x)i + c(x)j + d(x)k$ where a, b, c, d are some linear functionals on \mathbb{R}^4 . Being holomorphic, the quadratic map $x \mapsto A(x)x$ must satisfy the condition $A(ix)(ix) = -A(x)x$. From this condition it follows that

$$a(x) = b(ix), \quad b(x) = -a(ix), \quad c(x) = -d(ix), \quad d(x) = c(ix).$$

This means that the functionals $\alpha = a + bi$ and $\beta = d + ci$ must be holomorphic (i.e. complex linear). Since the map $x \mapsto A(x)x = (\alpha(x) + k\beta(x))x$ is holomorphic, the map $x \mapsto k\beta(x)x$ must be also holomorphic. If we multiply x by \sqrt{i} , then $k\beta(x)x$ gets multiplied by $-i$, but by bilinearity it must be multiplied by i . Hence $\beta = 0$ and $A(x) = \alpha(x)$ is a complex linear functional. \square

From this lemma it follows in particular that all geodesics of g lie in complex lines (since velocities and accelerations are proportional with some complex coefficient). Now we can use a complexified version of Beltrami's theorem (it follows from the results of Bochner [4], Otsuki and Tashiro [5]; for a detailed discussion and a sketch of a proof see Appendix 3):

Proposition 5 *Suppose that all germs of complex lines are totally geodesic surfaces with respect to some Hermitian metric on a part of \mathbb{C}^n . Then this metric is equivalent (up to a complex projective transformation) to a Fubini metric.*

This concludes the proof of the main theorem.

Complex families of circles

Consider a Kähler metric g defined in an open region $\Omega \subseteq \mathbb{C}^2$ and assume that all geodesics of g are arcs of circles. We saw that the exponential map of g at any point $p \in \Omega$ has the form $x \mapsto p + x + \frac{1}{2}A(x)x$ or $x \mapsto p + x + \frac{1}{2}xA(x)$ up to third order terms. Here A is some complex linear functional. Note that a complex projective transformation $x \mapsto p + (1 - \frac{1}{2}A(x))^{-1}x$ or $x \mapsto p + x(1 - \frac{1}{2}A(x))^{-1}$ has the same 2-jet and clearly takes all lines to circles. Therefore, the images of lines (through 0) under the above complex projective map are geodesics of g (through p). This is because a circle is determined by its velocity and acceleration at some point and hence by the 2-jet of some rectifying diffeomorphism. Thus the geodesics of g are point-wise rectifiable by means of complex projective transformations.

Suppose that we are given a family \mathcal{F} of curves in \mathbb{C}^2 (by a curve we mean a 1 dimensional closed submanifold). Let us say that the family \mathcal{F} is a *complete family of curves* in an open subset Ω of \mathbb{C}^2 if through each point of Ω in each direction there goes a curve from \mathcal{F} . A family \mathcal{F} is said to be *rectifiable* at some point $p \in \Omega$ if there exists a germ of diffeomorphism at p that takes each curve from \mathcal{F} passing through p to a straight line. A complete family \mathcal{F} of curves is called a *complex family of curves* in Ω if it is point-wise rectifiable in Ω by means of local diffeomorphisms holomorphic up to third order terms. Complete complex families of curves generalize the notion of geodesics with respect to a Kähler metric. As we saw a complex family of circles is point-wise rectifiable by means of complex projective transformations.

We have a local classification of all complete complex families of circles in Ω . Up to a complex projective transformation these are the following:

- Geodesics of Fubini metrics.
- A family outside the unit ball.
- Suspensions.

Let us describe the last 2 examples in detail.

A family of circles outside the unit ball. Inside the unit ball we have a model for the complex hyperbolic plane. The metric is given in coordinates (z_1, z_2) by

$$ds^2 = \frac{(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)(1 - z_1 \bar{z}_1 - z_2 \bar{z}_2) + (dz_1 \bar{z}_1 + dz_2 \bar{z}_2)(d\bar{z}_1 z_1 + d\bar{z}_2 z_2)}{(1 - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2}$$

All geodesics are circles (see Appendix 2). Note that this metric makes sense in the exterior of the unit ball as well. But in the exterior it will be no more positive definite. Nevertheless, geodesics make sense and they are circles. We get a complex family of circles that are not geodesics with respect to a (positive definite) Kähler metric (this is not obvious but not very hard to prove).

Suspensions. Let U be a domain in \mathbb{C} and \mathcal{F} a point-wise rectifiable family of circles in U (e.g. the set of geodesics in the Poincaré half-plane). Note that any point-wise rectifiable family in dimension 2 is complex (this follows from the result of Khovanskii [1]). We are going to define a complex family \mathcal{G} of circles in $U \times \mathbb{C}$ that will be called the *suspension* of \mathcal{F} . Each circle from \mathcal{G} must lie in some complex line. We will define \mathcal{G} on every complex line separately, and then prove that \mathcal{G} is point-wise rectifiable.

Take any complex line L in \mathbb{C}^2 . Then the projection π from $L \cap (U \times \mathbb{C})$ to U either maps everything to a point or is a linear conformal one-to-one map. In the first case (i.e. when L is “vertical”) define \mathcal{G} on L as the set of all real lines in L . In the second case the map π^{-1} clearly takes circles to circles. So define \mathcal{G} on L as the preimage of the set of all circles in U under the projection π .

Let us prove that the family \mathcal{G} thus constructed is a complex family of circles. Take a point $a \in U \times \mathbb{C}$. Suppose that \mathcal{F} can be rectified at the point $\pi(a)$ by some complex projective map $P = L_1/L_2$ where L_1 and L_2 are affine functions. It is easy to see that the map $(z, w) \mapsto (P(z), w/L_2(z))$ rectifies the family \mathcal{G} at a .

The proof of the above classification is not very hard. Nevertheless we will not give it here.

Appendix 1: Hermitian and Kähler metrics

Consider a Riemannian metric g in an open subset Ω of \mathbb{C}^n . This metric is called *Hermitian* if it is stable under the multiplication by i , i.e. $g(ix, iy) = g(x, y)$ for any 2 vectors x and y at the same point. With a Hermitian metric g one associates the differential (1, 1)-form $\omega(x, y) = g(ix, y)$ and a sesquilinear form (Hermitian inner product) $\langle X, Y \rangle = g(X, Y) - i\omega(X, Y)$. A metric g is said to be *Kähler* if $d\omega = 0$.

Let ∇^0 be the standard (flat) connection on \mathbb{C}^n . Denote by ∇ the Levi-Civita connection of g . Then for each pair of vector fields X and Y on Ω we have $\nabla_X Y = \nabla_X^0 Y + \Gamma(X, Y)$ where Γ is a symmetric \mathbb{R} -bilinear form at each point. The form Γ is called *Christoffel form*. In particular, the value of $\Gamma(X, Y)$ at a point p depends only on the values of X and Y at p (not on their derivatives). Let us recall the following fact:

Proposition 6 *A metric g is Kähler if and only if the corresponding covariant differentiation is complex linear, i.e., the Christoffel form is complex bilinear.*

PROOF. First assume that the metric is Kähler. Then for any 3 vector fields X, Y and Z we have

$$d\omega(X, Y, Z) = X\omega(Y, Z) - \omega([X, Y], Z) - \omega(Y, [X, Z]) = 0.$$

Here $[X, Y]$ denotes the commutator of the vector fields X and Y . Fix the values of X, Y and Z at some point p . We can always arrange that $\nabla_Y X = \nabla_Z X = 0$ at p by changing X in a neighborhood of p . Then $[X, Y] = \nabla_X Y - \nabla_Y X = \nabla_X Y$. Analogously, $[X, Z] = \nabla_X Z$. Recall that ∇_X depends on $X(p)$ only (not on the derivatives of X). Finally, we obtain

$$X\omega(Y, Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z).$$

On the other hand, by the compatibility of ∇ with g ,

$$X\omega(Y, Z) = Xg(iY, Z) = g(\nabla_X(iY), Z) + g(iY, \nabla_X Z).$$

Comparing our equations, we conclude that $g(\nabla_X(iY), Z) = g(i\nabla_X Y, Z)$. Since Z is arbitrary, $\nabla_X(iY) = i\nabla_X Y$, i.e., the covariant differentiation is complex linear.

The above argument can be reversed. If the covariant differentiation is complex linear, then $d\omega$ vanishes for all X, Y and Z such that $\nabla_Y X = \nabla_Z X = 0$ at some given point p . Since $X(p), Y(p)$ and $Z(p)$ can take arbitrary values, $d\omega = 0$ at p . But p is also arbitrary. Hence $d\omega = 0$ everywhere. \square

PROOF OF PROPOSITION 3. Indeed, the second differential of an exponential map coincides with Γ , but the latter is complex bilinear by Proposition 6. \square

Appendix 2: Fubini spaces

Fubini spaces are complex analogs of the classical geometries (Euclidean, Riemann, Lobachevsky).

Consider the complex space \mathbb{C}^{n+1} equipped with the pseudo-Hermitian form

$$H = Z_0\bar{Z}_0 + \alpha \sum_{j=1}^n Z_j\bar{Z}_j$$

where α is some real number. The *pseudosphere* is a hypersurface S given by the equation $H = 1$. Note that the pseudosphere is stable under the multiplication by complex numbers with absolute value 1, i.e., under the scalar $U(1)$ -action. The quotient space $F = S/U(1)$ is called a *Fubini space*.

Denote by C the cone where $H > 0$. Then the Fubini space can be also defined as the quotient C/\mathbb{C}^* (since the intersection of a \mathbb{C}^* -orbit with S is exactly a $U(1)$ -orbit). Hence, for $\alpha > 0$ we obtain the complex projective space $\mathbb{C}P^n$, for $\alpha = 0$ — the affine space \mathbb{C}^n and for $\alpha < 0$ — the complex hyperbolic space \mathbb{H}^n .

Let us introduce a Riemannian metric in a Fubini space. Suppose first that $\alpha \neq 0$. Then H induces a metric on S (for $\alpha < 0$, this metric will be negative so we should take it with sign minus) which is stable under the $U(1)$ -action. Hence a Fubini space F also inherits some metric. Namely, the distance between $U(1)$ -orbits is defined as the minimal distance from a point of one orbit to a point of the other orbit. For $\alpha = 0$, we should take the standard Euclidean metric on $\mathbb{C}^n = F$.

To get an affine model of a Fubini space F , it is enough to project it to the hyperplane $\{Z_0 = 1\}$. Namely, each point $x \in F$ can be viewed as a complex line in C . Take the intersection of this line with $\{Z_0 = 1\}$. Under this projection, F gets mapped to the whole hyperplane (for $\alpha \geq 0$) or to the interior of a ball (for $\alpha < 0$). In particular, for $\alpha > 0$ we get an affine chart of $\mathbb{C}P^n$. Metrics of Fubini spaces written down in the affine models are called the *Fubini metrics* on (parts of) \mathbb{C}^n .

Let us deduce the coordinate expressions of Fubini metrics for $\alpha > 0$. Take a vector $v \in T_x F$ at some point $x \in F$. Consider a lift X of x to C and a lift V of v looking out of X . We can always assume that $|X|^2 = 1$, i.e., $X \in S$ (all norms and inner products are with respect to the form H). Denote by W the projection of V to the orthogonal complement of X . Then the length of v with respect to the Fubini metric equals to the length of W with respect to H :

$$|v|^2 = \langle W, W \rangle = \langle V - \langle V, X \rangle X, V - \langle V, X \rangle X \rangle = \langle V, V \rangle - \langle X, V \rangle \langle V, X \rangle.$$

Now if X is arbitrary (not necessarily of unit length), then the formula for $|v|^2$ can be recovered by the homogeneity:

$$|v|^2 = \frac{\langle V, V \rangle \langle X, X \rangle - \langle X, V \rangle \langle V, X \rangle}{\langle X, X \rangle}.$$

The vector X can be regarded as the collection of homogeneous coordinates of the point x . In order to pass to affine coordinates, it is enough to put $X_0 = 1$, $V_0 = 0$ (X_0 and V_0 stand for zero-coordinates of X and V respectively). For $\alpha < 0$ the above expression is to be taken with negative sign.

Proposition 7 *All complex lines in Fubini metrics are totally geodesic surfaces. All geodesics are (parts of) circles.*

PROOF. Note that a Fubini metric is preserved under the action of (rather large) group of all H -unitary projective transformations. Each complex line is stable under a one-parametric subgroup of rotations around it. It follows that each complex line is a geodesic submanifold. On a coordinate line passing through the origin we have a classical geometry (standard Euclidean if $\alpha = 0$, spherical in central projection if $\alpha > 0$ or Lobachevsky in the Poincaré disk model if $\alpha < 0$). Clearly all geodesics inside this line are circles. Any complex line can be mapped to any other by an isometry. This concludes the proof. \square

A useful characterization of Fubini spaces was given by Bochner [4].

First recall the definition of holomorphic sectional curvature. Let g be a Hermitian metric in an open subset Ω of \mathbb{C}^n . Take a point $p \in \Omega$ and a vector ξ going out from this point. The vector ξ defines a germ of complex line. Consider the image of this germ under the exponential map of g . The image is a germ of 2-dimensional surface at the point p . Its Gauss curvature at p is denoted by $K(\xi)$ and is called the *holomorphic sectional curvature*. A metric is said to have *constant holomorphic sectional curvature* if $K(\xi)$ depends neither on the direction of ξ nor on the point p .

Theorem 8 (Bochner) *A Kähler metric g has constant holomorphic sectional curvature if and only if g is locally equivalent to a Fubini space via a holomorphic change of variables.*

Appendix 3: Complexified Beltrami's theorem

Proposition 5 is a complexified version of classical Beltrami's theorem [6]: if all geodesics are parts of straight lines, then the metric is locally equivalent to Eu-

clidean, Riemann or Lobachevsky. This complex version can be deduced from the results of Bochner [4], Otsuki and Tashiro [5]. Here we recall these results and also sketch another proof of Proposition 5 in dimension 2 which does not involve curvature considerations.

Lemma 9 *Let $\Gamma : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a homogeneous polynomial of degree 2 over reals (i.e., not necessarily holomorphic). Suppose that $\Gamma(v)$ is everywhere proportional to v with some complex coefficient $L(v)$. Then L is a complex-valued \mathbb{R} -linear function.*

PROOF. The coefficient L is a complex-valued function defined everywhere except perhaps 0. Since Γ is quadratic over reals, it satisfies the relation

$$\Gamma(v + w) + \Gamma(v - w) = 2(\Gamma(v) + \Gamma(w))$$

for all $v, w \in \mathbb{C}^n$. Substituting $L(u)u$ for $\Gamma(u)$, we obtain:

$$v(L(v + w) + L(v - w) - 2L(v)) + w(L(v + w) - L(v - w) - 2L(w)) = 0.$$

We can choose v and w to be linearly independent, so

$$L(v + w) + L(v - w) = 2L(v), \quad L(v + w) - L(v - w) = 2L(w).$$

If v and w are linearly dependent, this is also true due to the homogeneity of L . Hence the equations above hold for all v and w . They imply that L is \mathbb{R} -linear. \square

Proposition 10 *Consider a Hermitian metric g in an open subset Ω of \mathbb{C}^n . If all germs of complex lines lying in Ω are totally geodesic submanifolds, then the Christoffel form is equal to $\Gamma(v) = L(v)v$ where L is some complex linear functional (for a definition of the Christoffel form see Appendix 1).*

PROOF. Consider an arbitrary vector v at some point $x \in \Omega$ and a geodesic γ passing through x with velocity v . By the equation of geodesics, $\ddot{\gamma} + \Gamma(v, v) = 0$. But since the geodesic lies in some complex line, $\ddot{\gamma}$ is proportional to $\dot{\gamma} = v$ with some complex coefficient. Therefore, $\Gamma(v, v)$ is proportional to v . By lemma 9, $\Gamma(v, v) = L(v)v$. \square

Corollary 11 *Under the assumptions of Proposition 10 the metric g is Kähler.*

PROOF. By lemma 10, $\Gamma(v, v) = L(v)v$. By the symmetry of Γ , we have $\Gamma(v, w) = \frac{1}{2}(L(v)w + L(w)v)$.

Now we can use the Hermitian property: $g(X, X) = g(iX, iX)$. Apply the covariant differentiation ∇_Y to both sides of this relation:

$$g(\nabla_Y X, X) = g(\nabla_Y(iX), iX) = -g(i\nabla_Y(iX), X).$$

The standard connection ∇^0 is complex linear, hence $g(i\Gamma(Y, iX) + \Gamma(Y, X), X) = 0$. Since Y is arbitrary, it follows that $L(iX) = iL(X)$, i.e., L is complex linear. This means that Γ is \mathbb{C} -bilinear. By Proposition 6, g is Kähler in this case. \square

Consider 2 Hermitian metrics g' and g'' and denote the corresponding Levi-Civita connections by ∇' and ∇'' respectively. Recall that the difference $\Gamma(X) = \nabla''_X X - \nabla'_X X$ is a vector-valued quadratic form. The metrics g' and g'' are called *holomorphically projectively equivalent* if $\Gamma(x) = L(x)x$ for all vectors x where L is a complex linear functional (depending on point). Lemma 10 shows that if all complex lines are geodesic surfaces, then the metric is holomorphically projectively equivalent to the standard (flat) metric.

Otsuki and Tashiro proved [5] that a Hermitian metric that is holomorphically projectively equivalent to a Fubini metric, has constant holomorphic sectional curvature. By Bochner's theorem it is isometric to a Fubini space, an isometry being a holomorphic map. But a holomorphic map taking (locally) complex lines to complex lines is a complex projective transformation. Thus we obtain Proposition 5. Below we sketch a more straight-forward proof of it in dimension 2.

Suppose a metric g in an open subset Ω of \mathbb{C}^2 satisfies the conditions of Proposition 5. Choose a pair of constant linearly independent vector fields X and Y in Ω and compose the Gram determinant

$$G = G(X, Y) = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle \langle Y, X \rangle$$

where \langle, \rangle is the Hermitian inner product corresponding to g . Note that G does not essentially depend on X and Y . In fact, it is well defined as a function up to a positive constant factor.

Lemma 12 *The Hermitian metric $h = g/G^{2/3}$ is constant along each complex line. This means that for any vectors v and w of the same Euclidean length lying in the same complex line we have $h(v) = h(w)$.*

PROOF. Let X and Y be constant linearly independent vector fields in Ω . Then we have $XG(X, Y) = 3\text{Re}L(X)G(X, Y)$. On the other hand, $Xg(X, X) =$

$2\text{Re}L(X)g(X, X)$. It follows that $Xh = 0$. Since X is an arbitrary constant vector field, h must be constant along any real line. It remains to note that a Hermitian metric constant along any real line is also constant along any complex line. \square

Note that g can be recovered by h . Namely, if H is the Gram determinant of h , then $g = h/H^2$. It remains to describe all Hermitian metrics that are constant along any complex line. This is not difficult to accomplish. Any such metric considered as a function of a point x and a vector v out of x is a second degree polynomial in “complex momentum” v and “complex angular momentum” $x \wedge v$ (the wedge product is over complex numbers). One can readily verify that these metrics provide Fubini metrics (modulo complex projective transformations) after division by the square of their Gram determinants.

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