ROBUST SET-VALUED SOLUTIONS IN GAMES

Jörgen Weibull
Stockholm School of Economics and Ecole Polyéchnique

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Manuscript not yet available. Brings together and extends

1. Basu K. and J. Weibull (1991): "Strategy subsets closed under rational behavior", *Economics Letters*

2. Young P. (1993): "Evolution of conventions", *Econometrica*

3. Hurkens S. (1995): "Learning by forgetful players", *Games and Economic Behavior*

4. Young P. (1998): *Individual Strategy and Social Structure*, Princeton University Press

5. Asheim G., M. Voorneveld and J. Weibull (2009): "Epistemic robustness of sets closed under rational behavior", SSE WP.

1 John Nash's Ph D dissertation (1950)

1. His rationalistic/epistemic interpretation of equilibrium

2. His "mass action"/evolutionary interpretation of equilibrium

Q1: Are these informal claims correct?

Q2: What if we also ask for robustness, in both interpretations?

- The approach suggested here: set-valued solution concepts

2 Rationality $\not\Rightarrow$ equilibrium

- Rationality? Savage (1957)

- If players are rational, but they not know if the others are rational: Nash equilibrium does not follow

- Even if they know that the others are rational, but they do not know if the others know this: again Nash equilibrium does not follow...

- A stronger hypothesis: *common knowledge (CK)* of the game and all players' rationality [*Aumann-Brandenburger* (1995), *Lewis* (1969)]

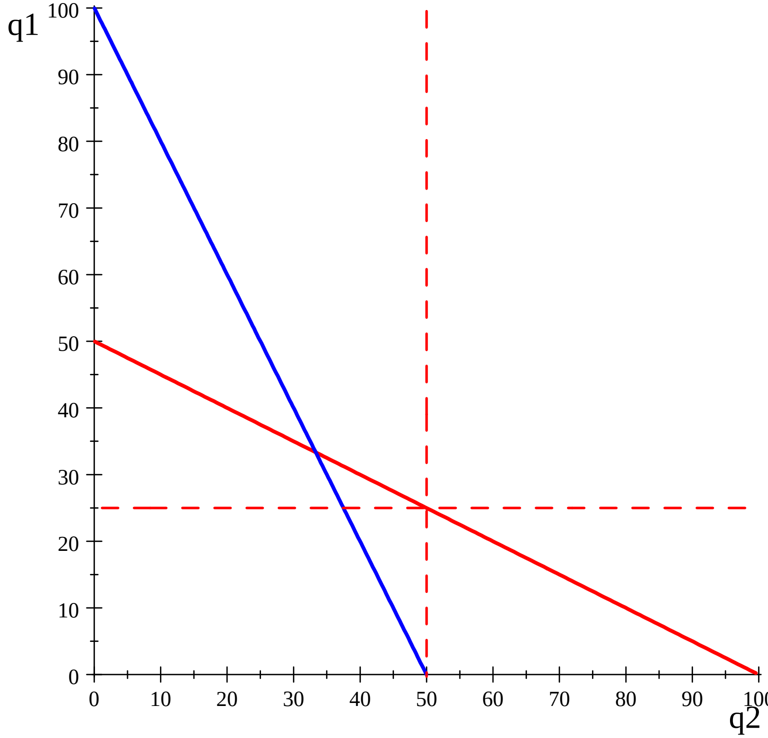
- In some games this implies equilibrium, in others it does not:

CK[rationality+game] $\not\Rightarrow$ Nash equilibrium

- Consider 1 positive ("wedding") and 3 negative examples ("funerals")!

2.1 A positive example

- Cournot duopoly with linear demand



2.2 Three negative examples

Example 1: A coordination game

	L	R
T	$2,2$	$0,0$
B	$0,0$	$1,1$

- **All** strategy profiles are compatible with CK[rationality,game]

Example 2: "matching pennies"

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

- **All** strategy profiles are compatible with CK[rationality,game]

Example 3: Game with unique equilibrium that is strict

	L	C	R
T	7, 0	2, 5	0, 7
M	5, 2	$\mathbf{3}, \mathbf{3}$	5, 2
B	0, 7	2, 5	7, 0

- **All** strategy profiles are compatible with CK[rationality,game]

3 Evolution $\not\Rightarrow$ equilibrium

Nash's (1950) "mass action" interpretation of equilibrium:

1. There is a large *population* of boundedly rational individuals/agents for each player role in the game

2. Every now and then, one individual from each such player population is randomly drawn to play the game, and each player has some empirical information about past play:

   "It is unnecessary to assume that the participants in a game have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.

To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population.

Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing.

... Thus the assumptions we made in this 'mass action' interpretation led to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point." (*John Nash's (1950) PhD. thesis*)

- In some games such stochastic evolution leads to Nash equilibrium, in some games it does not

- Re-consider the three negative examples!

4 Set-valued solution concepts

Domain of discussion: finite games in normal form, $G = (N, S, u)$, with mixed-strategy extensions $\tilde{G} = (N, \square(S), \tilde{u})$, where

- $N = \{1, ..., n\}$
- $S = \times_{i \in N} S_i$ and $u : S \to \mathbb{R}^n$
- $\Delta(S_i) = \left\{ x_i \in \mathbb{R}_+^{|S_i|} : \sum_{h \in S_i} x_{ih} = 1 \right\}$ unit simplex of mixed strategies
- $\square(S) = \times_{i \in N} \Delta(S_i)$ polyhedron of mixed-strategy profiles

- $\tilde{u} : \square(S) \to \mathbb{R}^n$ mathematical expectation of u under mixed-strategy profiles

Definition 4.1 *A* **point-valued solution concept** *for a class $\mathcal{G}$ of NF games is a correspondence φ that assigns a collection $\varphi(G)$ of mixed-strategy profiles to each game $G \in \mathcal{G}$. A point $x \in \square(S)$ is a* **solution** *under φ if $x \in \varphi(G)$.*

- Examples:

  Nash equilibrium [*Nash*, 1950]

  Perfect equilibrium [*Selten*, 1975]

  Proper equilibrium [*Myerson*, 1978]

Definition 4.2 *A* **set-valued solution concept** *for a class $\mathcal{G}$ of NF games is a correspondence ψ that assigns a collection $\psi(G)$ of sets of mixed-strategy profiles to each game $G \in \mathcal{G}$. A set $X \subseteq \square(S)$ is a* **solution** *under ψ if $X \in \psi(G)$.*

- Examples:

  Essential Nash equilibrium components [*Jiang* (1963)]

  Rationalizability [*Bernheim* (1984), *Pearce* (1984)]

  Strategically stable sets [*Kohlberg and Mertens* (1986)]

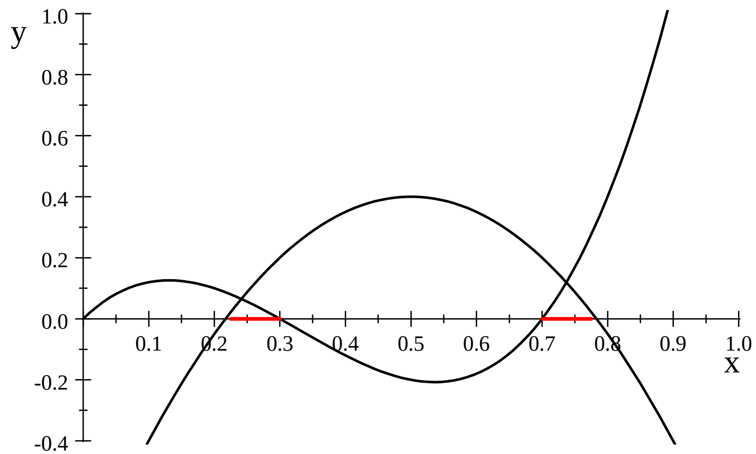
  Sets closed under rational behavior [*Basu and Weibull* (1991)]

5 Structure of the set of Nash equilibria

- The set of Nash equilibria:

$$X^{NE} = \{x \in \square(S) : \tilde{u}_i(x) - \tilde{u}_i(s_i, x_{-i}) \geq 0 \quad \forall i \in N,\ s_i \in S_i\}$$

- Finitely many polynomial inequalities: hence, a semi-algebraic set!



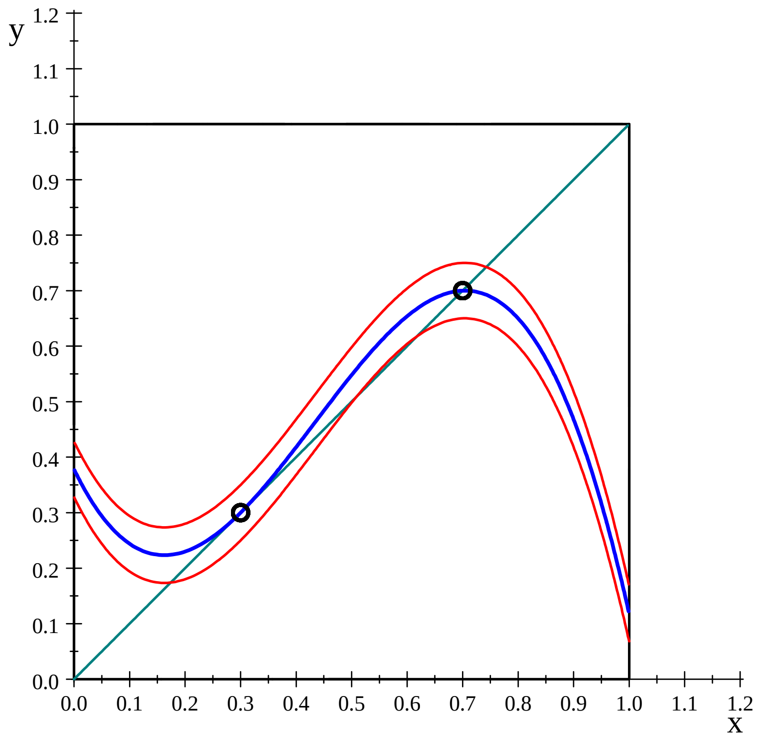
Proposition 5.1 *The set X^{NE} is the finite union of disjoint, closed and connected sets.*

- These subsets are called the *equilibrium components*

- For *generic extensive-form games*: payoffs are constant on each component [*Kreps and Wilson* (1982)]

5.1 Essential NE components

- *Essential fixed points* under continuous mappings [*Fort (1950)*]:



Definition 5.1 (Jiang, 1963) *An NE component* X *is* **essential** *if every nearby game* $G' = (N, S, u')$ *has some nearby NE: for all* $\varepsilon > 0$:

$$\left\|u' - u\right\| < \delta \quad \Rightarrow \quad \exists x^* \in X^{NE}\left(G'\right) \text{ with } \|x^* - X\| < \varepsilon$$

Proposition 5.2 (Jiang, 1963) $\exists$ *at least one essential component.*

Example 5.1 *Consider*

	a	b
a	$2,2$	$0,0$
b	$0,0$	$1,1$

Three Nash equilibria. However, the singleton $X = \{x^*\}$*, where* $x^* = \left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ *is the mixed NE, is highly non-robust - both epistemically & evolutionarily*

6 Epistemically robust solutions

[*Asheim, Voorneveld and Weibull (2009)*]

- Finite games $G = (N, S, u)$ with mixed-strategy extensions $\tilde{G} = (N, \square(S), \tilde{u})$

Epistemic model:

1. A compact Polish *type space* T_i for each player i. Let $T = \times_{i \in N} T_i$

2. *State space* $\Omega = \times_{i \in N} \Omega_i$, where $\Omega_i = S_i \times T_i$

3. To each type $t_i \in T_i$ is associated a Borel probability measure $\mu_i(t_i)$ over Ω_{-i} (not necessarily a product measure) and this mapping from types to beliefs is continuous and onto

[Approach close to *Mertens and Zamir* (1985); *Hu* (2007); *Brandenburger, Friedenberg and Keissler* (2008)]

Epistemic definitions:

- Let $B_i^p(E)$ be the event that **player i believes E with probability $\geq p$**.

  Formally: for each $p \in (0,1]$ and Borel measurable subset $E \subseteq \Omega$,

  $$B_i^p(E) = \{\omega \in \Omega : \mu_i(t_i(\omega))(E^{\omega_i}) \geq p\}$$

  where

  $$E^{\omega_i} = \{\omega_{-i} \in \Omega_{-i} : (\omega_i, \omega_{-i}) \in E\}$$

- Let C_i be player i's (rational) **choice correspondence**:

  $$C_i(t_i) = \beta_i\left(\mathrm{marg}_{S_{-i}}\mu_i(t_i)\right)$$

- Let $[R_i]$ be the event that **player i is rational**:

  $$[R_i] = \{\omega \in \Omega : s_i(\omega) \in C_i(t_i)\}$$

- For any set $Y = \times_{i \in N} Y_i$, with $Y_i \subset T_i \ \forall i$, let

$$C(Y) = \times_i C_i(Y_i)$$

- Let $[Y_i]$ be the event that **player i's type is in Y_i** :

$$[Y_i] = \{\omega \in \Omega : t_i(\omega) \in Y_i\}$$

Definition 6.1 *A product set $S^* = \times_{i \in N} S_i^*$ with $S_i^* \subset S_i \ \forall i$ is* **epistemically robust** *if $\exists\ \bar{p} < 1$ such that for each $p \in [\bar{p}, 1]$ $\exists Y$ with $C(Y) = S^*$ and*

$$B_i^p \left(\bigcap_{j \neq i} \Big([R_j] \cap [Y_j] \Big) \right) = [Y_i] \quad \forall i$$

Game-theoretic definitions:

- For each player i, let the (slightly generalized) pure-strategy best-reply correspondence $\beta_i : \mathbf{\Delta}\left(\times_{j\neq i} S_j\right) \rightrightarrows S_i$, defined by

$$\beta_i(\sigma_{-i}) = \left\{ s_i \in S_i : \tilde{u}_i(s_i, \sigma_{-i}) \geq \tilde{u}_i\left(s_i', \sigma_{-i}\right) \quad \forall s_i' \in S_i \right\}$$

Definition 6.2 (Basu and Weibull, 1991) *A product set* $S^* = \times_{i\in N} S_i^*$ *with* $S_i^* \subseteq S_i\ \forall i$ *is* **closed under rational behavior** *(*curb*) if*

$$\beta_i\left(\mathbf{\Delta}\left(\times_{j\neq i} S_j^*\right)\right) \subseteq S_i^* \quad \forall i$$

and **fixed** *under rational behavior (*furb*) if the inclusion is an identity.*

- We establish:

  Proposition: All curb sets are epistemically robust

  Proposition: No proper subset of a *minimal* curb set is epistemically robust

  Theorem: Epistemic robustness is equivalent with being furb

- Reconsider the earlier examples!

- Consider also

$$\begin{array}{cccc} & l & c & r \\ t & 3,1 & 1,2 & 0,0 \\ m & 0,3 & 2,1 & 0,0 \\ b & 5,0 & 0,0 & 6,3 \end{array}$$

  Note:

1. $x^* = \left(\left(\frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{4}, \frac{3}{4}, 0\right)\right)$ is a (perfect) Nash equilibrium

2. However, even if 2 would expect 1 to play x_1^*: 2 would be indifferent between l and c

3. If 1 believes that l and c are equally likely, then b is optimal for 1!

4. If 2 expects 1 to play b, then r is optimal for 2!

- Two curb sets but only one is *minimal*: $\{(b, r)\}$

7 Evolutionary robustness

- Stochastic population dynamics in the style of Nash's mass-action interpretation

1. Young (1993a): Evolution of conventions. The notion of stochastic stability

2. Young (1993b): Application to the Nash demand game

3. Hurkens (1995): Learning by forgetful players

4. Young (1998): Textbook with additional results

7.1 Young's model

- Domain: finite normal-form games $G = (N, S, u)$

- Model assumptions:

1. One population, of fixed size K, for each of the n player roles in G

2. Random matchings: one individual drawn from each player population

3. Each drawn individual observers a sample of size k from the last m rounds of play

4. The unperturbed process: play a best reply to the empirical distribution in your sample!

5. The ε-perturbed process: with probability $1-\varepsilon$ play a best reply to the empirical distribution in your sample, with probability $\varepsilon > 0$ play according to a fixed interior mixed strategy (mistakes, experimentations, new-beginners)

- A *history* h is a point in $H = S^m$.

- The *successor* of a history $h^t = (s(t-m), ..., s(t-1))$ is a history $h^{t+1} = (s(t-m+1), ..., s(t-1), s(t))$ for some $s(t) \in S$.

- For any $\varepsilon \geq 0$, this defines a *Markov chain* over the finite state space H

- For $\varepsilon > 0$: an *irreducible* and *aperiodic* Markov chain

- Hence, *ergodic* and admitting a *unique invariant distribution*, μ^ε

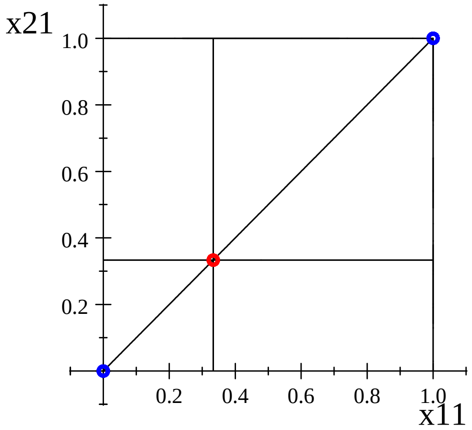
- As $\varepsilon \downarrow 0$, $\mu^\varepsilon \longrightarrow \mu^*$

Definition 7.1 (Young, 1993) *A state $h \in H$ is stochastically stable if $\mu^*(h) > 0$*

Example 7.1 *Coordination game:*

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Mixed-strategy polyhedron $\square(S)$:



Definition 7.2 *A finite normal-form game* $G = (I, S, u)$ *has property* **NDBR (non-degenerate best replies,** *if, for every player* i *and pure strategy* $s_i \in S_i$ *the set*

$$B_i(s_i) = \{x \in \square(S) : s_i \in \beta_i(x)\}$$

is either empty or $B_i(s_i) \cap [\square(S)]$ *contains an open set.*

- This is a generic property ("almost all games have it")

Theorem 7.1 (Young, 1998) *Let G be a finite game with the NDBR property. If k/m is sufficiently small, the unperturbed process converges with probability one to a minimal* curb *set. As $\varepsilon \downarrow 0$, the perturbed process puts arbitrarily large probability on the minimal* curb *set that has minimal stochastic potential.*

- Mathematics: Freidlin and Wintzell (1979, in Russian): *Random Perturbation of Dynamical Systems*, Springer Verlag.

- Reconsider earlier examples!

8 Relations to Nash equilibrium

Proposition 8.1 *For any finite game* $G = (N, S, u)$*:*

(i) $S^* = \times_{i \in N} S_i^*$ *maximal* furb $\Leftrightarrow$ T = *set of rationalizable pure-strategy profiles*

(ii) Minimal curb *set* $\exists$

(iii) Minimal curb $\Rightarrow$ furb

(iv) Every curb *set "absorbs a neighborhood:" if* S^* *is* curb *then* $\beta(B) \subseteq S^*$ *for some open neighborhood* B *of* $\Delta\left(\times_{j \neq i} S_j\right)$

(v) curb *sets "do not cut up" Nash equilibrium components: if* S^* *is* curb *and* X *a component of* X^{NE}*, then either* $X \subseteq \square(S^*)$ *or* $X \cap \square(S^*) = \varnothing$

(vi) Every curb *set contains an essential component of Nash equilibria*

(vii) Every curb *set contains a proper equilibrium*

- Recall that the mixed-strategy extension $\tilde{G}$ of any finite game G has at least one proper equilibrium (Myerson, 1978) and that every proper equilibrium induces a sequential equilibrium:

Proposition 8.2 (van Damme, 1983) *Let G be a finite normal-form game and let x^* be a proper equilibrium. For every finite extensive-form game Γ with perfect recall with normal form G, there exists a sequential equilibrium y^* in Γ that is realization-equivalent with x^*.*

9 Satisficing instead of maximizing

- *Herbert Simon (19579*: "...we must expect the firm's goals to be not maximizing profits, but attaining a certain level or rate of profit ..."

- "Satisficing"

Definition 9.1 *A set $S^* = \times_{i \in N} S_i^*$ with $\varnothing \neq S_i^* \subseteq S_i$ is* **closed under better replies** *if $\alpha\left(\square\left(S^*\right)\right) \subseteq S^*$, where*

$$\alpha_i\left(x\right) = \left\{s_i \in S_i : \tilde{u}_i\left(s_i, x_{-i}\right) \geq \tilde{u}_i\left(x\right)\right\}$$

Proposition 9.1 (Ritzberger and Weibull, 1995) *For any sign-preserving deterministic selection dynamic and any product set S^*: $\square\left(S^*\right)$ is asymptotically stable if and only if it is closed better replies.*

1. Ritzberger K. and J. Weibull (1995): "Evolutionary selection in normal-form games", *Econometrica*.

2. Weibull J. (1995): *Evolutionary Game Theory.* MIT Press.

3. Benaïm M. and J. Weibull (2003): "Deterministic approximation of stochastic evolution in games", *Econometrica*.

10 Conclusion

1. Do more set-wise analysis!

2. More work on the rationalistic/epistemic interpretation!

3. More work on the mass-action/evolutionary interpretation!

4. Apply!

11 References

References

[1] Aumann R. and A. Brandenburger (1995): "Epistemic conditions for Nash equilibrium", *Econometrica* 63, 1161-1180.

[2] Basu K. and J. Weibull (1991): "Strategy subsets closed under rational behavior", *Economics Letters* 36, 140-146.

[3] Bernheim D. (1984): "Rationalizable strategic behavior", *Econometrica* 52, 1007-1029.

[4] Fort M. (1950): "Essential and non essential fixed points", *American Journal of Mathematics* 72, 315-322.

[5] Hurkens S. (1995): "Learning by forgetful players", *Games and Economic Behavior* 11, 304-329.

[6] Jiang J.-H. (1963a): "Essential equilibrium points of n-person non-cooperative games", *Science Sinica* 12, 651-671.

[7] Jiang J.-H. (1963b): "Essential components of the set of fixed points in the multivalued mappings and its appliacation to the theory of games", *Science Sinica* 12, 951-964.

[8] Kohlberg E. and J.-F. Mertens (1986): "On the strategic stability of equilibria", *Econometrica* 54, 1003-1037.

[9] Kreps D. and R. Wilson (1982): "Sequential equilibria", *Econometrica* 50, 863-894.

[10] Myerson R.B. (1978): "Refinements of the Nash equilibrium concept", *International Journal of Game Theory* 7, 73-80.

[11] Nash J. (1950): "Non-cooperative games", Ph D thesis, Department of Mathematics, Princeton University.

[12] Pearce D. (1984): "Rationalizable strategic behavior", *Econometrica* 52, 1029-1050.

[13] Savage L. (1954): *The Foundations of Statistics*. Wiley, New York.

[14] Selten R. (1975): "Re-examination of the perfectness concept for equilibrium points in extensive games", *International Journal of Game Theory* 4, 25-55.

[15] Wu W. and J. Jian (1962): "Essential equilibrium points of n-person non-cooperative games", *Science Sinica* 11, 1307-1322.

[16] Young P. (1993a): "Evolution of conventions", *Econometrica 61*, 57-84.

[17] Young P. (1993b): "An evolutionary model of bargaining", *Journal of Economic Theory* 59, 145-168.

[18] Young P. (1998): *Individual Strategy and Social Structure*, Princeton University Press, Princeton.