Equivalence of candidate utility functions in electoral competition with probabilistic voting.

Alexei Zakharov, Constantin Sorokin

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Abstract

Most of the existing literature on electoral competition assumes that the candidates or political parties maximize either the probability of winning the election, or the expected share of seats in the parliament. There are a number of reasons to believe that this is a significant simplification of reality. We examine the properties of equilibria in an electoral competition game depending on the shape of the candidate utility functions. We demonstrate that, if shocks to voter utility are uncorrelated, then the equilibrium strategies of candidates do not depend on their utility functions, if the number of voters is very large. If the shocks to voter utilities are correlated, then the electoral equilibrium will generically depend on the shape of candidate utility functions, even as the number of voters tends to infinity. The analysis is extended to the case when there are costs associated with selecting a particular campaign strategy.

1 Introduction

In elections, the objectives of candidates are not often reduced to just winning.
Large margins of victory may induce a very large payoff; this may be true both in authoritarian and semi-authoritarian regimes (Simpser, 2008), or in parliaments where a supermajority may be required; in single-member constituencies, large margins of victory may serve as deterrents to high-quality challengers (Krasno and Green, 1988, Goodliffe, 2007) and therefore increase the probability of winning subsequent elections (Lazarus, 2009, Bonneau and Cann, 2014).

In Simpser (2013) it is argued that large victory margins for incumbents are a common feature of semiautocratic regimes, because they are beneficial to autocrats. A demonstration of overwhelming public support for one’s policies increases the rulers bargaining powers vis-a-vis business interests and trade unions, deters potential opposition from coordinating, and mitigates the pressure to share rents with other groups. Hence a common feature of semiautocratic or hybrid regimes is excessive electoral manipulation that often has the goal of distorting the information about the leader’s popularity.

A large margin of victory is needed (especially in countries without a prolonged tradition of a democratic power transfer) in order to initiate large-scale economic or political reform. The consequences may be dire for their initiator, unless he or she relies on the support of a large majority. A telling example is Chilean president Salvador Allende, who won the 1970 Presidential elections on top of a 36.63% plurality (with the runner-up receiving 35.29%); the broad socialist reforms that he authored provoked a coup in which he ultimately lost his life.

A loser who lost by a narrow margin may receive an additional reward — a “consolation prize” (Hojman (2004)). Finally, floor requirement, quotient formula, and district magnitude all affect the translation of votes into seats (Lijphart, 1990; Gallagher, 1992) even in “proportional representation” electoral systems. Coalition-building concerns further complicate the payoff functions of political parties (Snyder, Ting, and Ansolabehere, 2005; Laver and Shepsle, 1996; Schofield and Sened, 2006, among others). Anagol and Fujiwara (2014) have found runner-up effects in a large-sample study of elections in Brazil,
India, and Canada. They have shown that placing second (versus placing third) in a first-past-the-post election is associated with a significantly higher probability of running (and succeeding) in the subsequent election. In particular, placing second in a Brazilian mayoral election can increase the probability of running in the subsequent election by 9.4 percentage points, and the probability of winning by 8.3 percentage points.

A strong second-place result of an opposition candidate in a hybrid regime may grant him an extra degree of security from persecution. It is widely believed that a strong second-place showing by Russian anti-corruption campaigner Alexei Navalny in 2013 Moscow mayoral election allowed him to escape a real prison sentence on embezzlement charges (Oliphant, 2014).

So far, literature has given little consideration to the consequences that either consolation prizes or the benefits due to large winning margins may have on the political equilibrium. The two theoretical works that addressed this questions were earlier works by the authors.

Zakharov (2012) analyzed the effect of candidate preferences on their policy platforms if the voters are stochastic. The comparative statics were derived for a particularly simple case, when there were only two voters with different policy preferences and with different idiosyncratic preferences (or valences) for the candidates. In such a setting there are three election outcomes: either candidate winning by a two-candidate margin, and a close election, with each candidate receiving one vote. If the elections are winner-take-all (and the winner, in case of a close election, is decided by a fair coin toss), then the candidate payoffs should be linear in the number of votes. The same is true if these are proportional parliamentary elections with two competing parties, and the payoff of a party is equal to its number of seats.

However, the payoffs are no longer linear if there is a consolation prize to a loser of a coin toss, or if there is an extra benefit of winning by a margin of two votes versus winning by a coin toss. In the former case, each candidate will choose a policy position that is
closer to the ideal policy of the voter who has a higher valence for that candidate. In the latter case, each candidate will locate closer to the voter who idiosyncratically prefers the other candidate. The “mean voter theorem”, according to which candidates under stochastic voting should choose identical policy positions, holds only if the candidates payoffs are linear (such as if the elections are winner-take-all or proportional).

In a subsequent work, Zakharov and Sorokin (2014) generalized the result, showing that the mean-voter theorem in a two-candidate election with stochastic voters, one can only expect the candidates to choose identical policy positions only for a narrow class of voter’s probability of vote functions, unless the preferences of the candidates are symmetric around a 50/50 split in votes. In the latter case, an equilibrium in which both candidates select identical policy positions exists under a much broader class of probability-of-vote functions.

Here we are going to take our previous work one step further by assuming an infinite number of stochastic voters.1 We are going to do this by assuming an electoral competition game with some finite number of voters, then replicating the voters a multiple that tends to infinity. We demonstrate that the resulting limiting equilibrium depends on our assumption about the shocks to voter utility. In Section 2 we show that if the shocks are statistically independent across voters, then the resulting equilibrium is invariant with respect to our assumptions about the way how votes are transferred into utility. In Section 3 we consider shocks that are correlated. In that case different candidate utility functions will almost always result in different limiting equilibria. In Section 4, we look at the case when there are costs associated with choosing particular policy positions, and derive results similar to previous sections.

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1In stochastic voting, a component of voter utility is not observed by the candidates; hence, a voter’s decision, from a candidate’s point of view, is a random variable that depends continuously on candidate positions. If the candidate strategies constitute redistribution of a fixed budget, then greater shares accrue to those voters whose probability of vote is more responsive to changes in their allocated shares — for example, those who are $a$ priori indifferent between the two candidates, and whose stochastic
2 Independent voting

There are 2 candidates and a finite number of stochastic voters. The candidates engage in a one-shot game by choosing policy platforms $y_1, y_2$ from a compact policy space $X \subset \mathbb{R}^k$. The voter of type $i = 1, \ldots, n$ has policy preferences manifested in a twice continuously differentiable probability of voting (POV) function $p_i : X^2 \to [0, 1]$, where $p_i(y_1, y_2)$ is the probability that voter of type $i$ will support Candidate 1 given candidate policy platforms $y_1$ and $y_2$. For any given $y_1, y_2$, the vote of any one voter is statistically independent of the votes of all other voters. There are $m$ voters of each type. Let $p : X^2 \to [0, 1]^n$ be the profile of POV functions.

The payoff of a candidate depends on the share of votes she receives. Each candidate $j = 1, 2$ is endowed with a von Neumann-Morgenstern utility function:

$$u_j : [0, 1] \to [0, 1].$$

We assume that $u_j$ is a strictly increasing function and that $u_j(0) = 0$ and $u_j(1) = 1$. Denote by $F_m(x|y_1, y_2)$ the probability that the share of votes for Candidate 1 would be below $x$ given $m, y_1,$ and $y_2$. The payoff functions of the candidates are

$$U_{m1}(y_1, y_2) = \int_0^1 u_1(x)dF_m(x|y_1, y_2),$$

$$U_{m2}(y_1, y_2) = \int_0^1 u_2(1 - x)dF_m(x|y_1, y_2).$$

Denote by

$$G_m = \{\{1, 2\}, \{X, X\}, \{U_{m1}(y_1, y_2), U_{m2}(y_1, y_2)\}\}.$$  

the normal form electoral competition game given $m$. The utility functions of the candidates (4), (2) converge to a limit as the number of voters increases.
Theorem 1 Suppose that $u_1(\cdot)$ and $u_2(\cdot)$ are $k + 1$ times continuously differentiable. Then there exist $U_1(y_1, y_2)$ and $U_2(y_1, y_2)$ that are $C^k$ limits of $U_{m1}(y_1, y_2)$, $U_{m2}(y_1, y_2)$ as $m \to \infty$. Moreover,

$$U_1(y_1, y_2) = u_1\left(\frac{1}{n} \sum_{i=1}^{n} p_i(y_1, y_2)\right), \quad U_2(y_1, y_2) = u_2\left(1 - \frac{1}{n} \sum_{i=1}^{n} p_i(y_1, y_2)\right).$$

This result allows us to define the limit game

$$G = \langle\{1, 2\}, \{X, X\}, \{U_1(y_1, y_2), U_2(y_1, y_2)\}\rangle.$$ 

This game is strategically equivalent to the game where both candidates are expected voteshare maximizers, in the sense that the best-response functions of both players are invariant with respect to $u_1(\cdot)$, $u_2(\cdot)$:

Corollary 1 For all possible strictly increasing utility functions $u_1(\cdot)$, $u_2(\cdot)$ the limit games are strategically equivalent.

We have established the convergence of candidate payoff functions in games $G_m$ as the number of voters increases to a limit function that is invariant with respect to utility functions $u_j(\cdot)$. We now need to show the conditions under which equilibria in these games converge to an equilibrium in the limiting game $G$. The following statement is true:

Corollary 2 Let $u_1(\cdot)$, $u_2(\cdot)$ be $3$ times continuously differentiable candidates utility functions. Suppose that there is a sequence of games $\{G_k\}_{k \in \mathbb{N}}$ and a corresponding sequence of interior local (global) Nash equilibria converging to some point $(y_1^*, y_2^*)$ as $k \to \infty$. Suppose that there exists $a < 0$ such that $\langle D_j^2(U_j(y_1^*, y_2^*)) \cdot y_j, y_j \rangle < a ||y_j||^2$ for all $j = 1, 2$ and for all $y_j \neq 0$. Then $(y_1^*, y_2^*)$ is a local (global) Nash equilibrium of the limit game.

The condition that is required for the convergence of equilibria to an equilibrium in the limit game is that the second order conditions are uniformly bounded away from zero. We then derive the conditions for the reverse, when an equilibrium of the limit game implies the existence a sequence of equilibria in electoral competition games that converge to that equilibrium as the number of voters in each group increases to infinity.
Corollary 3 Let $u_1(\cdot), u_2(\cdot)$ be 3 times continuously differentiable candidates utility functions. Let $\{G_k\}_{k \in \mathbb{N}}$ be a sequence of games, and suppose that $(y_1^*, y_2^*)$ is an interior local (strictly global) Nash equilibrium of the limit game. Suppose that for all $j = 1, 2$ and for all $y_j \neq 0$ we have $\langle D_j^2(U_j(y_1^*, y_2^*)) \cdot y_j, y_j \rangle < a||y_j||^2$. Let the Jacobian of function $H : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$

$$H(y_1, y_2) = (D_1(U_1(y_1, y_2)), D_2(U_2(y_1, y_2)))$$

be nonzero at $(y_1^*, y_2^*)$. Then there exist $k_0 \in \mathbb{N}$ such that for all $k > k_0$, game $G_k$ has a local Nash (strictly global) equilibrium $(y_1^*, y_2^*)$. Moreover, the sequence $(y_{1k}, y_{2k})$ converges to $(y_1^*, y_2^*)$ as $k \to \infty$.

The condition that the Jacobian $H(y_1, y_2)$ is nonzero at equilibrium implies that players’ best-response functions intersect at equilibrium, instead of just touching each other.

3 Correlated voting

We have shown that the limit game does not depend on the shape of the utility functions $u_1(\cdot), u_2(\cdot)$. This result is a consequence of the fact that the variance of the vote shares of candidates converges to zero as the number of voters increases. This, in turn, follows from the assumption that the votes are statistically independent. As we will see, in a more general case the limit games will not be invariant with respect to utility functions of voters.

Suppose that there are two states on nature that occur with probabilities $q$ and $1 - q$. Let $p^l : X^2 \to [0, 1]^n$ be the profile of POV functions in state $l = 1, 2$. Assume that the votes are independent conditional on the state of nature $i = 1, 2$. Denote by $F_m^l(x|y_1, y_2)$ the probability that the share of votes for Candidate 1 would be below $x$ in state $l = 1, 2$. The expected payoff functions of the candidates will be, depending on $q$,

$$U_{m1}^q(y_1, y_2) = q \int_0^1 u_1(x)dF_m^1(x|y_1, y_2) + (1 - q)u_1(x)dF_m^2(x|y_1, y_2), \quad (4)$$
\[ U_{m2}^q(y_1, y_2) = q \int_0^1 u_2(1 - x) dF_m^1(x|y_1, y_2) + (1 - q) \int_0^1 u_2(1 - x) dF_m^2(x|y_1, y_2). \] (5)

Define the electoral competition game \( G^q_m \) similarly to (3).

The convergence of candidate utility functions converge to some limit as the number of voters of each type increases:

**Theorem 2** Suppose that \( u_1(\cdot) \) and \( u_2(\cdot) \) are \( k + 1 \) times continuously differentiable. Then there exist \( U_1^q(y_1, y_2) \) and \( U_2^q(y_1, y_2) \) that are \( C^k \) limits of \( U_{m1}^q(y_1, y_2) \), \( U_{m2}^q(y_1, y_2) \) as \( m \rightarrow \infty \).

Let \( G^q \) be the resulting limit game. We can establish the convergence of equilibria in games \( G^q_m \):

**Corollary 4** Let \( u_1(\cdot) \), \( u_2(\cdot) \) be 3 times continuously differentiable candidates utility functions. Then Corollaries 2 and 3 hold with respect to the sequence of games \( \{G^q_k\}_{k \in \mathbb{N}} \).

However, for \( q \in (0, 1) \), the limit games defined for different candidate utility functions \( u_1(\cdot), u_2(\cdot) \) will no longer be strategically equivalent. Moreover, we show that if for some pair of candidate utility functions the pair \( (y_1^*, y_2^*) \) is an equilibrium in the limit game, then for almost any other pair of utility functions it is not an equilibrium in the corresponding limit game.

First we introduce a regularity condition on two POV profiles.

**Definition.** A pair of probability of vote function profiles \( p^1, p^2 \) is regular if for all \( y_1, y_2 \) for all \( j = 1, 2 \), and for all \( q \in (0, 1) \) we have

\[
\left\| \left( \sum_{i=1}^n p^1_i(y_1, y_2) \right)' \right\| + \left\| \left( \sum_{i=1}^n p^2_i(y_1, y_2) \right)' \right\| \neq 0,
\]

and

\[
(1 - q) \left( \sum_{i=1}^n p^2_i(y_1, y_2) \right)' + q \left( \sum_{i=1}^n p^1_i(y_1, y_2) \right)' \neq 0 \quad (6)
\]
whenever
\[ \sum_{i=1}^{n} p^1_i(y_1, y_2) = \sum_{i=1}^{n} p^2_i(y_1, y_2). \] (7)

Denote by Ω the set of all possible strictly increasing $C^2$ functions such that $u(0) = 0$ and $u(1) = 1$. This set is convex and Borel.

Take $q \in [0, 1]$ and a pair of regular probability of vote function profiles $p^1, p^2$. Let $(y_1, y_2) \in int(X \times X)$. Denote by $V(y_1, y_2)$ be the set of all $(u'_1(\cdot), u'_2(\cdot)) \in \Omega \times \Omega$ such that first-order conditions for Nash equilibrium hold at $(y_1, y_2)$ in the limit game. We need to show that, we long is $q \in (0, 1)$, the set $V(y_1, y_2)$ is small even if the first-order conditions are satisfied at $(y_1, y_2)$ for some pair of candidate utility functions. We make our statement formal using the notion of finite shyness (see Anderson and Zame, 2000).

**Definition.** Let $X$ be a topological vector space and $C \subset X$ be a convex Borel subset of $X$. Let $\lambda_V$ Lebesgue measure on finite dimensional subspace $V$ of $X$. A Borel subset $E \subset C$ is **finitely shy** in $C$ if there is a finite-dimensional subspace $V \subset X$ such that $\lambda_V(C + a) > 0$ for some $a \in X$ and $\lambda_V(E + x) = 0$ for every $x \in X$.

The following statement holds:

**Theorem 3** If $q \in \{0, 1\}$ then either $V(y_1, y_2) = \emptyset$ or $V(y_1, y_2) = \Omega \times \Omega$. If $q \in (0, 1)$ then $V(y_1, y_2)$ is finitely shy in $\Omega \times \Omega$.

It follows that in the presence of correlated voting equilibrium (if it exists) is no longer invariant with respect to candidate utility functions.

### 3.1 Costly strategies

Now we assume that if candidates take positions $(y_1, y_2)$ they have to pay costs $c_1(y_1, y_2)$ and $c_2(y_1, y_2)$ respectively, with $c_i \in C^2[X \times X]$. In general, the equilibrium will now depend on the candidate utility functions, as the first-order conditions for candidate $i$ will...
depend on both $u_i$ and $c_i$. However, a version of corollaries 2 and 3 can be established. And again, there would be no ordinal equivalence between possible limit games, though for a simpler reason. We will clear this point with our next theorem.

**Theorem 4** Suppose that $(y_1, y_2)$ is a local Nash equilibrium in a limit game with utilities $u_1, u_2$; next suppose that $(\bar{u}_i)' \equiv (u_i)'$ in some $\epsilon$-neighbourhood of $(y_1, y_2)$. Then $(y_1, y_2)$ is a local Nash equilibrium in the limit game with utilities $\bar{u}_i$.

So, everything that matters for a local equilibrium in this case is the margin of expected utility at equilibrium share of votes.

## 4 Conclusion

Politicians competing for elected office face various incentives that are conditioned by their institutional environment; in particular, losing by a narrow margin, or winning by a large margin may incur additional benefits. We demonstrate that, in a large election and in the absence of campaign costs, such things will have an effect on the electoral equilibrium only if the random utility shocks received by voters are correlated. In the absence of such shocks, the consequence of the central limit theorem is that the relationship between candidate utility and candidate strategies will not depend on how votes are translated into candidate utility.

### Appendix

**Proof of corollary 2.**

The result can be easily obtained by taking the limit of first and second order conditions, we can do it due to theorem 1. Note that the SOC conditions are satisfied at the limit equilibrium since the SOC of games in the sequence are bounded away from zero.

**Proof of corollary 3.**
This proof is a bit more tricky, but still very straightforward. Consider the FOC 
\[ H(y_1, y_2) = (D_1(U_1), D_2(U_2)) \]
as a function of two “groups” of arguments: policy platforms 
\[ y = (y_1, y_2) \] and utility functions \[ U = (U_1(\cdot), U_1(\cdot)) \] then apply a Banach space version 
of implicit function theorem to them, taking candidate positions as one variable (with 
usual \( \mathbb{R}^2 \) metric) and the candidates’ payoff functions as the other (with \( C^2 \) metric, 
note that \( H \) is continuously differentiable with respect to \( U \)). Given the non-degeneracy 
conditions it would yield us the dependency of equilibrium positions from candidates 
utilities. Next, note that if equilibria second order conditions are bounded away from zero 
then for sufficiently \( C^2 \)-close candidate utilities (and thus sufficiently close equilibrium) 
it would also be the case.

**Proof of theorem 1.**

Let’s write down the player’s expected utility 
\[ U_{1m}(y_1, y_2) = \sum_{k=0}^{mn} \left[ \sum_{l_1 + \cdots + l_n = k} \prod_{i=1}^{m} C_{m}^{l_i} p_i^{l_i}(1 - p_i)^{m-l_i} \right] u_1(k/mn) \]
and take its derivative:

\[
D_1(U_{1m}(y_1, y_2)) = \sum_{j=1}^{j=n} mD_1(p_j) \\
\left\{ \sum_{k=1}^{mn} \left[ \sum_{l_1 + \cdots + l_n = k} \left( \prod_{i=1}^{m} \frac{(m - 1)!}{(l_j - 1)! (m - l_j)!} p_j^{l_j-1}(1 - p_j)^{m-l_j} \right) u_1(k/mn) \right] \\
- \sum_{k=0}^{mn-1} \left[ \sum_{l_1 + \cdots + l_n = k} \left( \prod_{i=1}^{m} \frac{(m - 1)!}{l_j!(m - 1 - l_j)!} p_j^{l_j}(1 - p_j)^{m-l_j-1} \right) u_1(k/mn) \right]\right\} \\
= \sum_{j=1}^{j=n} mD_1(p_j)
\]
\[
\left\{ \sum_{k=0}^{mn-1} \left\{ \sum_{\sum_{i \leq j} k_i = k, i \geq 0, i \leq m, k_j < m} \left( \prod_{i \neq j} C_{m_i}^{l_i} p_i (1 - p_i)^{m_i - l_i} \right) \frac{(m - 1)!}{l_j! (m - 1 - l_j)!} p_j (1 - p_j)^{m - l_j - 1} \right\} u_1(k^{1/mn}) - \sum_{k=0}^{mn-1} \left\{ \sum_{\sum_{i \leq j} k_i = k, i \geq 0, i \leq m, l_j < m} \left( \prod_{i \neq j} C_{m_i}^{l_i} p_i (1 - p_i)^{m_i - l_i} \right) \frac{(m - 1)!}{l_j! (m - 1 - l_j)!} p_j (1 - p_j)^{m - l_j - 1} \right\} u_1(k/mn) \right\} = \sum_{j=1}^{j=n} m D_1(p_j)
\]

Note that

\[ m(u_1(k^{1/mn}) - u_1(k/mn)) = \frac{1}{n}(u_1(k/mn))' + \frac{1}{2mn^2} (u_1(k/\theta/mn))'' \]

therefore

\[ D_1(U_{1m}(y_1, y_2)) = \frac{1}{n} \sum_{j=1}^{j=n} D_1(p_j) \]

\[ \sum_{k=0}^{mn-1} \left\{ \sum_{\sum_{i \leq j} k_i = k, i \geq 0, i \leq m, l_j < m} \left( \prod_{i \neq j} C_{m_i}^{l_i} p_i (1 - p_i)^{m_i - l_i} \right) C_{m-1}^{l_j} p_j (1 - p_j)^{m - l_j - 1} \right\} (u_1(k/mn))' \]

\[ + O\left(\frac{1}{m}\right). \quad (8) \]

The last equity holds due to the fact that \((u_1(\cdot))''\) is bounded and

\[ \sum_{k=0}^{mn-1} \left\{ \sum_{\sum_{i \leq j} k_i = k, i \geq 0, i \leq m, l_j < m} \left( \prod_{i \neq j} C_{m_i}^{l_i} p_i (1 - p_i)^{m_i - l_i} \right) C_{m-1}^{l_j} p_j (1 - p_j)^{m - l_j - 1} \right\} \equiv \]

\[ \equiv (p_j + (1 - p_j))^{m-1} \prod_{i=1, i \neq j}^{n} (p_i + (1 - p_i))^m \equiv 1 \]
Now we are ready to apply the law of large numbers. Note that if

$$\forall \varepsilon_1, \varepsilon_2 > 0, \exists m'_0 \text{ s.t. } \forall m' \in \mathbb{N}, m > m'_0$$

$$\sum_{k=0}^{k \leq m'} \left( \frac{1}{n} \sum_{j=1}^{n} p_j - \varepsilon_1 \right) \left[ \sum_{l_1+\cdots+l_n=k}^{l_i \geq 0, l_i \leq m'} \left( \prod_{i=1}^{n} C_{m'P_i}^{l_i} (1 - p_i)^{m'-l_i} \right) \right] +$$

$$+ \sum_{k \geq m'}^{k \geq m'} \left( \frac{1}{n} \sum_{j=1}^{n} p_j - \varepsilon_1 \right) \left[ \sum_{l_1+\cdots+l_n=k}^{l_i \geq 0, l_i \leq m'} \left( \prod_{i=1}^{n} C_{m'P_i}^{l_i} (1 - p_i)^{m'-l_i} \right) \right] < \varepsilon_2$$

then

$$\forall \varepsilon_1, \varepsilon_2 > 0, \exists m_0 \text{ s.t. } \forall m \in \mathbb{N}, m > m_0$$

$$\sum_{k=0}^{k \leq m} \left( \frac{1}{n} \sum_{j=1}^{n} p_j - \varepsilon_1 \right) \left[ \sum_{l_1+\cdots+l_n=k}^{l_i \geq 0, l_i \leq m, l_j < m} \left( \prod_{i=1}^{n} C_{m'P_i}^{l_i} (1 - p_i)^{m-l_i} \right) C_{m-1P_j}^{l_j} (1 - p_j)^{m-l_j-1} \right] +$$

$$+ \sum_{k \geq m}^{k \geq m} \left( \frac{1}{n} \sum_{j=1}^{n} p_j - \varepsilon_1 \right) \left[ \sum_{l_1+\cdots+l_n=k}^{l_i \geq 0, l_i \leq m, l_j < m} \left( \prod_{i=1}^{n} C_{m'P_i}^{l_i} (1 - p_i)^{m-l_i} \right) C_{m-1P_j}^{l_j} (1 - p_j)^{m-l_j-1} \right] < \varepsilon_2$$

The intuition for this claim is straightforward — one missing voter can’t foil the law of large numbers. Thus we obtain that

$$D_1(U_1m(y_1, y_2)) \rightarrow \left( u_1 \left( \sum_{j=1}^{n} p_j / n \right) \right)' \sum_{j=1}^{n} D_1(p_j).$$

Observe that this is exactly the derivative of $U_1$. The convergence for higher derivatives can be obtained by iterating the argument above.

**Proof of Theorem 3.**

The first claim of the theorem is already established above, to prove the second one we need to select a finite-dimensional subspace $H$ of $C^2$ such that for some $\hat{u}$ we have
\[ \mu_H(H + \hat{u} \cap \Omega) > 0 \] but for all \( \hat{u} \) we have \( \mu_H(H + \hat{u} \cap V(\bar{y}_1, \bar{y}_2)) = 0 \). First let’s write down the FOC; denote \( P = \sum_{i=0}^{n} p_i(\bar{y}_1, \bar{y}_2) \) and \( P^q = \sum_{i=0}^{n} p_i^q(\bar{y}_1, \bar{y}_2) \), from now on they are just constants:

\[
\begin{align*}
(1 - q)u'(P)D_1(P) + qu'(P^q)D_1(P^q) &= 0, \\
(1 - q)u'_2(1 - P)D_2(P) + qu'_2(1 - P^q)D_2(P^q) &= 0.
\end{align*}
\]

Take \( H(\alpha) = \{ u_1(\alpha) = u_2(\alpha) | u(x) = \alpha x^{k} + (1 - \alpha)x \} \), clearly, if \( \alpha \in (0, 1) \) \( H(\alpha) \subset \Omega \).

However, let’s substitute \( H(\alpha) + \hat{u} \) into FOC:

\[
\begin{align*}
(1 - q)(k\alpha P^{k-1} + 1 - \alpha + \hat{u}'(P))D_1(P) + \\
+q(k\alpha(P^q)^{k-1} + 1 - \alpha + \hat{u}'(P^q))D_1(P^q) &= 0, \\
(1 - q)(k\alpha(1 - P)^{k-1} + 1 - \alpha + \hat{u}'_2(1 - P))D_2(P) + \\
+q(k\alpha(1 - P^q)^{k-1} + 1 - \alpha + \hat{u}'_2(1 - P^q))D_2(P^q) &= 0.
\end{align*}
\]

Suppose that the equations above hold for two different \( \alpha \)'s. Denote by \( \beta \) difference between them: \( \beta = \alpha^1 - \alpha^2 \). By subtracting FOC for \( \alpha'' \) from FOC for \( \alpha' \) we obtain:

\[
\begin{align*}
(1 - q)(k\beta P^{k-1} - \beta)D_1(P) + \\
+q(k\beta(P^q)^{k-1} - \beta)D_1(P^q) &= 0, \\
(1 - q)(k\beta(1 - P)^{k-1} - \beta)D_2(P) + \\
+q(k\beta(1 - P^q)^{k-1} - \beta)D_2(P^q) &= 0.
\end{align*}
\]

Since \( \beta \neq 0 \) we have:

\[
\begin{align*}
(1 - q)(kP^{k-1} - 1)D_1(P) + \\
+q(k(P^q)^{k-1} - 1)D_1(P^q) &= 0, \\
(1 - q)(1 - kP^{k-1})D_2(P) + \\
+q(1 - k(P^q)^{k-1})D_2(P^q) &= 0.
\end{align*}
\]

If \( P = P^q \) we have \( (1 - q)(kP^{k-1} - 1)D_1(P) + q(k(P^q)^{k-1} - 1)D_1(P^q) \neq 0 \), so the equations above cannot hold. Note that in the equations above only constants remained (given the
equilibrium point), therefore they has to be satisfied for any $k$ — if not, we will take that very $k$ and obtain the result needed. But recall that $D_1(P)$ and $D_1(P)$ can’t be zero simultaneously due to regularity condition and $P \neq P^q$ for the same reason, therefore we can boil down\(^3\) the first group of equations to

\[ k(aP^{k-1} + (P^q)^{k-1}) = b \]

which clearly can’t hold for all $k$ — its left hand size isn’t constant under assumptions we made.

**Proof of Theorem 4.**

The proof is rather intuitive: just take the FOC denote $P = \sum_{i=0}^n p_i(\bar{y}_1, \bar{y}_2)$ and $P^q = \sum_{i=0}^n p^q_i(\bar{y}_1, \bar{y}_2)$

\[
\begin{cases}
  u'_1(\sum_{i=0}^n p_i(y_1, y_2))D_1(\sum_{i=0}^n p_i(y_1, y_2)) = c'_1(y_1, y_2), \\
  -u'_2(1 - \sum_{i=0}^n p_i(y_1, y_2))D_2(\sum_{i=0}^n p_i(y_1, y_2)) = c'_2(y_1, y_2).
\end{cases}
\]

From the expressions above it’s clear: all that matters is the marginal utility around the equilibrium. If it remains the same the local equilibrium stands, if it changes then the equilibrium can stand still only by coincidence.

**Proof of Theorem 2.**

We have $D_1(P_l) = -D_2(P_l)$ whenever $y_1 = y_2$. Since $D_1(P_N) = -\sum_{l=0}^{N-1} D_1(P_l)$, from (??) we have

\[ D_2(U_2) = D_1(P_N) + \sum_{l=0}^{N-1} (1 - u^2_{N-l})D_1(P_l) = 0. \tag{9} \]

This equation is equivalent to (??) when (??) holds.

Now suppose that the voters are equally biased at convergent positions, so $p_l(y_1, y_2) = p$. We have

\[ D_1(P_l) = \sum_{i=1}^N D_1(p_i) \times W_i = GW_i, \tag{10} \]

\(^3\)by taking linear combination of different equations
where
\[ W_i = p^{l-1}(1 - p)N-1 \left( (1 - p)C_i^{N-1} - pC_i^{N-1} \right). \] (11)

As \( D_1(p_i) = -D_2(p_i) \), the first order conditions can be rewritten as
\[ D_1(U_1) = G \sum_{l=0}^{N} u_i^l W_l = 0 \] (12)
\[ D_2(U_2) = G \sum_{l=0}^{N} u_i^l W_{N-l} = 0. \] (13)

Thus we have the system \( G = 0 \) of \( k \) equations in \( k \) unknowns.

It remains to be shown that (11) is satisfied somewhere. Fix \( z \in \mathbb{R}^k \). Let there be \( \alpha > 0 \) such that for all \( r \in \mathbb{R}^2, |r| = 1 \), we have \( r \cdot D_1(p_i) < 0 \) for all \( y_1 = y_2 = z + \alpha r \), for all \( l \). One must show that we have \( r \cdot D_1(U_1) < 0 \) and \( r \cdot D_2(U_2) < 0 \) for all \( y_1 = y_2 = z + \alpha r \).

We have
\[ r \cdot D_1(U_1) = r \cdot \sum_{l=0}^{N} u_i^l D_1(P_l) = \sum_{i=1}^{N} r \cdot D_1(p_i)V_i^1, \] (14)
where
\[ V_i^1 = \sum_{l=1}^{N} u_i^l \left\{ \sum_{S \ni i, |S| = l} \left[ \prod_{k \in S} p_k \prod_{k \notin S} (1 - p_k) \right] - \sum_{S \ni i, |S| = l} \left[ \prod_{k \notin S, k \neq i} (1 - p_k) \prod_{k \in S} p_k \right] \right\} \] (15)

Fix \( i = 1 \). We have
\[ V_1^1 = \sum_{j=1}^{N-1} \left\{ (u_{N+1-j}^1 - u_{N-j}^1) \sum_{S \subset N-\{1\}, |S| = N-j} \left( \prod_{k \in S} p_k \prod_{k \notin S} (1 - p_k) \right) \right\}. \] (16)

As \( u_i^l \geq u_{i-1}^l \), with strict inequality a least for one \( l \), we have \( V_i^1 > 0 \) for all \( i = 1, \ldots, N \). Hence, \( r \cdot D_1(U_1) < 0 \). Denote by \( B_{z\alpha} \) the ball of radius \( \alpha \) centered on \( z \). Let \( F_\beta(y) = y + \beta D_1(y, y) \) for \( \beta \geq 0 \). As \( D_1(p_i(y, y)) \) are bounded, so is \( D_1(U_1(y, y)) \). Because \( r \cdot D_1(U_1(y, y)) < 0 \) for all \( y = z + \alpha r \), there exists \( \beta > 0 \) such that \( F_\beta(B_{z\alpha}) \subset B_{z\alpha} \). From Brouwer fixed-point theorem there exists \( y \in B_{z\alpha} \) such that \( F(y) = y \), or \( D_1(U_1(y, y)) = 0 \).
References


Chris W. Bonneau and Damon M. Cann. 2014. Institutions, War Chests, and Candidate Deterrence. Unpublished manuscript.


