Title: Maps of several variables of finite total variation. II. E. Helly-type pointwise selection principles

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Abstract: Given $a=(a_1, \ldots, a_n), b=(b_1, \ldots, b_n) \in \mathbb{R}^n$ with $a < b$ componentwise and a map $f$ from the rectangle $I_a^b = [a_1, b_1] \times \cdots \times [a_n, b_n]$ into a metric semigroup $M=(M,d,+)$, denote by $\text{TV}(f, I_a^b)$ the Hildebrandt-Leonov total variation of $f$ on $I_a^b$, which has been recently studied in [V.V., V.~Chistyakov, Yu.V.~Tretyachenko, Maps of several variables of finite total variation, J.~Math.~Anal.~Appl., this issue (2010), submitted].

The following Helly-type pointwise selection principle is proved:
If a sequence $\{f_j\}_{j \in \mathbb{N}}$ of maps from $I_a^b$ into $M$ is such that the closure in $M$ of the set $\{f_j(x)\}_{j \in \mathbb{N}}$ is compact for each $x \in I_a^b$ and $C \equiv \sup_{j \in \mathbb{N}} \text{TV}(f_j, I_a^b)$ is finite, then there exists a subsequence of $\{f_j\}_{j \in \mathbb{N}}$ which converges pointwise on $I_a^b$ to a map $f$ such that $\text{TV}(f, I_a^b) \leq C$.

A variant of this result is established concerning the weak pointwise convergence when values of maps lie in a reflexive Banach space $(M, \|\cdot\|)$ with separable dual $M^*$. 

\[ \text{TV}(f, I_a^b) \leq C \]
Dear Editors of Journal of Mathematical Analysis and Applications,

Attached please find the second part of the paper by

V. V. Chistyakov and Yu. V. Tretyachenko
"Maps of several variables of finite total variation. II.
E. Helly-type pointwise selection principles"

submitted for publication in your journal
(this paper is a continuation of part I).

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This paper may be appropriate for Associate Editor
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Sincerely,
V. V. Chistyakov
Maps of several variables of finite total variation. II.
E. Helly-type pointwise selection principles✩

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Abstract

Given \(a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n) \in \mathbb{R}^n\) with \(a < b\) componentwise and a map \(f\) from the rectangle \(I^b_a = [a_1, b_1] \times \cdots \times [a_n, b_n]\) into a metric semigroup \(M = (M, d, +)\), denote by \(TV(f, I^b_a)\) the Hildebrandt-Leonov total variation of \(f\) on \(I^b_a\), which has been recently studied in [V. V. Chistyakov, Yu. V. Tretyachenko, Maps of several variables of finite total variation. I, J. Math. Anal. Appl., this issue (2010), submitted]. The following Helly-type pointwise selection principle is proved: If a sequence \(\{f_j\}_{j \in \mathbb{N}}\) of maps from \(I^b_a\) into \(M\) is such that the closure in \(M\) of the set \(\{f_j(x)\}_{j \in \mathbb{N}}\) is compact for each \(x \in I^b_a\) and \(C \equiv \sup_{j \in \mathbb{N}} TV(f_j, I^b_a)\) is finite, then there exists a subsequence of \(\{f_j\}_{j \in \mathbb{N}}\), which converges pointwise on \(I^b_a\) to a map \(f\) such that \(TV(f, I^b_a) \leq C\). A variant of this result is established concerning the weak pointwise convergence when values of maps lie in a reflexive Banach space \((M, \| \cdot \|)\) with separable dual \(M^*\).

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1. Introduction to part II

This paper is a continuation of [14]. Its purpose is to establish two variants of a Helly-type pointwise selection principle for maps of several real variables taking their values in a metric semigroup and a reflexive separable Banach space. Part of the results of this paper were announced in [7] without proofs.

The classical Helly selection principle ([16]) states that a bounded sequence of real valued functions on the closed interval, which is of uniformly bounded (Jordan) variation, contains a pointwise convergent subsequence whose limit is a function of bounded variation. This theorem and its recent generalizations for real valued functions and metric space valued maps of one real variable ([4, 6, 8, 10, 11, 12, 13]) have numerous applications in different branches of Analysis (e.g., [3, 8, 15, 17, 20] and references therein).

As it was already mentioned in the Introduction in [14], extensions of the Helly theorem to functions and maps of several real variables depend upon notions of (bounded) variation used for these maps, which generalize different aspects of the classical Jordan variation of univariate functions. Up till now it is known ([17, 19]) that only the approach due to Vitali-Hardy-Krause gives the notion of variation for real valued functions of several variables such that a complete analogue of the Helly theorem holds with respect to the pointwise convergence of extracted subsequences. This counterpart of Helly’s theorem is based on the notion of a (totally) monotone real valued function of several variables [5, 17, 21] and an appropriate generalization of Jordan’s decomposition theorem when a function of bounded variation is represented as the difference of two monotone functions.

The main difficulty that we overcome in this paper is that for metric semigroup valued maps there is no counterpart of Jordan’s decomposition theorem, and we have to develop a completely different technique (the Helly-type selection principle for functions of two real variables given in [2] plays the role of the induction basis in the proof of our first main result).

The paper is organized as follows. In Section 2 we formulate our two main results, Theorems 1 and 2. In Sections 3 and 4 we collect all main ingredients and auxiliary known facts needed for their proofs. Finally, in Section 5 we prove Theorems 1 and 2.

2. Main results

Throughout this paper we adopt definitions and make use of notations from [14]. In particular, the basic rectangle $I^b_a$ with $a, b \in \mathbb{R}^n, a < b$, and
a metric semigroup \((M, d, +)\) are fixed throughout the paper. Recall that \(BV(I_a^b; M)\) designates the space of all maps \(f : I_a^b \to M\) of finite (or bounded) total variation ([14, equality (2.3)]). Also, recall that a sequence \(\{f_j\} \equiv \{f_j\}_{j \in \mathbb{N}}\) of maps from \(I_a^b\) into \(M\) is said: (a) to converge pointwise on \(I_a^b\) to a map \(f : I_a^b \to M\) if \(d(f_j(x), f(x)) \to 0\) as \(j \to \infty\) for all \(x \in I_a^b\); (b) to be pointwise precompact (on \(I_a^b\)) provided the closure in \(M\) of the set \(\{f_j(x)\}_{j \in \mathbb{N}}\) is compact for all \(x \in I_a^b\).

Our first main result, to be proved in Section 5, is the following Helly-type pointwise selection principle in the space \(BV(I_a^b; M)\), which looks quite classically (which is not at all the case with its proof):

**Theorem 1.** A pointwise precompact sequence \(\{f_j\}\) of maps from the rectangle \(I_a^b\) into a metric semigroup \((M, d, +)\) such that

\[
C \equiv \sup_{j \in \mathbb{N}} TV(f_j, I_a^b) \text{ is finite} \quad (2.1)
\]

contains a subsequence which converges pointwise on \(I_a^b\) to a map \(f \in BV(I_a^b; M)\) such that \(TV(f, I_a^b) \leq C\).

This result contains as particular cases the results of [17, III.6.5] and [18] \((n = 2\) and \(M = \mathbb{R}\)), [19] \((n \in \mathbb{N}\) and \(M = \mathbb{R}\)) and [2] \((n = 2\) and \(M\) is a metric semigroup).

Our second main result (Theorem 2 below) is concerned with a weak analogue of Theorem 1 taking into account certain specific features when the values of maps under consideration lie in a reflexive separable Banach space.

Let \((M, \| \cdot \|)\) be a normed linear space over the field \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\) and \(M^*\) be its dual, i.e., \(M^* = L(M; \mathbb{K})\), the space of all continuous linear functionals on \(M\). It is well-known that \(M^*\) is a Banach space under the norm \(\|u^*\| = \sup\{\|u^*(u)\| : u \in M \text{ and } \|u\| \leq 1\}, u^* \in M^*\). The natural duality between \(M\) and \(M^*\) is determined by the bilinear functional \(\langle \cdot, \cdot \rangle : M \times M^* \to \mathbb{K} \) defined by \(\langle u, u^* \rangle = u^*(u)\) for all \(u \in M\) and \(u^* \in M^*\), so that \(|\langle u, u^* \rangle| \leq \|u\| \cdot \|u^*\|\), where \(\| \cdot \|\) is the absolute value in \(\mathbb{K}\). Recall that a sequence \(\{u_j\} \subset M\) converges weakly in \(M\) to an element \(u \in M\) (in symbols, \(u_j \overset{w}{\to} u\) in \(M\)) if \(\langle u_j, u^* \rangle \to \langle u, u^* \rangle\) in \(\mathbb{K}\) as \(j \to \infty\) for all \(u^* \in M^*\); if this is the case then it is known that \(\|u\| \leq \lim\inf_{j \to \infty} \|u_j\|\).

Since a normed linear space \((M, \| \cdot \|)\) is a metric semigroup, the notions of the Vitali-type \(n\)-th variation, \(|\alpha|\)-th variation for \(0 \neq \alpha \leq 1\) and the total variation of a map \(f : I_a^b \to M\) are introduced as in [14] with respect to the induced metric \(d(u, v) = \|u - v\|, u, v \in M\).
Theorem 2. Suppose \((M, \| \cdot \|)\) is a reflexive separable Banach space with separable dual \(M^*\) and \(\{f_j\}\) is a sequence of maps from \(I_a^b\) into \(M\). If \(\{f_j\}\) satisfies condition (2.1) from Theorem 1 and

\[
c(x) \equiv \sup_{j \in \mathbb{N}} \|f_j(x)\| \text{ is finite for all } x \in I_a^b, \tag{2.2}
\]

then there exists a subsequence of \(\{f_j\}\), again denoted by \(\{f_j\}\), and a map \(f \in \text{BV}(I_a^b; M)\) satisfying \(\text{TV}(f, I_a^b) \leq C\) such that

\[
f_j(x) \rightharpoonup f(x) \quad \text{in } M \text{ for all } x \in I_a^b. \tag{2.3}
\]

This theorem will be proved in Section 5. It is an extension of a weak selection principle from [3, Chapter 1, Theorem 3.5] for maps of bounded Jordan variation of one real variable. More comments and remarks on Theorems 1 and 2 can be found in Section 5.

3. Ingredients of the proofs of the main results

In this section we collect main ingredients of the proof of Theorem 1. The first three of them are already established in [14] as Theorems 1, 2 and 3.

Now we recall several definitions and results for real valued functions. A function \(g : I_a^b \to \mathbb{R}\) is said to be totally monotone (cf., e.g., [9, Part II, Section 3], [19]) if, given \(0 \neq \alpha \leq 1\) and \(x, y \in I_a^b\) with \(x \leq y\), we have:

\[
(−1)^{|α|} \sum_{0 \leq θ \leq α} (−1)^{|θ|} g(x + θ(y − x)) \geq 0. \tag{3.1}
\]

For real valued functions the sum in (3.1) (with no factor \((-1)^{[α]}\)) is called the \([α]-th\) mixed difference (in the sense of Vitali, Hardy and Krause) of \(g_α^x\) on the rectangle \(I_a^b[α]\) and denoted by \(\text{md}_1[α](g_α^x, I_a^b[α])\) (however, note the difference with [14, equality (2.6)] in the general case). In this case the Vitali \(n\)-th variation \(V_n(g, I_a^b)\) of \(g\) on \(I_a^b\) is defined as in [14, equality (2.2)] with the mixed difference at the right-hand side replaced by \(\text{md}_n(g, I_a^b[x[σ−1]])\). The other definitions related to the bounded variation context remain the same as in [14], and so, we keep the same notation for real valued functions as well.

Denote by \(\text{Mon}(I_a^b; \mathbb{R})\) the set of all totally monotone real valued functions on \(I_a^b\). It is known (e.g., [9, 19]) that if \(g \in \text{Mon}(I_a^b; \mathbb{R})\), then \(g \in \text{BV}(I_a^b; \mathbb{R})\), the value at the left-hand side of (3.1) is equal to \(V[α](g_α^x, I_a^b[α]), g(x) \leq g(y)\) and \(\text{TV}(g, I_a^b) = g(y) − g(x)\) for all \(x, y \in I_a^b\) with \(x \leq y\).
The following Helly-type selection principle in the class \( \text{Mon}(I^b_a; \mathbb{R}) \) is due to Leonov [19, Lemma 3] (for totally monotone functions of two variables it was established in [17, III.6.5] and [18, Theorem 3.1]):

**Theorem A.** An infinite uniformly bounded family of totally monotone functions on \( I^b_a \) contains a sequence, which converges pointwise on \( I^b_a \) to a function from \( \text{Mon}(I^b_a; \mathbb{R}) \).

It was shown in [19, Corollary 2] that the linear space \( \text{BV}(I^b_a; \mathbb{R}) \) equipped with the norm \( \| g \| = |g(a)| + \text{TV}(g, I^b_a) \), \( g \in \text{BV}(I^b_a; \mathbb{R}) \), is a Banach space. This assertion was refined in [9, Part I, Theorem 1]: the space \( \text{BV}(I^b_a; \mathbb{R}) \) is a Banach algebra with respect to the norm \( \| \cdot \| \), and \( \| g \cdot g' \| \leq 2^a \| g \| \cdot \| g' \| \) for all \( g, g' \in \text{BV}(I^b_a; \mathbb{R}) \).

Theorem A implies a Helly-type selection principle in the space \( \text{BV}(I^b_a; \mathbb{R}) \) [19, Theorem 4]: an infinite family of functions from \( \text{BV}(I^b_a; \mathbb{R}) \), which is bounded under the norm \( \| \cdot \| \), contains a pointwise convergent sequence, whose (pointwise) limit belongs to the space \( \text{BV}(I^b_a; \mathbb{R}) \). The crucial observation in the proof of this result is that, given \( g \in \text{BV}(I^b_a; \mathbb{R}) \), if we set \( \nu_g(x) = \text{TV}(g, I^b_a) \) and \( \pi_g(x) = \nu_g(x) - g(x), x \in I^b_a \), then ([19, Theorem 3]) the functions \( \nu_g \) and \( \pi_g \) belong to \( \text{Mon}(I^b_a; \mathbb{R}) \), and Jordan’s decomposition holds: \( g = \nu_g - \pi_g \) on \( I^b_a \); then Theorem A applies to the uniformly bounded families of functions \( \{ \nu_g \} \) and \( \{ \pi_g \} \) in the standard way.

Now let us consider the case of maps \( f : I^b_a \to M \) of finite total variation valued in a metric semigroup \( (M, d, +) \). Clearly, there is no counterpart of Jordan’s decomposition for these maps, and so, in order to prove Theorem 1, we ought to argue in a completely different way. It will be seen later that, along with the already mentioned ingredients, the following two theorems will be of significance in the proof of Theorem 1 (in a certain sense all six ingredients alluded to above replace the “real valued” arguments involving Jordan’s decomposition).

**Theorem B.** If \( f \in \text{BV}(I^b_a; M) \), \( x, y \in I^b_a \), \( x \leq y \), and \( 0 \neq \gamma \leq 1 \), then

\[
\sum_{0 \neq \alpha \leq \gamma} V_{[\alpha]}(f^x_y, I^b_a) = \text{TV}(f, I_x^{x+y}(y-x)) 
\leq \text{TV}(f, I_x^{x+y}(y-x)) - \text{TV}(f, I^b_a). \tag{3.2}
\]

**Theorem C.** If \( f \in \text{BV}(I^b_a; M) \) and if we set \( \nu_f(x) = \text{TV}(f, I^b_a), x \in I^b_a \), then for \( \nu_f : I^b_a \to \mathbb{R} \), called the total variation function of \( f \), we have: \( \nu_f \in \text{Mon}(I^b_a; \mathbb{R}) \) and \( \text{TV}(\nu_f, I^b_a) = \text{TV}(f, I^b_a) \).
These two theorems are extensions to maps of several variables of two more properties of the Jordan variation for maps of one variable; in this case (3.2) is, actually, the equality known as the *additivity* of Jordan’s variation (cf. property (a) in the Introduction in [14]). On the other hand, Theorem B is a counterpart of Chistyakov’s inequality [9, Part II, Lemma 8] and Theorem C is a generalization of Theorem 3 from [19] and Corollary 11 from [9, Part II] given for $M = \mathbb{R}$. For metric semigroup valued maps of two variables cf. [2, inequalities (11), (13) and Theorem 1].

The proof of Theorem B is identical with the proof of Lemma 8 from [9, Part II] and the proof of Theorem C is identical with the proofs of Lemma 9 and Corollaries 10 and 11 from [9, Part II] when $M = \mathbb{R}$, and so, they are omitted. However, it is to be noted that these proofs rely on: (i) the property of *additivity* of $|\alpha|$-th variation $V_{|\alpha|}$ for each $0 \neq \alpha \leq 1$ (re-established in [14] as Theorem 1); (ii) equality (3.2) from [9, Part I, Lemma 5] (re-established in [14] as Lemma 1); and (iii) Lemma 7 from [9, Part I]. The last item (iii) is not verified yet, and it is our intention now to formulate its counterpart for metric semigroup valued maps as Theorem 3 below.

Since the total variation (2.3) in [14] is defined via truncated maps with the base at the point $a$, the next theorem addresses a counterpart of Chistyakov’s equality [9, Part I, Lemma 7] exhibiting the relationship between the mixed difference $\text{md}_{|\alpha|}(f^a_x, I^y_x | \alpha)$ and certain mixed differences of maps $f^a_\beta$ with the base at $a$ for some $0 \neq \beta \leq 1$.

**Theorem 3.** If $f : I^b_a \to M$, $0 \neq \alpha \leq 1$ and $x, y \in I^b_a$ with $x \leq y$, then

$$\text{md}_{|\alpha|}(f^x_a, I^y_x | \alpha) \leq \sum_{0 \leq \beta \leq 1} \text{md}_{|\beta|}(f^a_\beta, f^{x+\alpha(y-x)}_{a+\alpha(x-a)} | \beta).$$

The proof of Theorem 3 will be given in the next section.

4. **Proof of Theorem 3**

In order to prove Theorem 3, we need an auxiliary Lemma 1, which plays the same role as Lemma 8 in [14].
Lemma 1. Given a map \( h : \mathcal{A}_0 \to M \) and a multiindex \( \alpha \in \mathcal{A}_0 \), we have:

1. if \( 1 - \alpha \) is even, then the following two equalities hold:
   \[
   \sum_{\text{ev } \theta \leq \alpha} h(1 - \alpha + \theta) + \sum_{\text{od } \beta \leq 1 - \alpha} \sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta) = \sum_{\text{ev } \beta \leq 1 - \alpha} \sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta), \quad (4.1)
   \]
   \[
   \sum_{\text{od } \theta \leq \alpha} h(1 - \alpha + \theta) + \sum_{\text{od } \beta \leq 1 - \alpha} \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta) = \sum_{\text{ev } \beta \leq 1 - \alpha} \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta), \quad (4.2)
   \]

and if \( 1 - \alpha \) is odd, then the following two equalities hold:

\[
\sum_{1 - \alpha \leq \text{ev } \theta \leq 1} h(\theta) + \sum_{\text{ev } \beta \leq 1 - \alpha} \sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta) = \sum_{\text{od } \beta \leq 1 - \alpha} \sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta), \quad (4.3)
\]
\[
\sum_{1 - \alpha \leq \text{od } \theta \leq 1} h(\theta) + \sum_{\text{ev } \beta \leq 1 - \alpha} \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta) = \sum_{\text{od } \beta \leq 1 - \alpha} \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta). \quad (4.4)
\]

PROOF. It suffices to establish equality (4.8) from the proof of Lemma 8 in [14] (cf. also Step 0 in that proof). We divide the proof into four steps.

Suppose that \( 1 - \alpha \) is even.

1. Let us prove (4.1). If \( \alpha = 1 \), then \( 1 - \alpha = 0 \) is even, and equality (4.1) is equivalent to the identity \( \sum_{\text{ev } \theta \leq 1} h(\theta) = \sum_{\text{ev } \theta \leq 1} h(\theta) \). If \( \alpha = 0 \) and if \( 1 - \alpha = 1 \) is even, then (4.1) can be written as

\[
h(1) + \sum_{\text{od } \beta \leq 1} \sum_{\text{ev } \theta \leq \beta} h(\theta) = \sum_{\text{ev } \beta \leq 1} \sum_{\text{ev } \theta \leq \beta} h(\theta),
\]

which was established in [14, equality (4.6)] for even \( \gamma = 1 \). Thus, in what follows we assume that \( \alpha \neq 0, 1 \), i.e., \( 0 < |\alpha| < n \).

We have \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \), where \( \mathcal{L}_1 = \{ 1 - \alpha + \theta' : \exists \text{ ev } \theta' \leq \alpha \} \) (so that \( 1 - \alpha \in \mathcal{L}_1 \)) and \( \mathcal{L}_2 = \{ \text{ ev } : \exists \text{ odd } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \) (so that \( 0 \in \mathcal{L}_2 \)), and \( \mathcal{R} = \{ \text{ ev } : \exists \text{ odd } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \), i.e., \( \mathcal{R} = \{ \text{ ev } : \theta \leq 1 \} \). We are going to show that \( \mathcal{L} = \mathcal{R} \). This equality follows immediately from the definition of \( \mathcal{R} \) and the following two assertions:

\[
\theta \in \mathcal{L}_1 \iff \theta \text{ is even and } \alpha \lor \theta = 1, \quad (4.5)
\]
\[
\theta \in \mathcal{L}_2 \iff \theta \text{ is even and } \alpha \lor \theta \neq 1, \quad (4.6)
\]

where \( \alpha \lor \theta = \max \{ \alpha, \theta \} = \alpha + \theta - \alpha \theta \); in particular, (4.5) and (4.6) imply that \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are disjoint. Let us prove (4.5). If \( \theta \in \mathcal{L}_1 \), then \( \theta = 1 - \alpha + \theta' \).
for some even $\theta' \leq \alpha$ and, since $1 - \alpha$ is even and $|\theta| = |1 - \alpha| + |\theta'|$, then $\theta$ is even, $\theta \leq (1 - \alpha) + \alpha = 1$ and

$$\alpha \vee \theta = \alpha + (1 - \alpha + \theta') - \alpha(1 - \alpha + \theta') = 1 + \theta' - \alpha \theta' = 1.$$  

Conversely, if $\theta$ is even and $\alpha \vee \theta = 1$, then $\alpha \vee \theta = 1 + \alpha \theta \geq 1$. Setting $\theta' = \alpha + \theta - 1$, we find $\theta = 1 - \alpha + \theta'$, where $|\theta'| = |\alpha| + |\theta| - n = |1 - \alpha| - 1$ is even and $\theta' \leq \alpha$, and so, $\theta \in \mathcal{L}_1$. Now we establish (4.6). If $\theta \in \mathcal{L}_2$, then $\theta$ is even and there exists odd $\beta \leq 1 - \alpha$ s.t. $\theta \leq \alpha + \beta$, and so, $\alpha \leq \alpha + \beta$ and $\theta \leq \alpha + \beta$ imply $\alpha \vee \theta \leq \alpha + \beta$. Since $\beta$ is odd, $1 - \alpha$ is even and $\beta \leq 1 - \alpha$, we have $|\beta| < |1 - \alpha| = n - |\alpha|$. It follows that

$$|\alpha \vee \theta| \leq |\alpha + \beta| = |\alpha| + |\beta| < |\alpha| + (n - |\alpha|) = n,$$

and so, $\alpha \vee \theta \neq 1$. Conversely, if $\theta$ is even and $\alpha \vee \theta \neq 1$, then there exists $i \in \{1, \ldots, n\}$ s.t. $\alpha_i = 0$ and $\theta_i = 0$. Setting $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_i = 0$ and $\beta_j = 1 - \alpha_j$ if $j \neq i$, we find $\beta \leq 1 - \alpha$, $|\beta| = |1 - \alpha| - 1$ is odd and $\theta \leq \alpha + \beta$, and so, $\theta \in \mathcal{L}_2$.

In order to calculate the values $L(\theta)$ and $R(\theta)$ for $\theta \in \mathcal{L} = \mathcal{R}$, we note that, given $0 \leq \beta \leq 1 - \alpha$, we have:

$$\theta \leq \alpha + \beta \quad \text{is equivalent to} \quad (1 - \alpha)\theta \leq \beta. \quad (4.7)$$

In fact, condition $0 \leq \beta \leq 1 - \alpha$ is equivalent to condition $\alpha \beta = 0$:

$$0 \leq \beta \leq 1 - \alpha \iff \beta(1 - \alpha) = \beta \iff \beta - \alpha \beta = \beta \iff \alpha \beta = 0,$$

and so, if $\theta \leq \alpha + \beta$, then $(1 - \alpha)\theta \leq (1 - \alpha)(\alpha + \beta) = (1 - \alpha)\alpha + \beta - \alpha \beta = \beta$, and if $(1 - \alpha)\theta \leq \beta$, then $\theta - \alpha \theta \leq \beta$, and so, $\theta \leq \alpha \theta + \beta \leq \alpha + \beta$.

Given $\theta \in \mathcal{R}$, by virtue of (4.7), we find

$$R(\theta) = |\{\text{even } \beta : \beta \leq 1 - \alpha \text{ and } \theta \leq \alpha + \beta\}| = |\{\text{even } \beta : (1 - \alpha)\theta \leq \beta \leq 1 - \alpha\}|.$$

If $\theta \in \mathcal{L}_1$, then there exists a unique even $\theta' \leq \alpha$ s.t. $\theta = 1 - \alpha + \theta'$, and so, since $\theta \notin \mathcal{L}_2$, then $L(\theta) = 1$. At the same time,

$$(1 - \alpha)\theta = (1 - \alpha)(1 - \alpha + \theta') = (1 - \alpha)^2 + (1 - \alpha)\theta' = 1 - \alpha,$$

and so, by the above, $R(\theta) = 1$ as well. Suppose now that $\theta \in \mathcal{L}_2$. Then, by (4.6), $1 \neq \alpha \vee \theta = \alpha + \theta - \alpha \theta = \alpha + (1 - \alpha)\theta$ or $(1 - \alpha)\theta \neq 1 - \alpha$, and so, taking into account (4.7) and Lemma 2(b) from [14] we find that

$$L(\theta) = |\{\text{odd } \beta : \beta \leq 1 - \alpha \text{ and } \theta \leq \alpha + \beta\}| = |\{\text{odd } \beta : (1 - \alpha)\theta \leq \beta \leq 1 - \alpha\}|$$

is equal to $R(\theta)$. 

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In the rest of the proof we exhibit only the essential ingredients and differences.

2. Let us establish (4.2). If \( \alpha = 1 \), we get an identity, and if \( \alpha = 0 \) and \( 1 = 1 - \alpha \) is even, we get equality (4.7) from [14] with even \( \gamma = 1 \), and so, we suppose that \( 0 < |\alpha| < n \). We have \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \), where \( \mathcal{L}_1 = \{1 - \alpha + \theta' : \exists \text{odd } \theta' \leq \alpha \} \) and \( \mathcal{L}_2 = \{\text{odd } \theta : \exists \text{odd } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \), and \( \mathcal{R} = \{\text{odd } \theta : \exists \text{even } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \), which, actually, is \( \mathcal{R} = \{\text{odd } \theta : \theta \leq 1 \} \). We need to verify only that \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are nonempty: the rest of the proof of (4.2) (including (4.5) and (4.6)) is the same as in Step 1 where ‘even \( \theta' \)’ is replaced by ‘odd \( \theta' \).

Since \( \alpha \neq 0 \), there exists \( i \in \{1, \ldots, n\} \) s.t. \( \alpha_i = 1 \), and so, if we set \( \theta' = (\theta'_1, \ldots, \theta'_n) \) with \( \theta'_i = 1 \) and \( \theta'_j = 0 \) if \( j \neq i \), then \( |\theta'| = 1 \) is odd and \( \theta' \leq \alpha \). It follows that \( 1 - \alpha + \theta' \in \mathcal{L}_1 \).

Since \( \alpha \neq 1 \), there exists \( i \in \{1, \ldots, n\} \) s.t. \( \alpha_i = 0 \), and so if we set \( \beta = (\beta_1, \ldots, \beta_n) \) with \( \beta_i = 0 \) and \( \beta_j = 1 - \alpha_j \) if \( j \neq i \), then \( |\beta| = |1 - \alpha| - 1 \) is odd and \( \beta \leq 1 - \alpha \). Given \( k \in \{1, \ldots, n\}, k \neq i \), setting \( \theta = (\theta_1, \ldots, \theta_n) \) with \( \theta_k = 1 \) and \( \theta_j = 0 \) if \( j \neq k \), we find \( |\theta| = 1 \) is odd and \( \theta \leq \alpha + \beta \), and so, \( \theta \in \mathcal{L}_2 \).

Assume now that \( 1 - \alpha \) is odd. Note that \( \alpha \neq 1 \).

3. Let us prove (4.3). If \( \alpha = 0 \) and \( 1 = 1 - \alpha \) is odd, then (since ev \( \theta = 1 \) cannot hold in the first sum at the left of (4.3)) equality (4.3) is equivalent to [14, equality (4.6)] with odd \( \gamma = 1 \). Thus, we assume that \( |\alpha| > 0 \).

We have \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \), where \( \mathcal{L}_1 = \{\text{even } \theta : 1 - \alpha \leq \theta \leq 1 \} \) and \( \mathcal{L}_2 = \{\text{even } \theta : \exists \text{even } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \), and \( \mathcal{R} = \{\text{even } \theta : \exists \text{odd } \beta \leq 1 - \alpha \text{ s.t. } \theta \leq \alpha + \beta \} \), and so, \( \mathcal{R} = \{\text{even } \theta : \theta \leq 1 \} \). We have to show that \( \mathcal{L} = \mathcal{R} \).

First, we show that \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are nonempty. Since \( \alpha \neq 0, \alpha_i = 1 \) for some \( i \in \{1, \ldots, n\} \), and so, setting \( \theta = (\theta_1, \ldots, \theta_n) \) with \( \theta_i = 1 \) and \( \theta_j = 1 - \alpha_j \) if \( j \neq i \), we find that \( 1 - \alpha \leq \theta \leq 1 \) and \( |\theta| = |1 - \alpha| + 1 \) is even, whence \( \theta \in \mathcal{L}_1 \). Now, since \( \alpha \neq 1 \), \( \alpha_i = 0 \) for some \( i \in \{1, \ldots, n\} \), and if we set \( \beta = (\beta_1, \ldots, \beta_n) \) with \( \beta_i = 0 \) and \( \beta_j = 1 - \alpha_j \) if \( j \neq i \), then we find that \( |\beta| = |1 - \alpha| - 1 \) is even, \( \theta = 0 \) is even and \( 0 \leq \alpha + \beta \), and so, \( 0 \in \mathcal{L}_2 \).

Second, we assert that (4.5) and (4.6) hold; this will imply that \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are disjoint and \( \mathcal{L} = \mathcal{R} \). In order to prove (4.5), we let \( \theta \in \mathcal{L}_1 \). Then \( \theta \) is even and \( 1 - \alpha \leq \theta \leq 1 \), and so,

\[
\alpha \lor \theta = \alpha + \theta - \alpha \theta = \alpha + (1 - \alpha) \theta = \alpha + (1 - \alpha) = 1.
\]
Conversely, if $\theta$ is even and $\alpha \vee \theta = 1$, then $\alpha + \theta - \alpha \theta = 1$, and so, $(1 - \alpha)\theta = 1 - \alpha$ implying $1 - \alpha \leq \theta$ and $\theta \in \mathcal{L}_1$. The proof of (4.6) follows the same lines as in Step 1 if 'odd $\beta$' is replaced by 'even $\beta$'.

Given $\theta \in \mathcal{R}$, taking into account (4.7), we have $R(\theta) = \{\{\text{odd $\beta$}: (1 - \alpha)\theta \leq \beta \leq 1 - \alpha\}\}$. If $\theta \in \mathcal{L}_1$, then $\theta \notin \mathcal{L}_2$, and so, $L(\theta) = 1$; in this case $1 - \alpha \leq \theta$, and so, $(1 - \alpha)\theta = 1 - \alpha$ and $R(\theta) = 1$. Now if $\theta \in \mathcal{L}_2$, then $\alpha \vee \theta \neq 1$, and so, $(1 - \alpha)\theta \neq 1 - \alpha$ and, by virtue of [14, Lemma 2(b)], the value $L(\theta) = \{\{\text{even $\beta$}: (1 - \alpha)\theta \leq \beta \leq 1 - \alpha\}\}$ is equal to $R(\theta)$.

Finally, we establish (4.4). If $\alpha = 0$ and $1 = 1 - \alpha$ is odd, we get equality (4.7) from [14] with odd $\gamma = 1$. Assume that $|\alpha| > 0$. We have $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1 = \{\text{odd $\theta$}: 1 - \alpha \leq \theta \leq 1\}$ (and so, $1 - \alpha \in \mathcal{L}_1$) and $\mathcal{L}_2 = \{\text{odd $\theta$}: \exists \text{even $\beta$} \leq 1 - \alpha$ s.t. $\theta \leq \alpha + \beta\}$, and $\mathcal{R} = \{\text{odd $\theta$}: \exists \text{odd $\beta$} \leq 1 - \alpha$ s.t. $\theta \leq \alpha + \beta\}$, and so, $\mathcal{R} = \{\text{odd $\theta$}: \theta \leq 1\}$. That $\mathcal{L}_2$ is nonempty can be seen as follows. Since $\alpha \neq 1$, $\alpha_i = 0$ for some $i \in \{1, \ldots, n\}$, and so, if we set $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_i = 0$ and $\beta_j = 1 - \alpha_j$ if $j \neq i$, then $\beta \leq 1 - \alpha$ and $|\beta| = |1 - \alpha| - 1$ is even. Now, since $\alpha \neq 0$, $\alpha_k = 1$ for some $k \neq i$. If we set $\theta = (\theta_1, \ldots, \theta_n)$ with $\theta_k = 1$ and $\theta_j = 0$ if $j \neq k$, then $|\theta| = 1$ is odd and $\theta \leq \alpha + \beta$, and so, $\theta \in \mathcal{L}_2$. Assertion (4.5) with 'even $\theta$ is odd' replaced by 'odd $\theta$ is odd' is established as in Step 3, while the proof of (4.6) follows the same lines as in Step 1 with 'odd $\beta$' replaced by 'even $\beta$'. It follows that $\mathcal{L} = \mathcal{R}$. The proof completes with the last paragraph of Step 3. 

**Proof of Theorem 3.** The inequality (actually, equality) is clear if $\alpha = 1$, and so, we assume that $\alpha \neq 1$. The mixed difference at the left-hand side of the inequality is given by (2.6) from [14, Lemma 1], while given $\alpha \leq \beta \leq 1$, noting that $\alpha \beta = \alpha$ and applying equality (2.5) from [14, Lemma 1] we get the following expression for the mixed difference at the right-hand side (cf. [9, Part I, expression (3.7)]):

$$md|_{|\beta|}(f^a, I_{a + \alpha(x-a)}^{x+a(y-x)}) = d\left(\sum_{\text{ev } \theta \leq \beta} h(\theta), \sum_{\text{od } \theta \leq \beta} h(\theta)\right),$$

where $h(\theta) = f(a + (\alpha \vee \theta)(x-a) + \alpha\theta(y-x))$ and $\alpha \vee \theta = \alpha + \theta - \alpha \theta$. Changing the summation multiindex $\beta \mapsto \beta - \alpha$ in the sum at the right of the inequality in Theorem 3, we find that it is equivalent to

$$d(u, v) \leq \sum_{0 \leq \beta \leq 1 - \alpha} d\left(\sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta), \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta)\right).$$
where
\[ u = \sum_{\text{ev } \theta \leq \alpha} f(x + \theta(y - x)) \quad \text{and} \quad v = \sum_{\text{od } \theta \leq \alpha} f(x + \theta(y - x)). \]

Setting
\[ u(\beta) = \sum_{\text{ev } \theta \leq \alpha + \beta} h(\theta) \quad \text{and} \quad v(\beta) = \sum_{\text{od } \theta \leq \alpha + \beta} h(\theta) \quad \text{if} \quad 0 \leq \beta \leq 1 - \alpha, \]

the last inequality can be rewritten as
\[
\sum_{0 \leq \beta \leq 1 - \alpha} d(u, v) \leq \sum_{|\beta|=1-\alpha} \sum_{|\beta|=j-1} d(u(\beta), v(\beta)). \tag{4.8}
\]

In order to establish (4.8), we will apply Lemma 7 from [14] with \( m = |1 - \alpha| + 1 = n - |\alpha| + 1 \) and
\[
u_j = \sum_{|\beta|=j-1} u(\beta) \quad \text{and} \quad v_j = \sum_{|\beta|=j-1} v(\beta) \quad \text{if} \quad 1 \leq j \leq m.
\]

Suppose that we have already verified equalities (4.3) and (4.5) from [14]. Then by [14, Lemma 7], we get inequality (4.2) from [14], where, by virtue of [14, inequality (3.2)],
\[
d(u_j, v_j) = d \left( \sum_{|\beta|=j-1} u(\beta), \sum_{|\beta|=j-1} v(\beta) \right) \leq \sum_{|\beta|=j-1} d(u(\beta), v(\beta)), \quad 1 \leq j \leq m.
\]

Summing over \( j = 1, \ldots, m \) and taking into account [14, inequality (4.2)], we arrive at (4.8):
\[
d(u, v) \leq \sum_{j=1}^{m} d(u_j, v_j) \leq \sum_{j=1}^{m-1} \sum_{|\beta|=j} d(u(\beta), v(\beta)).
\]

Assume that \( 1 - \alpha \) is even; then \( m \) is odd. Let us verify the first equality in [14, (4.3)]. For this, we apply equality (4.1) and calculate the first sum at the left-hand side of (4.1). Given even \( \theta \leq \alpha \), we have \( 1 - \alpha + \theta \in \mathcal{L}_1 \) (cf. Step 1 in the proof of Lemma 1), and so, by (4.5), \( \alpha \lor (1 - \alpha + \theta) = 1 \) and \( \alpha(1 - \alpha + \theta) = \theta \), so that the definition of \( h(1 - \alpha + \theta) \) implies
\[
\sum_{\text{ev } \theta \leq \alpha} h(1 - \alpha + \theta) = \sum_{\text{ev } \theta \leq \alpha} f(x + \theta(y - x)) = u.
\]
Applying equality (4.1), we get:

\[
\begin{align*}
  u + \sum_{i=1}^{\frac{m-1}{2}} u_{2i} &= u + \sum_{i=1}^{\frac{1-\alpha}{2}} \sum_{|\beta|=2i-1} u(\beta) = u + \sum_{\text{od } \beta \leq 1-\alpha} u(\beta) \\
  &= \sum_{\text{ev } \beta \leq 1-\alpha} u(\beta) = \sum_{i=0}^{\frac{1-\alpha}{2}} \sum_{|\beta|=2i} u(\beta) = \sum_{i=0}^{\frac{(1-\alpha)+2}{2}} u_{2i+1} \\
  &= \sum_{i=1}^{\frac{(1-\alpha)+2}{2}} u_{2i-1} = \sum_{i=1}^{\frac{(m+1)/2}{2}} u_{2i-1},
\end{align*}
\]

and the first equality in [14, (4.3)] follows. In a similar manner we find that the first sum at the left-hand side of (4.2) is equal to \(v\), and, by virtue of (4.2), the calculations above show that the second equality in [14, (4.3)] holds as well.

Now suppose that \(1 - \alpha\) is odd, and so, \(m\) (defined above) is even. In order to verify the first equality in [14, (4.5)], we calculate the first sum at the left-hand side of (4.3). Given even \(\theta\) with \(1 - \alpha \leq \theta \leq 1\), we have (cf. Step 3 in the proof of Lemma 1) \(\theta \in \mathcal{L}\) and \(\alpha \land \theta = 1\). Moreover (cf. [9, Part I, assertion (3.9)]), there exists a unique \(\theta' \in \mathcal{A}_0\) s.t. \(\theta' \leq \alpha\) and \(\theta = 1 - \alpha + \theta'\) (define \(\theta'\) by \(\theta' = \alpha + \theta - 1\)). Since \(|\theta'| = |\alpha| + |\theta| - n = |\theta| - |1 - \alpha|\) and \(1 - \alpha\) is odd, then \(\theta'\) is odd, and \(\alpha \theta = \alpha(1 - \alpha + \theta') = \theta'\). It follows that \(h(\theta) = f(x + \theta'(y - x))\). Changing the summation multiindex \(\theta \mapsto \theta'\) in the first sum at the left of (4.3), we get:

\[
\begin{align*}
  \sum_{1-\alpha \leq \text{ev } \theta \leq 1} h(\theta) &= \sum_{\text{od } \theta' \leq \alpha} f(x + \theta'(y - x)) = v.
\end{align*}
\]

Applying equality (4.3), we find

\[
\begin{align*}
  \sum_{i=1}^{\frac{m}{2}} u_{2i} &= \sum_{i=1}^{\frac{(1-\alpha)+1}{2}} \sum_{|\beta|=2i-1} u(\beta) = \sum_{\text{od } \beta \leq 1-\alpha} u(\beta) \\
  &= v + \sum_{\text{ev } \beta \leq 1-\alpha} u(\beta) = v + \sum_{i=0}^{\frac{(1-\alpha)-1}{2}} \sum_{|\beta|=2i} u(\beta) = v + \sum_{i=0}^{\frac{(1-\alpha)}{2}} u_{2i+1} \\
  &= v + \sum_{i=1}^{\frac{(1-\alpha)+1}{2}} u_{2i-1} = v + \sum_{i=1}^{\frac{m}{2}} u_{2i-1},
\end{align*}
\]

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which proves the first equality in [14, (4.5)]. Similarly, the first sum at the left-hand side of (4.4) is equal to $u$, and, by virtue of (4.4), the calculations above prove the second equality in [14, (4.5)].

This completes the proof of Theorem 3.

5. Proofs of Theorems 1 and 2

Now we are in a position to prove Theorems 1 and 2.

**Proof of Theorem 1.** We divide the proof into four steps for clarity.

1. We apply the induction argument on the dimension $n$ of the basic rectangle $I_a^b \subset \mathbb{R}^n$. For $n = 1$ Theorem 1 was established in [6, Theorem 5.1] (and refined in [4, Theorem 1] and [8, Theorem 1.3]) in the case when $(M, d)$ is an arbitrary metric space, and for $n = 2$ it was proved in [2, Theorem 2]. Now, suppose that $n \geq 3$ and Theorem 1 is already established for domain rectangles of dimension $\leq n - 1$.

Given $j \in \mathbb{N}$, we let $\nu_j$ be the total variation function of $f_j$ on $I_a^b$, i.e., $\nu_j(x) = TV(f_j, I_a^b)$ for all $x \in I_a^b$. By Theorem C and condition (2.1), the sequence $\{\nu_j\} \subset Mon(I_a^b; \mathbb{R})$ is uniformly bounded (by $C$), and so, by Theorem A, there exist a subsequence of $\{\nu_j\}$ and the corresponding subsequence of $\{f_j\}$, again denoted as the whole sequences $\{\nu_j\}$ and $\{f_j\}$, respectively, and a function $\nu \in Mon(I_a^b; \mathbb{R})$ s.t.

$$\lim_{j \to \infty} \nu_j(x) = \nu(x) \text{ for all } x \in I_a^b. \quad (5.1)$$

It is known ([1], [17, III.5.4], [21]) that the set of discontinuity points of any totally monotone function on $I_a^b \subset \mathbb{R}^n$ lies on at most a countable set of hyperplanes of dimension $n - 1$ parallel to the coordinate axes. Given $i \in \{1, \ldots, n\}$, denote by $Z_i$ the union of the set of all rational points of the interval $[a_i, b_i]$, the two-point set $\{a_i, b_i\}$ and the set of those points $z_i \in [a_i, b_i]$, for which the hyperplane

$$H_i(z_i) = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{z_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n] \quad (5.2)$$

contains points of discontinuity of $\nu$. Clearly, the sets $Z_i \subset [a_i, b_i]$ are countable and dense in $[a_i, b_i]$, and so, we may assume that $Z_i = \{z_i(k)\}_{k=1}^{\infty}$.

2. In order to apply the induction hypothesis, we need an estimate on the $(n - 1)$-dimensional total variation of any function $f = f_j$ from the sequence
\{f_j\} ‘over the hyperplane’ (5.2) in the sense to be made precise below. This is done as follows.

Let us fix \(i \in \{1, \ldots, n\}\) and set \(1^i = (1, \ldots, 1, 0, 1, \ldots, 1)\), where 0 is the \(i\)-th coordinate of \(1^i\) and the other coordinates of \(1^i\) are equal to 1. Note that \(|1^i| = n - 1\). Given \(z_i \in \mathbb{Z}_i\), we put

\[
\overline{\alpha} = \overline{\alpha}(z_i) = (a_1, \ldots, a_{i-1}, z_i, a_{i+1}, \ldots, a_n) .
\]  

(5.3)

The map \(f^\overline{\alpha}_i : J^b_a[1^i] \to M\) with the base at \(\overline{\alpha}\), truncated by \(1^i\), is defined on the \((n-1)\)-dimensional rectangle \(J^b_a[1^i] \subset \mathbb{R}^{n-1}\) and given by: if \(x \in J^b_a\), then \(x[1^i] \in J^b_a[1^i]\) and

\[
f^\overline{\alpha}_i(x[1^i]) = f(\overline{\alpha} + 1^i(x - \overline{\alpha})) = f(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n).
\]  

(5.4)

The \((n-1)\)-dimensional total variation of \(f^\overline{\alpha}_i\) on \(J^b_a[1^i]\) is equal to

\[
TV_{n-1}(f^\overline{\alpha}_i, J^b_a[1^i]) = \sum_{0 \neq \alpha \leq 1} V_{|\alpha|}(f^\overline{\alpha}_i)^{|1^i|}_\alpha, (I^b_a[1^i]|\alpha),
\]  

(5.5)

where the summation is taken over \((n-1)\)-dimensional multiindices \(\alpha = (\alpha_1, \ldots, \alpha_{n-1})\) s.t. \(0_{n-1} \neq \alpha \leq 1_{n-1}\), i.e., \(\alpha \in \mathcal{A}(n-1)\) (this is the only instance and exception when the summation is over \((n-1)\)-dimensional multiindices). Given \(\alpha \in \mathcal{A}(n-1)\), we set \(\overline{\alpha} = (\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_i, \ldots, \alpha_{n-1})\), where 0 occupies the \(i\)-th place, and note that \(\alpha = |\overline{\alpha}|\). We have

\[
(f^\overline{\alpha}_i)^{|1^i|}_\alpha = f^\overline{\alpha}_i\quad \text{on}\quad (I^b_a[1^i]|\alpha) = I^b_a[\overline{\alpha}] = I^b_a[|\overline{\alpha}|].
\]

In fact, given \(x \in I^b_a\), we find \(x[|\overline{\alpha}|] = (x[1^i]|\alpha)\) and

\[
(f^\overline{\alpha}_i)^{|1^i|}_\alpha(x[|\overline{\alpha}|]) = (f^\overline{\alpha}_i)^{|1^i|}_\alpha((x[1^i]|\alpha) = f^\overline{\alpha}_i((a[1^i]) + (\overline{\alpha}[1^i])(x[1^i] - a[1^i]))
\]

\[
= f^\overline{\alpha}_i([a + \overline{\alpha}(x - a)][1^i])
\]

\[
= f(\overline{\alpha} + 1^i[a + \overline{\alpha}(x - a) - \overline{\alpha}]).
\]  

(5.6)

Since \(\overline{\alpha} + 1^i(a - \overline{\alpha}) = \overline{\alpha}\) and \(1^i\overline{\alpha} = \overline{\alpha}\), we get

\[
\overline{\alpha} + 1^i[a + \overline{\alpha}(x - a) - \overline{\alpha}] = \overline{\alpha} + 1^i(a - \overline{\alpha}) + 1^i\overline{\alpha}(x - a)
\]

\[
= \overline{\alpha} + \overline{\alpha}(x - a) = \overline{\alpha} + \overline{\alpha}(x - \overline{\alpha}),
\]  

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and so, the value (5.6) is equal to
\[ f(\bar{\alpha} + \alpha (x - \bar{\alpha})) = f_{\bar{\alpha}}(x|\bar{\alpha}). \]
It follows that the \(|\alpha|\)-th variation at the right-hand side of (5.5) is equal to
\[ V_{\alpha}((f_{\bar{\alpha}})^{1|1}_a, (I^b_a|1^i)\alpha) = V_{\bar{\alpha}}(f_{\bar{\alpha}}, I^b_{\bar{\alpha}}|\bar{\alpha}). \]
Noting that the set \(\mathcal{A}(n - 1)\) is bijective to the set of those \(\bar{\alpha} \in \mathcal{A}(n)\), for which \(0 \neq \bar{\alpha} \leq 1^i\), and applying Theorem B with \(x = \bar{\alpha}, y = b\) and \(\gamma = 1^i\), we get:
\[
TV_{n-1}(f_{\bar{\alpha}}, I^b_a|1^i) = \sum_{0 \neq \bar{\alpha} \leq 1^i} V_{\bar{\alpha}}(f_{\bar{\alpha}}, I^b_{\bar{\alpha}}|\bar{\alpha}) = TV(f, I^b_{\bar{\alpha}}) \\
\leq TV(f, I^b_{\bar{\alpha}}) - TV(f, I^b_a) \leq TV(f, I^b_a). (5.7)
\]
Thus, given \(j \in \mathbb{N}\) and \(i \in \{1, \ldots, n\}\), setting back \(f = f_j\), by virtue of (5.3), (5.7) and (2.1), we find, for all \(z_i \in Z_i\) and \(\bar{\alpha} = \bar{\alpha}(z_i)\):
\[
TV_{n-1}((f_j^{\bar{\alpha}(z_i)}, I^b_a|1^i) \leq C < \infty. (5.8)
\]
3. Now, we make use of the diagonal processes. For \(i = 1\) and \(z_1 = z_1(1) = z_1(1) \in Z_1\) the sequence \(((f_j^{\bar{\alpha}(z_1(1))})_{j=1}^{\infty} = ((f_j^{\bar{\alpha}(z_1(1))})_{j=1}^{\infty}\) satisfies the uniform estimate (5.8) on the rectangle \(I^b_a|1^1\) of dimension \(n-1\) and, since each map from this sequence is of the form (5.4) with \(z_i = z_1 = z_1(1)\), then it follows from the assumptions of Theorem 1 that the sequence under consideration is pointwise precompact on \(I^b_a|1^1\). By the induction hypothesis, the sequence \(\{f_j\}\) contains a subsequence, denoted by \(\{f_j^{1}\}\), s.t. \((f_j^{1}\bar{\alpha}(z_1(1)))\) converges pointwise on \(I^b_a|1^1\) to a map from \(I^b_a|1^1\) into \(\mathcal{M}\) of \((n-1)\)-dimensional finite total variation on \(I^b_a|1^1\). Since, by (5.4),
\[
(f_j^{1}\bar{\alpha}(z_1(1)))(x_2, \ldots, x_n) = (f_j^{1}\bar{\alpha}(z_1(1)))(x|1^1) = f_j^{1}(z_1(1), x_2, \ldots, x_n)
\]with \(x = (x_1, \ldots, x_n) \in I^b_a\) and \(x_i \in [a_i, b_i]\) for \(i \in \{2, \ldots, n\}\), then the pointwise convergence above means, actually, that the sequence \(\{f_j\}\) converges pointwise on the hyperplane \(H_1(z_1(1)) = \{z_1(1)\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]\).
Inductively, if \(k \geq 2\) and a subsequence \(\{f_{j}^{k-1}\}_{j=1}^{\infty}\) of \(\{f_j\}\), which is pointwise convergent on \(U_{l=1}^{k-1}H_1(z_1(l))\), is already chosen, then the sequence \(\{(f_{j}^{k-1}\bar{\alpha}(z_1(k))\}_{j=1}^{\infty}\) satisfies the uniform estimate (5.8) on the rectangle \(I^b_a|1^1\),
where $f_j$ is replaced by $f_j^{k-1}$ and $\mathbf{a}(z_i)$—by $\mathbf{a}(z_1(k))$. Moreover, since, as above, the sequence is pointwise precompact on $I^b_a[1^1]$, then, by the induction hypothesis, there exists a subsequence $\{f_j^k\}_{j=1}^\infty$ of $\{f_j^{k-1}\}_{j=1}^\infty$ s.t. $\{f_j^k\}_{j=1}^\infty$ converges pointwise on $I^b_a[1^1]$ as $j \to \infty$ to a map from $I^b_a[1^1]$ into $M$ of $(n-1)$-dimensional finite total variation on $I^b_a[1^1]$. Again, as above, this pointwise convergence means that the sequence $\{f_j^k\}_{j=1}^\infty$ converges pointwise on the hyperplane $H_1(z_1(k))$ and, as a consequence, on the set $\bigcup_{l=1}^k H_1(z_1(l))$ as well. We infer that the diagonal sequence $\{f_j^k\}_{j=1}^\infty$ extracted in the last paragraph again by $\{f_j\}$. Then we let $i = 2$, $z_2 = z_i(1) = z_2(1) \in Z_2$ and, beginning with the sequence $\{(f_j^k)_{1}^{\mathbf{a}(z_1(1))}\}_{j=1}^\infty = \{(f_j^k)_{1}^{\mathbf{a}(z_2(1))}\}_{j=1}^\infty$, apply the above arguments of this step. Doing this, we will end up with a diagonal sequence, a subsequence of the original sequence $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on $H_1(Z_1) \cup H_2(Z_2)$. Now suppose that for some $i \in \{2, \ldots, n - 1\}$ we have already extracted a (diagonal) subsequence of $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on the set $H_1(Z_1) \cup \cdots \cup H_{i-1}(Z_{i-1})$. Then we let $z_i = z_i(1) \in Z_i$ and apply the above arguments of this step to the sequence $\{(f_j^k)_{1}^{\mathbf{a}(z_1(1))}\}_{j=1}^\infty$: a subsequence of the original sequence $\{f_j\}$ converges pointwise on the set $H_1(Z_1) \cup \cdots \cup H_i(Z_i)$. In this way after finitely many steps we obtain a subsequence of the original sequence $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on the set $H(Z) = \bigcup_{i=1}^n H_i(Z_i)$.

4. Finally, let us show that the sequence $\{f_j\}$ from the end of Step 3 converges at each point $y \in I^b_a \setminus H(Z)$. Note that $y$ is a point of continuity of the function $\nu$ from (5.1) s.t. its coordinates $a_i < y_i < b_i$ are irrational for all $i \in \{1, \ldots, n\}$. Due to the density of $H(Z)$ in $I^b_a$, the continuity of $\nu$ at $y$ and properties of totally monotone functions, given $\varepsilon > 0$, there exists $x = x(\varepsilon) \in H(Z)$ with $x < y$ s.t. $0 \leq \nu(y) - \nu(x) \leq \varepsilon$. By virtue of (5.1), choose a number $j_0(\varepsilon) \in \mathbb{N}$ s.t. $|\nu_j(y) - \nu(y)| \leq \varepsilon$ and $|\nu(x) - \nu_j(x)| \leq \varepsilon$
for all \( j \geq j_0(\varepsilon) \). By [14, Theorem 2] and Theorem B with \( \gamma = 1 \), for all \( j \geq j_0(\varepsilon) \) we have:

\[
d(f_j(x), f_j(y)) \leq TV(f_j, I_\nu^y) \leq TV(f_j, I_\nu^y) - TV(f_j, I_\nu^x) = \nu_j(y) - \nu_j(x) \\
\leq |\nu_j(y) - \nu(y)| + (\nu(y) - \nu(x)) + |\nu(x) - \nu_j(x)| \leq 3\varepsilon.
\]

Since \( x \in H(Z) \) and, as it was shown in Step 3, the sequence \( \{f_j(x)\}_{j=1}^\infty \) is convergent in \( M \), it is Cauchy, and so, there exists a number \( j_1(\varepsilon) \in \mathbb{N} \) s.t. \( d(f_j(x), f_j'(x)) \leq \varepsilon \) for all \( j \geq j_1(\varepsilon) \) and \( j' \geq j_1(\varepsilon) \). It follows that if \( J(\varepsilon) = \max\{j_0(\varepsilon), j_1(\varepsilon)\} \), \( j \geq J(\varepsilon) \) and \( j' \geq J(\varepsilon) \), then we have:

\[
d(f_j(y), f_j'(y)) \leq d(f_j(y), f_j(x)) + d(f_j(x), f_j'(x)) + d(f_j'(x), f_j'(y)) \\
\leq 3\varepsilon + \varepsilon + 3\varepsilon = 7\varepsilon.
\]

Thus, the sequence \( \{f_j(y)\}_{j=1}^\infty \) is Cauchy in the metric space \( M \), and so, since it is also precompact by the assumption, it is convergent in \( M \).

It follows from here and the end of Step 3 that the sequence \( \{f_j(y)\}_{j=1}^\infty \) converges in \( M \) at each point \( y \in (I_\nu^b \setminus H(Z)) \cup H(Z) = I_\nu^b \), i.e., the sequence \( \{f_j\} \), which is a subsequence of the original sequence \( \{f_j\} \), converges pointwise on \( I_\nu^b \). Let us denote the pointwise limit of \( \{f_j\} \) by \( f : I_\nu^b \to M \). Then, by virtue of [14, Theorem 3] and assumption (2.1), we find

\[
TV(f, I_\nu^b) \leq \liminf_{j \to \infty} TV(f_j, I_\nu^b) \leq C,
\]

and so, \( f \in BV(I_\nu^b, M) \).

This completes the proof of Theorem 1. \( \square \)

**Remark 5.1.** In Theorem 1 the precompactness of the sets \( \{f_j(x)\}_{j=1}^\infty \) at all points \( x \in I_\nu^b \) cannot be replaced by the closedness and boundedness even at a single point of \( I_\nu^b \). The corresponding examples for maps of one variable are constructed in [4, Section 3], [6, Section 5] and [8, Section 1] and can be easily adapted for maps of several variables.

**Proof of Theorem 2.** The proof is adapted for the situation under consideration from the proof of Theorem 7 from [10].

1. In this step we show that, given \( j \in \mathbb{N} \) and \( u^* \in M^* \), we have:

\[
TV((f_j(\cdot), u^*), I_\nu^b) \leq TV(f_j, I_\nu^b)\|u^*\| \leq C\|u^*\|, \quad (5.9)
\]
where the function \( \langle f_j(\cdot), u^* \rangle : I^b_0 \to \mathbb{K} \) is given by \( \langle f_j(\cdot), u^* \rangle(x) = \langle f_j(x), u^* \rangle, \) \( x \in I^b_0, \) and \( C \) is the constant from (2.1).

In fact, given \( 0 \neq \alpha \leq 1 \) and \( x, y \in I^b_0 \) with \( x < y, \) by virtue of (2.5) from [14, Lemma 1] where \( d(u, v) \) is replaced by the absolute value \( |u - v| \) in \( \mathbb{K} \) and later on—by the norm in \( M, \) we get:

\[
\text{md}_{|\alpha|}(\langle f_j(\cdot), u^* \rangle^a, I^y_x |\alpha) = \left| \sum_{0 \leq \theta \leq \alpha} (-1)^{|\theta|} \langle f_j(a + \alpha(x - a) + \theta(y - x)), u^* \rangle \right| \\
= \left| \sum_{0 \leq \theta \leq \alpha} (-1)^{|\theta|} f_j(a + \alpha(x - a) + \theta(y - x)), u^* \rangle \right| \\
\leq \left| \sum_{0 \leq \theta \leq \alpha} (-1)^{|\theta|} f_j(a + \alpha(x - a) + \theta(y - x)) \right| \cdot \|u^*\| \\
= \text{md}_{|\alpha|}(\langle f_j \rangle^a, I^y_x |\alpha)\|u^*\|.
\]

It follows that if \( \mathcal{P} = \{x[\sigma] \}_{\sigma=0}^{\kappa} \) is a net partition of \( I^b_0, \) then \( \mathcal{P}[\alpha] = \{x[\sigma][\alpha] \}_{\sigma=0}^{\kappa|\alpha} \) is a net partition of \( I^b_0|\alpha, \) and so, setting \( x = x[\sigma - 1] \) and \( y = x[\sigma] \) in the calculations above, we find

\[
\sum_{1 \leq \sigma \leq \kappa|\alpha} \text{md}_{|\alpha|}(\langle f_j(\cdot), u^* \rangle^a, I^x_{x[\sigma-1]} |\alpha) \leq \sum_{1 \leq \sigma \leq \kappa|\alpha} \text{md}_{|\alpha|}(\langle f_j \rangle^a, I^x_{x[\sigma-1]} |\alpha)\|u^*\| \\
\leq V_{|\alpha|}(\langle f_j \rangle^a, I^b_0 |\alpha)\|u^*\|,
\]

the summation over \( \sigma|\alpha \) being taken only over those coordinates \( \sigma_i \) in the range \( 1 \leq \sigma_i \leq \kappa_i \) with \( i \in \{1, \ldots, n\}, \) for which \( \alpha_i = 1. \) Since \( \mathcal{P} \) is an arbitrary partition of \( I^b_0, \) we get:

\[
V_{|\alpha|}(\langle f_j(\cdot), u^* \rangle^a, I^b_0 |\alpha) \leq V_{|\alpha|}(\langle f_j \rangle^a, I^b_0 |\alpha)\|u^*\|,
\]

and so, inequality (5.9) follows from the definition of the total variation.

Moreover, by virtue of (2.2), we have:

\[
|\langle f_j(x), u^* \rangle| \leq \|f_j(x)\| \cdot \|u^*\| \leq c(x)\|u^*\|, \quad x \in I^b_0, \quad u^* \in M^*,
\]

and so, the sequence \( \{\langle f_j(\cdot), u^* \rangle\}_{j=1}^{\infty} \) of functions from \( I^b_0 \) into (metric semigroup) \( \mathbb{K} \) is pointwise bounded on \( I^b_0 \) and, hence, pointwise precompact for each \( u^* \in M^*. \)

Taking this and (5.9) into account and applying Theorem 1 to the sequence \( \{\langle f_j(\cdot), u^* \rangle\}_{j=1}^{\infty} \) for any given \( u^* \in M^*, \) we extract a subsequence of
\{f_j\}, denoted by \{f_{j,v}\} (which depends on \(u^*\) in general), and find a function 
\(g_{u^*} \in \text{BV}(I_a^b; \mathbb{K})\) satisfying \(\text{TV}(g_{u^*}, I_a^b) \leq C\|u^*\|\) s.t. \(\langle f_{j,v^*}(x), u^* \rangle \to g_{u^*}(x)\) in \(\mathbb{K}\) as \(j \to \infty\) for all \(x \in I_a^b\).

2. Making use of the diagonal process and the separability of \(\mathcal{M}^*\), let us get rid of the dependence of \(\{f_{j,v}\}\) on \(u^* \in \mathcal{M}^*\). Let \(\{u_k^*\}_{k=1}^\infty\) be a countable dense subset of \(\mathcal{M}^*\). By Step 1, for \(u^* = u_k^*\) we get a subsequence \(\{f_j^{(1)}\} = \{f_{j,v^*}\}_{j=1}^\infty\) of the original sequence \(\{f_j\}\) and a function 
\(g_{u_k^*} \in \text{BV}(I_a^b; \mathbb{K})\) satisfying \(\text{TV}(g_{u_k^*}, I_a^b) \leq C\|u_k^*\|\) s.t. \(\langle f_j^{(1)}(x), u_k^* \rangle \to g_{u_k^*}(x)\) in \(\mathbb{K}\) for all \(x \in I_a^b\). Inductively, if \(k \geq 2\) and a subsequence \(\{f_j^{(k-1)}\}_{j=1}^\infty\) of \(\{f_j\}\) is already chosen, then by virtue of (5.9) and (5.10), we have:

\[
\text{TV}((f_j^{(k-1)}(\cdot), u_k^*), I_a^b) \leq C\|u_k^*\|
\]

and \(|\langle f_j^{(k-1)}(x), u_k^* \rangle| \leq c(x)\|u_k^*\|, x \in I_a^b\), for all \(j \in \mathbb{N}\). By Theorem 1, applied to the sequence \(\{(f_j^{(k-1)}(\cdot), u_k^*)\}_{j=1}^\infty\) of \(\{f_j^{(k-1)}\}_{j=1}^\infty\) and a function \(g_{u_k^*} \in \text{BV}(I_a^b; \mathbb{K})\) satisfying \(\text{TV}(g_{u_k^*}, I_a^b) \leq C\|u_k^*\|\) s.t. \(\langle f_j^{(k)}(x), u_k^* \rangle \to g_{u_k^*}(x)\) in \(\mathbb{K}\) as \(j \to \infty\) for all \(x \in I_a^b\). Then the diagonal sequence \(\{f_j^{(j)}\}_{j=1}^\infty\), again denoted by \(\{f_j\}\), is a subsequence of the original sequence and satisfies the condition:

\[
\langle f_j(x), u_k^* \rangle \to g_{u_k^*}(x) \quad \text{as} \quad j \to \infty \quad \text{for all} \quad x \in I_a^b \quad \text{and} \quad k \in \mathbb{N}. \tag{5.11}
\]

3. Now, given \(u^* \in \mathcal{M}^*\) and \(x \in I_a^b\), let us show that the sequence \(\{(f_j(x), u^*)\}_{j=1}^\infty\) is Cauchy in \(\mathbb{K}\). Taking into account (5.11) we may assume that \(u^* \neq u_k^*\) for all \(k \in \mathbb{N}\). Let \(\varepsilon > 0\) be arbitrary. By the density of \(\{u_k^*\}_{k=1}^\infty\) in \(\mathcal{M}^*\), there exists \(k = k(\varepsilon) \in \mathbb{N}\) s.t. \(\|u^* - u_k^*\| \leq \varepsilon/(1 + 4c(x))\). By (5.11), there exists \(j_0 = j_0(\varepsilon) \in \mathbb{N}\) s.t. \(|\langle f_j(x), u_k^* \rangle - \langle f_j^*(x), u_k^* \rangle| \leq \varepsilon/2\) for all \(j \geq j_0\) and \(j' \geq j_0\). It follows that for such \(j\) and \(j'\) we have:

\[
|\langle f_j(x), u^* \rangle - \langle f_j^*(x), u^* \rangle| \leq |\langle f_j(x) - f_j^*(x), u^* - u_k^* \rangle| + |\langle f_j(x), u_k^* \rangle - \langle f_j^*(x), u_k^* \rangle| \leq \|f_j(x) - f_j^*(x)\| \cdot \|u^* - u_k^*\| + \frac{\varepsilon}{2} \leq 2c(x)\frac{\varepsilon}{1 + 4c(x)} + \frac{\varepsilon}{2} \leq \varepsilon.
\]

Thus, \(\{(f_j(x), u^*)\}_{j=1}^\infty\) is Cauchy in \(\mathbb{K}\) and, hence, there exists an element of \(\mathbb{K}\) denoted by \(g_{u^*} \in \mathbb{K}\) s.t. \(\langle f_j(x), u^* \rangle \to g_{u^*}(x)\) in \(\mathbb{K}\) as \(j \to \infty\). In other words,
we have shown that for each \( u^* \in M^* \) there exists a function \( g_{u^*} : I_a^b \to \mathbb{K} \) satisfying (cf. [14, Theorem 3] and (5.9))

\[
\text{TV}(g_{u^*}, I_a^b) \leq \liminf_{j \to \infty} \text{TV}(\langle f_j(\cdot), u^* \rangle, I_a^b) \leq C\|u^*\|
\]

(and so, \( g_{u^*} \in \text{BV}(I_a^b; \mathbb{K}) \)) and

\[
\lim_{j \to \infty} \langle f_j(x), u^* \rangle = g_{u^*}(x) \quad \text{in } \mathbb{K} \text{ for all } x \in I_a^b \text{ and } u^* \in M^*. \tag{5.12}
\]

4. Let us prove (2.3), i.e., \( f_j(x) \) converges weakly in \( M \) as \( j \to \infty \) for all \( x \in I_a^b \). By the reflexivity of \( M \), we have \( f_j(x) \in M = M^{**} \equiv L(M^*; \mathbb{K}) \) for all \( j \in \mathbb{N} \). Define the functional \( G_x : M^* \to \mathbb{K} \) by \( G_x(u^*) = g_{u^*}(x) \) for all \( u^* \in M^* \). By virtue of (5.12), we get

\[
\lim_{j \to \infty} \langle f_j(x), u^* \rangle = G_x(u^*) \quad \text{for all } u^* \in M^*,
\]

i.e., the sequence \( \{f_j(x)\}_{j=1}^\infty \subset L(M^*; \mathbb{K}) \) converges pointwise on \( M^* \) to the operator \( G_x : M^* \to \mathbb{K} \). By the Banach-Steinhaus uniform boundedness principle, \( G_x \in L(M^*; \mathbb{K}) = M \) and \( \|G_x\| \leq \text{lim inf}_{j \to \infty} \|f_j(x)\| \). Setting \( f(x) = G_x \) for all \( x \in I_a^b \), we find that \( f : I_a^b \to M \) and

\[
\lim_{j \to \infty} \langle f_j(x), u^* \rangle = G_x(u^*) = \langle G_x, u^* \rangle = \langle f(x), u^* \rangle \quad \text{in } \mathbb{K} \tag{5.13}
\]

for all \( u^* \in M^* \) and \( x \in I_a^b \), and so, \( f_j(x) \overset{w}{\to} f(x) \) in \( M \) as \( j \to \infty \) for all \( x \in I_a^b \), which proves (2.3).

5. It remains to show that \( f \in \text{BV}(I_a^b; M) \) and \( \text{TV}(f, I_a^b) \leq C \). By (5.13), we have: if \( x, y \in I_a^b \) with \( x < y \) and \( 0 \neq \alpha \leq 1 \), then

\[
\sum_{0 \leq \theta \leq \alpha} (-1)^{\theta} f_j(a+\alpha(x-a)+\theta(y-x)) \overset{w}{\to} \sum_{0 \leq \theta \leq \alpha} (-1)^{\theta} f(a+\alpha(x-a)+\theta(y-x))
\]

in \( M \) as \( j \to \infty \), and so, by virtue of (2.5) from [14, Lemma 1] and the remarks preceding Theorem 2,

\[
\text{md}_{|\alpha|}(f^\alpha, I_x^y | \alpha) = \left\| \sum_{0 \leq \theta \leq \alpha} (-1)^{\theta} f(a+\alpha(x-a)+\theta(y-x)) \right\| \leq \liminf_{j \to \infty} \left\| \sum_{0 \leq \theta \leq \alpha} (-1)^{\theta} f_j(a+\alpha(x-a)+\theta(y-x)) \right\| \leq \liminf_{j \to \infty} \text{md}_{|\alpha|}(\langle f_j \alpha, I_x^y | \alpha \rangle).
\tag{5.14}
\]

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Arguing as in Step 2 of the proof of Theorem 3 from [14], making use of the inequality (5.14), which coincides with [14, inequality (5.2)], and taking into account (2.1), we get:

$$TV(f, I^b_a) \leq \liminf_{j \to \infty} TV(f_j, I^b_a) \leq C.$$ 

This completes the proof of Theorem 2.

**Remark 5.2.** Note that instead of condition (2.2) in Theorem 2 we may assume only that the value $c(a) = \sup_{j \in \mathbb{N}} \|f_j(a)\|$ is finite. In fact, it follows from [14, Theorem 2] and condition (2.1) that, given $x \in I^b_a$ and $j \in \mathbb{N}$,

$$\|f_j(x)\| \leq \|f_j(a)\| + \|f_j(x) - f_j(a)\| \leq c(a) + TV(f_j, I^a_x) \leq c(a) + C.$$ 

**References**


