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On inflection points, monomial curves, and hypersurfaces containing projective curves*

S. L'vovsky

Profsoyuznaya ul. 20/9, kv.162, Moscow 117292, Russia, (e-mail: serge@ium.ips.ras.ru)

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1. Introduction

Let $X \subset \mathbb{P}^n$ be a projective curve. In this paper we try to find an upper bound for the number of linearly independent hypersurfaces $Z \supset X$ of given degree m and to investigate the borderline cases.

Maybe the first result in this direction is due to Castelnuovo. To wit, his well-known lemma (see, for instance, [8, Ch. 4, Sect. 3]) says that if $n(n-1)/2$ linearly independent quadrics pass through $d \geq 2n+3$ points in uniform position in $\mathbb{P}^n$, then these points lie on a rational normal curve. As an immediate consequence, one can see that

Theorem 1.1 (Castelnuovo). *If $n(n-1)/2$ linearly independent quadrics pass through a non-degenerate curve $C \subset \mathbb{P}^n$, then C is a rational normal curve.*

By induction on dimension, one can derive the following result from this theorem:

Theorem 1.2. *If $X \subset \mathbb{P}^n$ is a non-degenerate irreducible projective variety and $c = \operatorname{codim} X$, then X is contained in at most $c(c+1)/2$ linearly independent quadrics; the equality is attained iff the Δ-genus of X is 0.*

The condition "Δ-genus of X is zero" means that $\deg X = c+1$; see [10, Section 3] for complete classification of such varieties.

The above mentioned lemma of Castelnuovo was generalized to the case of hypersurfaces of arbitrary degree by J. Harris [10, Section 1, Lemma]:

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Lemma 1.3 (J. Harris). *Any $d \geq kn+1$ points in general position in $\mathbb{P}^n$ impose at least $kn+1$ linear conditions on hypersurfaces of degree k. If $d > kn+1$ (resp. $kn+2$ if $k=2$), then these points impose exactly $kn+1$ conditions iff they lie on a rational normal curve in $\mathbb{P}^n$.*

This lemma implies the following

Theorem 1.4. *If $X \subset \mathbb{P}^n$ is a non-degenerate irreducible projective curve and $m \geq 2$, then $h^0(\mathcal{I}_X(m)) \leq \binom{m+n}{n} - mn - 1$. The equality is attained iff X is a rational normal curve.*

On the other hand, as early as 1894 Fano [6] generalized the lemma of Castelnuovo in another direction: he proved that if exactly $n(n-1)/2-1$ linearly independent quadrics pass through $d \geq 2n+5$ points in uniform position in $\mathbb{P}^n$, then these points lie on a linearly normal curve of arithmetic genus 1, which is cut out by these quadrics. Nowadays this result was rediscovered by D. Eisenbud and J. Harris [11, Proposition 3.20] (see also [3, Theorem 3.1]). This immediately implies the following

Theorem 1.5 (Fano, Eisenbud, Harris). *If $n(n-1)/2-1$ linearly independent quadrics pass through a non-degenerate curve $C \subset \mathbb{P}^n$, then C is either a rational normal curve or a linearly normal curve of arithmetic genus 1.*

Induction on dimension yields

Theorem 1.6. *Suppose that $X \subset \mathbb{P}^n$ is a non-degenerate irreducible projective variety and put $c = \operatorname{codim} X$. If X is contained in exactly $c(c+1)/2-1$ linearly independent quadrics, then $\deg X = \operatorname{codim} X$ and smooth one-dimensional linear sections of X are linearly normal elliptic curves.*

The varieties whose one-dimensional linear sections are linearly normal elliptic curves are also completely classified (see, for instance, [7]).

Finally, F. L. Zak proposed new proofs and generalizations of these facts in a recent unpublished paper [17]. Zak's proofs make extensive use of secant varieties.

The aim of the present paper is to propose a new method of proof of Theorems 1.4 and 1.5 and some similar results. Using this method, we give new proofs of Theorems 1.4 and 1.5. Moreover, we generalize Theorem 1.5 to the case of hypersurfaces of arbitrary degree:

Theorem 1.7. *Suppose that $X \subset \mathbb{P}^n$, $n \geq 2$, is a non-degenerate irreducible projective curve and $m \geq 2$. If $h^0(\mathcal{I}_X(m)) < \binom{m+n}{n} - mn - 1$, then $h^0(\mathcal{I}_X(m)) \leq \binom{m+n}{n} - m(n+1)$. The equality is attained iff $\deg X = n+1$ and the genus of X equals 1.*

Finally, we obtain some results with less compact statements (see below).

It should be noted that actually Fano, Eisenbud and Harris proved much more that we reprove here. The author would like to stress that it is the method of proof that claims to novelty, not the final results.

Our method of proof of the above results is based on the study of inflection points of the curve X. It involves two ideas. The first one is that the presence of an inflection point imposes an upper bound on the number of hypersurfaces of degree m containing X (Proposition 3.2). For the case of canonical linear system, this idea was used by R. O. Buchweitz [2]. The second idea is to compare the Hilbert polynomials of X and of the monomial curve C^A, where $A = (a_0, \dots, a_n)$ is the vanishing sequence at the point $p \in X$ (see Section 2 for the definition). It turns out that those Hilbert polynomials coincide once the values of the Hilbert function agree in some degree at least as large as the maximal degree of generators of the homogeneous ideal of C^A.

In the above results, inflection points appeared in proofs rather than in statements. If we assume that the curve X has an inflection point of the prescribed type, then the upper bound on the number of hypersurfaces of given degree containing X can sometimes be made explicit, and something can be said about the borderline cases.

Theorem 1.8. *Let $X \subset \mathbb{P}^n$ be an irreducible non-degenerate curve. Suppose that $p \in X$ is a smooth point and denote by $(0 = a_0, a_1, \dots, a_n)$ the vanishing sequence at p. Put $\delta = \max_{1 \le i < j \le n}\{(a_i - a_{i-1}) + (a_j - a_{j-1})\}$, and denote by L the number of gaps of the semigroup generated by $0, a_n - a_{n-1}, \dots, a_n - a_1, a_n$. Suppose that $m \ge \delta$ is an integer. Then:*

(i) $h^0(\mathcal{I}_X(m)) \le \binom{m+n}{m} - a_n \cdot m - 1 + L$;

(ii) If the equality is attained, then $\deg X = a_n$, *the arithmetic genus of X equals L, and the homogeneous ideal of X is generated by its elements of degree $\le m$.*

Remark 1.9. If $C = C^A \subset \mathbb{P}^n$ is the monomial curve corresponding to the vanishing sequence $A = (0, a_1, \dots, a_n)$ (cf. Section 2), then the bound (i) in theorem 1.8 is attained for the point $p = (1 : 0 \dots : 0) \in C$. Hence, this bound cannot be sharpened uniformly for all curves having an inflection point with the vanishing sequence $(0, a_1, \dots, a_n)$.

Our proofs require some study of monomial curves and their defining equations. In Section 5 we obtain, as a by-product, some results on this subject. All these results can be restated in purely combinatorial terms (see Corollaries 5.9 and 5.10). For example, we prove that if $0 = a_0 < a_1 < \dots < a_n$ is an increasing sequence of integers such that g.c.d. of $(a_1, \dots, a_n)$ equals 1, then conductor of the additive semigroup in $\mathbb{Z}$ generated by $a_0, \dots, a_n$ is at most $(\delta - 2)a_n + 1$, where $\delta = \max_{1 \le i < j \le n}\{(a_i - a_{i-1}) + (a_j - a_{j-1})\}$. This bound is admittedly not sharp (for example, if a_1 and a_2 are relatively prime, then conductor is at most $(a_1 - 1)(a_2 - 1)$; cf. [12]), but it has the advantage of being applicable to any sequence $(a_1, \dots, a_n)$ with g.c.d.$(a_1, \dots, a_n) = 1$.

Unfortunately, proofs of these combinatorial results are ultimately based on the heavy cohomological machinery from the paper [9]. It would be interesting to find more elementary proofs.

Monomial curves (and semigroups in $\mathbb{N}$, which are closely related to the latter) have been studied by many authors (cf. [1], [2], [13], and references therein). Corollary 5.9 strengthens Theorem 4.1 from [1]. Hilbert functions of semigroup algebras were considered in [2] and [14].

The paper is organized as follows. In Section 2 we recall some well-known definitions and results on inflection points. Section 3 contains the combinatorial results that will be used in our proofs of Theorems 1.4 and 1.5. In Section 4 we prove our main degeneration result. Section 5 contains auxiliary results on equations defining monomial curves, while Section 6 is devoted to the final proofs.

Some results from this paper were posted at publications.math.duke.edu/alg-geom/9504002.

Notation and conventions

We work over an algebraically closed field of arbitrary characteristic.

If a and b are integers, then $(a;b) = \{n \in \mathbb{Z} \mid a < n < b\}$ and $[a;b] = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$.

For any subset $X \subset \mathbb{P}^n$, denote by $\langle X \rangle$ its linear span. A projective subvariety $X \subset \mathbb{P}^n$ is called *non-degenerate* iff $\langle X \rangle = \mathbb{P}^n$.

If $X \subset \mathbb{P}^n$ is a closed subscheme, then $\mathscr{I}_X \subset \mathscr{O}_{\mathbb{P}^n}$ is its ideal sheaf.

By $v_m : \mathbb{P}^n \to \mathbb{P}^{\binom{m+n}{n}-1}$ we denote the m-th Veronese mapping.

The set of non-negative integers will be denoted by $\mathbb{N}$. If $0 < a_1 < \ldots < a_n$ is an increasing sequence of positive integers, then $\langle a_1 \ldots a_n \rangle \subset \mathbb{N}$ denotes the semigroup generated by $0, a_1, \ldots, a_n$. If g.c.d. of $a_1, \ldots, a_n$ equals 1, then

$$\langle a_1 \ldots a_n \rangle \supseteq d + \mathbb{N} \tag{1}$$

for some $d \geq 0$. In that case, the minimal d for which (1) holds is called *conductor* of the semigroup, and any positive integer not contained in the semigroup is called its *gap*.

Abusing the language, we will sometimes write "$s \in \mathscr{F}$" instead of "s is a local section of a sheaf $\mathscr{F}$".

The notation $\mathbb{P}(V)$ means $\operatorname{Proj}\operatorname{Sym}(V^*)$.

2. Preliminaries

In this section we chiefly recall some well-known definitions.

Let C be a smooth projective curve, $\mathscr{L}$ a line bundle on C, $V \subset H^0(\mathscr{L})$ a linear subspace. The pair $(\mathscr{L}, V)$ will be referred to as a *linear system* on C. Consider a morphism of sheaves $a : V \otimes \mathscr{O}_C \to \mathscr{L}$ defined by the formula $v \otimes \varphi \mapsto \varphi v$, where $v \in V$ and $\varphi \in \mathscr{O}_C$. Following [9], put $\mathscr{M}_V = \ker a$; if a is epimorphic (i. e. the linear system has no base points), then $\mathscr{M}_V$ is a locally free sheaf of rank $\dim V - 1$ and degree $-\deg \mathscr{L}$. If the linear system

$(\mathscr{L}, V)$ has no base points and $f: C \to \mathbb{P}(V^*)$ is the mapping associated to it, them $\mathscr{M}_V = f^*\Omega^1_{\mathbb{P}(V^*)}$.

If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^n$, then it is called t-regular if $H^i(\mathscr{F}(t-i)) = 0$ for all $i > 0$ (cf. [16, Lecture 14]). A closed subscheme $X \subset \mathbb{P}^n$ is called t-regular if its ideal sheaf is t-regular. If $X \subset \mathbb{P}^n$ is t-regular, then it is t'-regular for all $t' \geq t$, and the homogeneous ideal of X is generated by its components of degree $\leq t$ (loc. cit.)

Let $(\mathscr{L}, V)$ be a linear system on a curve C, and suppose that $\dim V = n+1$. For any point $p \in C$, there exists a basis $s_0, \dots, s_n$ of V such that $\mathrm{ord}_p(s_0) < \mathrm{ord}_p(s_1) < \dots < \mathrm{ord}_p(s_n)$. Denote $\mathrm{ord}_p(s_i)$ by $a_i(p)$. The sequence $a_0(p), a_1(p), \dots, a_n(p)$ is called *vanishing sequence of* $(\mathscr{L}, V)$ *at* p. For any $s \in V$, the number $\mathrm{ord}_p(s)$ coincides with one of $a_j(p)$'s. The number $w(p) = \sum_{0 \leq i \leq n}(a_i(p) - i)$ is called *weight* of p with respect to the linear system $(\mathscr{L}, V)$. If $w(p) \neq 0$, one says that p is an *inflection point*. If p is not an inflection point, then $a_i(p) = i$ for all i, and vice versa.

We will use the following well-known

Proposition 2.1. *Suppose that no point of C is an inflection point with respect to the linear system $(\mathscr{L}, V)$. Then this linear system defines an isomorphism of C onto a rational normal curve in $\mathbb{P}^n$.*

If $X \subset \mathbb{P}^n = \mathrm{Proj}\, k[X_0, \dots, X_n]$ is an irreducible curve and $\nu: C \to X$ is its normalization, then one may consider the linear system $(\mathscr{L}, V)$ on C, where $\mathscr{L} = \nu^*\mathscr{O}_C(1)$ and V is spanned by $\nu^*(X_j)$'s.

Lemma 2.2. *In the above setting, suppose that $(a_0, \dots, a_n)$ is the vanishing sequence of $(\mathscr{L}, V)$ at a point $p \in C$. Then g.c.d. of $(a_0, \dots, a_n)$ equals* 1.

Proof. Denote g.c.d. of $(a_0, \dots, a_n)$ by d. Then $\mathrm{ord}_p(\nu^*(f))$ is divisible by d for any $f \in \mathscr{O}_{X,\nu(p)}$. Since the rings $\nu^*\mathscr{O}_{X,\nu(p)}$ and $\mathscr{O}_{C,p}$ have the same field of fractions, we arrive at a contradiction if we assume that $d > 1$.

For a non-negative integer n, denote by $\mathscr{A}_n$ the set of strictly increasing sequences of $n+1$ non-negative integers:

$$\mathscr{A}_n = \{\, (a_0, \dots, a_n) \mid 0 \leq a_0 < \dots < a_n \,\} .$$

For any $A \in \mathscr{A}_n$ one can consider the monomial curve $C^A \subset \mathbb{P}^n$, which is the projective curve with $(t^{a_0} : t^{a_1} : \dots : t^{a_n})$ a generic point. Denote by S^A the homogeneous coordinate ring of C^A, and by $J^A \subset R = k[X_0, \dots, X_n]$ its homogeneous ideal.

Actually, S^A has the structure of a bigraded ring. To wit, for a monomial $\xi = X_0^{k_0} \dots X_n^{k_n} \in R$ put $\mathrm{wt}^A(\xi) = a_0k_0 + a_1k_1 + \dots + a_nk_n$ and call this number *A-weight* of ξ. The decomposition $R = \bigoplus R^j_m$, where $R^j_m \subset R$ is the k-subspace spanned by the monomials of degree m and A-weight j, defines a bigrading on the ring R; the ideal J^A is bihomogeneous with respect to this bigrading, whence the ring S^A is bigraded as well.

A linear combination of monomials of the same A-weight is called A-*quasihomogeneous* polynomial. The natural projection of R onto the space of quasihomogeneous polynomials of A-weight j will be denoted by $\mathrm{pr}^j : R \to \sum_m R^j_m$.

3. On Hilbert functions of monomial curves

For any integer $k \geq 1$ and any $A = (a_0, \dots, a_n) \in \mathscr{A}_n$, put

$$v_k(A) = \{ a_{i_1} + \dots + a_{i_k} \mid 0 \leq i_1 \leq \dots \leq i_k \leq n \} .$$

Sometimes we will assume that $v_0(A) = \{0\}$.

Some simple properties of curves C^A and their homogeneous rings are gathered in the following

Proposition 3.1. *Let $A = (a_0, \dots, a_n)$ and $B = (b_0, \dots, b_n)$ be increasing sequences of non-negative integers.*

(i) $\dim S^A_k$ *equals the cardinality of* $v_k(A)$.
(ii) If there exist constants p and q such that $b_i = p \cdot a_i + q$ for all i, then $C^A = C^B$ and $S^A \cong S^B$.
(iii) If there exists a constant p such that $b_i = p - a_{n-i}$ for all i, then $C^A = C^B$ and $S^A \cong S^B$.

The following easy observation plays the key role in the sequel.

Proposition 3.2. *If $(\mathscr{L}, V)$ is a linear system on a smooth curve C, where* $\dim \mathscr{L} = n+1$, *and if $A = (a_0, \dots, a_n)$ is the vanishing sequence at a point $p \in C$, then* $\dim \operatorname{im} \left(\operatorname{Sym}^m V \to H^0(\mathscr{L}^{\otimes m})\right) \geq \dim S^A_m$ *for any $m \geq 1$.*

Now let us find out in what cases $\dim S^A_m$ is small.

Proposition 3.3. *Suppose that $A \in \mathscr{A}_n$. Then*

(i) $\dim S^A_m \geq mn + 1$;
(ii) $\dim S^A_m = mn + 1$ *if and only if A is an arithmetic progression.*
(iii) If $\dim S^A_m > mn + 1$, *then* $\dim S^A_m \geq m(n+1)$.
(iv) Suppose that $n \geq 3$. Then $\dim S^A_m = m(n+1)$ *if and only if either $a_n - a_{n-1} = 2(a_j - a_{j-1})$ for all $j \in [1; n-1]$ or $a_1 - a_0 = 2(a_j - a_{j-1})$ for all $j \in [2; n]$.*

Proof. For $1 \leq i, j \leq n$, put $b_{ij} = (m-i)a_0 + a_j + (i-1)a_n \in v_m(A)$. It is clear that the following chain of inequalities holds:

$$\begin{array}{lrl} & ma_0 < & b_{11} < b_{12} < \dots < b_{1n} < \\ (2) & & b_{21} < b_{22} < \dots < b_{2n} < \\ & & \dots\dots\dots\dots\dots\dots\dots \\ & & b_{m1} < b_{m2} < \dots < b_{mn} . \end{array}$$

Since this chain contains $mn + 1$ elements of $v_m(A)$, assertion (i) follows from Proposition 3.1(i). The "if" part of assertion (ii) (resp. (iv)) is clear since, by virtue of Proposition 3.1, one may assume that $A = (0, 1, \ldots, n)$ (resp. $(0, 1, \ldots, n-1, n+1)$).

To proceed, we need the following

Lemma 3.4. *(i) If $(b_{k,n-1}; b_{k+1,1}) \cap v_m(A) = \{b_{kn}\}$ for some $k \in [1; m-1]$, then $a_n - a_{n-1} = a_1 - a_0$.*

(ii) If $a_{j+1} - a_j = a_1 - a_0$ for some $j \in [2; n-1]$ and $(b_{k,j-1}; b_{k,j+1}) \cap v_m(A) = \{b_{kj}\}$ for some $k \in [1; m-1]$, then $a_j - a_{j-1} = a_1 - a_0$.

Proof of the lemma. To prove (i), observe that the number $x = (m-k-1)a_0 + a_1 + a_{n-1} + (k-1)a_n \in v_m(A)$ belongs to the interval $(b_{k,n-1}; b_{k+1,1})$; thus $x = b_{kn}$, whence $a_n - a_{n-1} = a_1 - a_0$.

To prove (ii), observe that the hypothesis implies the equality

$$b_{k,j+1} = (m-k-1)a_0 + a_1 + a_j + (k-1)a_n \ .$$

Hence, the number $x = (m-k-1)a_0 + a_1 + a_{j-1} + (k-1)a_n$ belongs to the interval $(b_{k,j-1}; b_{k,j+1})$; thus $x = b_{kj}$, whence $a_j - a_{j-1} = a_1 - a_0$.

To prove assertion (iii) of Proposition 3.3 together with the "only if" part of (ii), it suffices to prove that A is an arithmetic progression whenever $\dim S_m^A < m(n+1)$. Indeed, in this case at most $m-2$ elements of $v_m(A)$ are not contained in the chain (2). Since for $k \in [1; m-1]$ the intervals $(b_{k,n-1}; b_{k+1,1})$ are disjoint, it follows that $(b_{k,n-1}; b_{k+1,1}) \cap v_m(A) = \{b_{kn}\}$ for some $k \in [1; m-1]$, whence $a_n - a_{n-1} = a_1 - a_0$ by virtue of Lemma 3.4(i). The same argument shows that for each $i \in [2; n-1]$ there exists a number $k_i \in [1; n]$ such that $(b_{k_i,i-1}; b_{k_i,i+1}) \cap v_m(A) = \{b_{k_i,i}\}$. Applying Lemma 3.4(ii) $n-2$ times, we see that $a_{n-1} - a_{n-2} = \ldots = a_1 - a_0$, i.e. A is an arithmetic progression.

It remains to prove the "only if" part of (iv). First we do it for $m = 2$:

Lemma 3.5. *Suppose that $A \in \mathscr{A}_n$, where $n \geq 3$, and $\dim S_2^A = 2n+2$. Then either $a_n - a_{n-1} = 2(a_j - a_{j-1})$ for all $j \in [1; n-1]$ or $a_1 - a_0 = 2(a_j - a_{j-1})$ for all $j \in [2; n]$.*

Proof of the lemma. For any $A = (a_0, \ldots, a_n) \in \mathscr{A}_n$, put $A' = (a_0, \ldots, a_{n-1}) \in \mathscr{A}_{n-1}$ and $A'' = (a_1, \ldots, a_n) \in \mathscr{A}_{n-1}$. Observe that

$$\dim S_2^A \geq \dim S_2^{A'} + 2 \qquad \text{and } \dim S_2^A \geq \dim S_2^{A''} + 2 \ . \tag{3}$$

Indeed, since A is a strictly increasing sequence, we see that $v_2(A) \setminus v_2(A') \supset \{a_{n-1}+a_n, 2a_n\}$ and $v_2(A) \setminus v_2(A'') \supset \{a_0+a_1, 2a_0\}$, whence the desired inequality.

Now we proceed by induction on n. For $n = 3$, the proof is straightforward. Suppose now that the lemma is proved for all $A \in \mathscr{A}_k$, where $k < n$, and that $n > 3$. If $A \in \mathscr{A}_n$ and $\dim S_2^A = 2n+2$, then (3) implies that $\dim S_2^{A'} \leq 2n$ and $\dim S_2^{A''} \leq 2n$. Moreover, we see that one of these inequalities is actually an equality (if this is not the case, then part (ii) of Proposition 3.3, which we have already proved, implies that A' and A'' are arithmetic progressions, whence A is an arithmetic progression and $\dim S_2^A = 2n+1$, contrary to the assumption).

From now on we will assume that $\dim S_2^{A'} = 2n$ (the argument for the case when $\dim S_2^{A''} = 2n$ is the same if one mirrors the indices with respect to $n/2$). Observe that $\{a_{n-1}+a_n, 2a_n\} \cap v_2(A') = \emptyset$; it follows from our assumptions that $v_2(A) = v_2(A') \cup \{a_{n-1}+a_n, 2a_n\}$, whence $a_{n-2}+a_n \in v_2(A')$. The only element of $v_2(A')$ that is *a priori* not less than $a_{n-2}+a_n$, is $2a_{n-1}$. Hence, $a_{n-2}+a_n = 2a_{n-1}$.

If we apply the induction hypothesis to A', we see that either $a_1, \dots, a_{n-1}$ is an arithmetic progression with step d and $a_1 - a_0 = 2d$, or $a_0, \dots, a_{n-2}$ is an arithmetic progression with step d and $a_{n-1} - a_{n-2} = 2d$. In the first case it is clear that $a_1, \dots, a_n$ is an arithmetic progression with the same step d, and we are done. In the second case it is easy to show that $\dim S_2^A = 2n+3$, which contradicts the hypothesis. This completes the proof of the lemma.

To prove (iv) in full generality, we may assume that $m > 2$. Denote by $\mathscr{E}$ the set of elements of $v_m(A)$ that do not belong to the chain (2). Suppose that $\operatorname{card} v_m(A) = m(n+1)$, i.e. $\operatorname{card} \mathscr{E} = m-1$. I claim that each of the $m-1$ intervals $(b_{k,n-1}; b_{k+1,1})$ contains exactly one element of $\mathscr{E}$. Indeed, assume the converse; then Lemma 3.4(i) implies that $a_n - a_{n-1} = a_1 - a_0$. Moreover, arguing as in the proof of assertion (iii) above, we see that for each $i \in [2; n-1]$ there exists a number $k_i \in [1; n]$ such that $(b_{k_i,i-1}; b_{k_i,i+1}) \cap v_m(A) = \{b_{k_i,i}\}$; applying Lemma 3.4(ii) $n-2$ times we see that A is an arithmetic progression – a contradiction.

Hence, for each $k \in [1; m-1]$ there exists a number $x_k \in (b_{k,n-1}; b_{k+1,1})$, $x_j \neq b_{kn}$, and each of the intervals $(b_{i-1,1}; b_{i+1,1})$, $i \in [2; n-1]$ does not contain elements of $v_m(A)$. Now Lemma 3.4(ii) implies that A is an arithmetic progression whenever $a_n - a_{n-1} = a_1 - a_0$. Hence, $a_n - a_{n-1} \neq a_1 - a_0$, and for any $k \in [1; n-1]$ the number $(m-k-1)a_0 + a_1 + a_{n-1} + (k-1)a_n \in v_m(A)$ belongs to the interval $(b_{k,n-1}; b_{k+1,1})$ and does not coincide with b_{kn}. This shows that $x_k = (m-k-1)a_0 + a_1 + a_{n-1} + (k-1)a_n$. If is clear that either $b_{k,n-1} < x_k < b_{kn}$ for all $k \in [1; n-1]$ (this is the case if $a_1 - a_0 < a_n - a_{n-1}$), or $b_{kn} < x_k < b_{k+1,1}$ (this is the case if $a_1 - a_0 > a_n - a_{n-1}$).

In the first case, there are exactly $2n+2$ elements of $v_m(A)$ in the interval $[b_{m-2,n}; b_{mn}]$. Since a number $x \in v_m(A)$ belongs to $[b_{m-2,n}; b_{mn}]$ whenever $x = (m-2)a_n + y$ with $y \in v_2(A)$, we see that $\dim S_2^A = 2n+2$, and the conclusion about A follows from Lemma 3.5.

In the second case the argument is analogous if we consider $v_m(A) \cap [ma_0; b_{1n}]$. This completes the proof.

Remark 3.6. In Corollary 5.9 below we give an explicit upper bound for the number m_0 starting with which $\dim S_m^A$ becomes a linear function of m.

By way of an amusing application, we show how Proposition 3.3(iv) yields a proof of the following well-known

Proposition 3.7. *Suppose that the characteristic is zero. If $C \subset \mathbb{P}^n$ is an elliptic curve of degree $n+1$, then C has exactly $(n+1)^2$ distinct inflection points, and the weight of each of these points equals 1 (or, equivalently, the vanishing sequence is of the form $(1, \dots, n-1, n+1)$).*

Proof. It follows immediately from the Plücker formula [5, Proposition 1.1] that $\sum_{p\in C} w(p) = (n+1)^2$; to prove that the $(n+1)^2$ inflection points are distinct, we show that $w(p) = 1$ for any inflection point $p \in C$. To that end, denote by A the vanishing sequence at p; since C is contained in $n(n-1)/2 - 1$ linearly independent quadrics, we see that $h_C(2) \le 2n+2$, whence $\dim S_2^A \le 2n+2$ by virtue of Proposition 3.1. Since $a_0 = 0$ and $a_1 = 1$, we see, by Proposition 3.3(iv), that $A = (0, 1, \dots, n-1, n+1)$ and $w(p) = 1$.

Remark 3.8. Of course, this fact can be established directly by Riemann-Roch. The $(n+1)^2$ inflection points are points of order $n+1$ on the elliptic curve.

4. Deformations to monomial curves

Let $(\mathscr{L}, V)$ be a linear system on a smooth curve C, where $\dim V = n+1$. Denote by S the homogeneous coordinate ring of the curve $\overline{\phi(C)} \subset \mathbb{P}^n$, where ϕ is the rational mapping defined by our linear system. One has $S = R/I$, where $R = k[X_0, \dots, X_n]$ and I is the homogeneous ideal of $\overline{\phi(C)}$.

Fix a point $p \in C$, and denote by $A = (a_0, \dots, a_n)$ the vanishing sequence at p. For any $j, m \ge 0$, put $F^j R_m = \sum_{l\ge j} R_m^l$ and $F^j R = \sum_m F^j R_m$ (cf. the end of Section 2). Filtration F induces filtrations on I and S; both of these filtrations will be also denoted by F. The associated graded ring $\operatorname{gr} S$ equals $\bigoplus_{i,m}(F^i S_m / F^{i+1} S_m)$; thus, $\operatorname{gr} S$ becomes a bigraded ring.

Lemma 4.1. *In the above setting,* $\operatorname{pr}^j(F^j I_m) \subset J^A \cap R_m^j$.

Proof. Fix an isomorphism $\psi: \mathscr{L}_p \to \mathscr{O}_p$ and a generator $\pi \in \mathfrak{m}_p$. One may choose a basis $(s_0, \dots, s_n)$ in V in such a way that $\psi(s_i) \equiv \pi^{a_i} \bmod \mathfrak{m}_p^{a_i+1}$. Observe that $G(s_0, \dots, s_n) \equiv \operatorname{pr}^j(G)(1, \dots, 1) \cdot \pi^j \pmod{\pi^{j+1}}$ for any $G \in F^j R$, whence $\operatorname{pr}^j(G)(1, \dots, 1) = 0$ whenever $G \in F^j I$. Since $\operatorname{pr}^j(G)$ is quasihomogeneous, we see that $\operatorname{pr}^j(G)$ vanishes on C^A, which proves the lemma.

Proposition 4.2. *There is a natural epimorphism $\lambda: \operatorname{gr} S \to S^A$, where $\operatorname{gr} S$ is the associated graded ring of S with respect to the filtration F and $A = (a_0, \dots, a_n)$ is the vanishing sequence of $(\mathscr{L}, V)$ at p. This homomorphism respects the bigradings on $\operatorname{gr} S$ and S^A.*

Proof. In view of Lemma 4.1, the homomorphism

$$\varphi_m^j : F_p^j S_m / F_p^{j+1} S_m \to (S^A)_m^j$$

that acts by the formula $G(s_0, \dots, s_n) \mapsto \operatorname{pr}^j(G) \bmod J^A$, where $G \in F^j R_m$, is well defined. It is clear that $\lambda = \bigoplus_{m,j} \varphi_m^j$ is the desired epimorphism.

Corollary 4.3. $\dim S_m \ge \dim S_m^A$.

This corollary suggests the following

Definition 4.4. *A linear system $(\mathscr{L}, V)$ on a curve C is called m-extremal at a point $p \in C$ if* $\dim S_m = \dim S_m^A$, *where A is the vanishing sequence of $(\mathscr{L}, V)$ at p.*

Corollary 4.5. *In the above setting, a linear system $(\mathscr{L}, V)$ is m-extremal iff* $\mathrm{pr}^j(F^j I_m) = J_m^A \cap R^j$ *for all j.*

Here's how the definition of m-extremality works:

Proposition 4.6. *Let $(\mathscr{L}, V)$ be a linear system on a smooth curve C, $p \in C$ a point, and $A \in \mathscr{A}_n$ the vanishing sequence of $(\mathscr{L}, V)$ at p. If $(\mathscr{L}, V)$ is m-extremal at p and $\sum_{t \geq m} J_t^A = RJ_m^A$, then the linear system $(\mathscr{L}, V)$ is t-extremal at p for all $t \geq m$.*

Proof. We resume the notation from the proof of Proposition 4.2. Denote by $\lambda_t : S_t \to S_t^A$ the t-th component of λ. We are to show that λ_t is injective whenever $t \geq m$. To that end, assume that $0 \neq \xi \in \ker \lambda_t$. Choose a polynomial $G \in R_t$ such that $\xi = G(s_0, \dots, s_n)$ and $G \in F^j R_t \setminus F^{j+1} R_t$ with the greatest possible j. Since $\lambda(\xi) = 0$, we see that $\mathrm{pr}^j(G) \in J_t^A$; the hypothesis implies that $\mathrm{pr}^j G = \sum P_i Q_i$, where all the Q_i's lie in J_m^A and are quasihomogeneous. Hence, $G = \sum P_i Q_i + G_1$, where $G_1 \in F^{j+1} R_t$. Since λ_m is injective, we see that $Q_i(s_0, \dots, s_n) = 0$ for all i, whence $\xi = G_1(s_0, \dots, s_n)$, contrary to our choice of j. This contradiction completes the proof.

Another application of the notion of m-extremality is the following

Proposition 4.7. *If a linear system $(\mathscr{L}, V)$ on a curve C is m-extremal at a point p with vanishing sequence A and if $\sum_{t \geq m} J_t^A = RJ_m^A$, then $\sum_{t \geq m} I_t = RI_m$.*

Proof. Since $F^j I_t = 0$ for $j \gg 0$, the proposition will follow once we have proved that $F^j I_t \subseteq RI_m + F^{j+1} I_t$ for any $t \geq m$ and any j. To prove these inclusions, suppose that $f \in F^j I_t$. Then $\mathrm{pr}^j(f) \in J_t^A \cap R^j$ by virtue of Lemma 4.1. Since $J_t^A = R_{t-m} J_m^A$ by hypothesis, we see that $pr^j(f) = \sum g_\alpha h_\alpha$, where $g_\alpha \in R_{t-m}^{j-n_\alpha}$ and $h_\alpha \in R^{n_\alpha} \cap J_m^A$. Since our linear system is m-extremal, Corollary 4.5 implies that $h_\alpha = \mathrm{pr}^{n_\alpha}(\tilde{h}_\alpha)$, where $\tilde{h}_\alpha \in F^{n_\alpha} I_m$. Hence,

$$\mathrm{pr}^j(f) = \sum g_\alpha \,\mathrm{pr}^{n_\alpha}(\tilde{h}_\alpha) = \mathrm{pr}^j\left(\sum g_\alpha h_\alpha\right);$$

it is clear that $f - \sum g_\alpha \tilde{h}_\alpha \in F^{j+1} I_t$, and $\sum g_\alpha h_\alpha \in RI_m$ by construction. This completes the proof of the proposition.

The following proposition says that a projective curve X can be deformed to its associated monomial curve provided that "many" hypersurfaces of given degree contain X.

Proposition 4.8. *Suppose that the linear system $(\mathscr{L}, V)$ is m-extremal at a point $p \in C$ for some m and denote by A the vanishing sequence of $(\mathscr{L}, V)$ at p. If $\sum_{j \geq m} J_m^A = RJ_m^A$, then degree and arithmetic genus of $X = \phi(C)$ are equal to those of C^A.*

Proof. Proposition 4.6 shows that $(\mathscr{L}, V)$ is t-extremal for all $t \geq m$. Hence, Hilbert polynomials of X and C^A coincide.

Degree and arithmetic genus of the rational curve C^A are easily computable. First observe that, in view of Proposition 3.1, one may assume that $A = (a_0, \dots, a_n)$, where $a_0 = 0$ and g.c.d. of a_j's equals 1.

Proposition 4.9. *Denote by $\nu: \mathbb{P}^1 \to \mathbb{P}^n$ the morphism that sends $(1 : t)$ to $(t^{a_0} : t^{a_1} : \dots : t^{a_n})$. Under the above assumptions on A, $\nu(\mathbb{P}^1) = C^A$, ν is an isomorphism on $\mathbb{P}^1 \setminus \{0, \infty\}$, $\nu^* \mathscr{O}_{\mathbb{P}^n}(r) = \mathscr{O}_{\mathbb{P}^1}(ra_n)$ for all r, and degree of C^A equals a_n.*

Proof. The first assertion is obvious. It follows from the hypothesis that there exist integers $c_1, \dots, c_n$ such that $\sum c_i a_i = 1$. The rational map $\psi: C^A \to \mathbb{A}^1$ which sends $(1 : x_1 : \dots : x_n)$ to $\prod x_i^{c_i}$ is the inverse to the restriction of ν to $\mathbb{P}^1 \setminus \{0, \infty\}$. Since $\nu^* \mathscr{O}_{\mathbb{P}^n}(1) = \mathscr{O}_{\mathbb{P}^1}(a_n)$ and ν is birational, we see that $\deg C^A = a_n$.

Computation of the arithmetic genus of C^A will be a particular case of the following

Proposition 4.10. *Suppose that $A = (0, a_1, \dots, a_n)$ is an abstract vanishing sequence such that g.c.d. of a_j's equals 1, and that $X = C^A \subset \mathbb{P}^n$ is the corresponding monomial curve. Put $\hat{A} = (0, b_1, \dots, b_n) \in \mathscr{A}_n$, where $b_j = a_n - a_{n-j}$. If $r \in \mathbb{N}$, then*

$$\dim H^1(X, \mathscr{O}_X(r)) = \operatorname{card}(\gamma(A) \setminus [0; ra_n]) + \operatorname{card}(\gamma(\hat{A}) \setminus [0; ra_n]) + \operatorname{card}(\gamma(A) \cap \mu_{ra_n}(\gamma(\hat{A})) \cap [0; ra_n]); \tag{4}$$

$$\dim H^1(\mathscr{I}_X(r)) = \operatorname{card}([0; ra_n] \setminus (v_r(A) \cup \gamma(A) \cup \mu_{ra_n}(\gamma(\hat{A})))) , \tag{5}$$

where $\gamma(B)$ stands for the set of gaps of the semigroup generated by a vanishing sequence B and the map $\mu_d: \mathbb{Z} \to \mathbb{Z}$ acts by the formula $r \mapsto d - r$.

Proof. We will use the notation from the previous proposition. Till the rest of the proof $H^0(\mathscr{O}_{\mathbb{P}^1}(ra_n))$ will be identified with the space of polynomials in t of degree $\leq ra_n$. Consider an exact sequence

$$0 \to \mathscr{L} \to \nu_* \nu^* \mathscr{L} \to \mathscr{Q} \to 0 ; \tag{6}$$

Since ν is an isomorphism on $\mathbb{P}^1 \setminus \{0, \infty\}$ by Proposition 4.9, we see that $\operatorname{supp}(\mathscr{Q}) \subseteq \{\nu(0), \nu(\infty)\}$ and $\mathscr{Q} \cong \nu_* \mathscr{O}_{\mathbb{P}^1}/O_X$. The stalk of the latter sheaf at $\nu(0)$ is isomorphic to $k[[t]]/k[[t^{a_1}, \dots, t^{a_n}]] \cong k^{\gamma(A)}$, and its stalk at $\nu(\infty)$ is isomorphic to $k[[s]]/k[[s^{b_1}, \dots, s^{b_n}]] \cong k^{\gamma(\hat{A})}$, where $s = t^{-1}$. It is clear from the sequence (6) that $H^1(X, \mathscr{L}) \cong \operatorname{coker}(\varphi: H^0(\nu_* \nu^* \mathscr{L}) \to H^0(\mathscr{Q}))$. Identifying source (resp. target) of φ with the space of polynomials in t of degree $\leq ra_n$ (resp. $k^{\gamma(A)} \oplus k^{\gamma(\hat{A})}$), put $\varphi(f) = (\alpha(f), \beta(f)) \in k^{\gamma(A)} \oplus k^{\gamma(\hat{A})}$. It is easily seen that $\alpha(f) \in k^{\gamma(A)}$ is the family $\{p_g\}_{g \in \gamma(A)}$, where p_g is the coefficient of f at t^g, and

$\beta(f) \in k^{\gamma(\hat{A})}$ is the family $\{q_g\}_{g\in\gamma(\hat{A})}$, where q_g is the coefficient of $t^{ra_n}f(t^{-1})$ at t^g. Now it is clear that $h^1(\mathscr{L}) = \dim\operatorname{coker}\varphi$ equals the right-hand side of (4).

To prove (5), observe that $H^1(\mathscr{I}_X(r)) \cong \operatorname{coker} f$, where $f: H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(X, \mathscr{L})$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} & & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) & = & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) & \longrightarrow & 0 \\ & & \Big\downarrow f & & \Big\downarrow g & & \Big\downarrow \\ 0 & \longrightarrow & H^0(X, \mathscr{L}) & \longrightarrow & H^0(\mathbb{P}^1, \nu^*\mathscr{L}) & \xrightarrow{\varphi} & k^{\gamma(A)} \oplus k^{\gamma(\hat{A})}, \end{array}$$

where the lower row is H^0 of the exact sequence (6). Since $\operatorname{coker} g \cong k^{[0;ra_n]\setminus v_r(A)}$, this diagram yields the exact sequence

$$0 \longrightarrow \operatorname{coker} f \longrightarrow k^{[0;ra_n]\setminus v_r(A)} \xrightarrow{\bar{\varphi}} k^{\gamma(A)} \oplus k^{\gamma(\hat{A})}.$$

Let $p \in k^{[0;ra_n]\setminus v_r(A)}$ be a family $\{p_t\}_{t\in[0;ra_n]\setminus v_r(A)}$. It is clear that $\bar{\varphi}(p) = (\alpha, \beta) \in k^{\gamma(A)} \oplus k^{\gamma(\hat{A})}$, where $\alpha = \{u_t\}_{t\in\gamma(A)}$ with $u_t = p_t$ whenever $t \le ra_n$ and 0 otherwise, and $\beta = \{v_t\}_{t\in\gamma(\hat{A})}$ with $v_t = ra_n - t$ whenever $t \le ra_n$ and 0 otherwise. Since $\operatorname{coker} f \cong \ker\bar{\varphi}$, Eq. (5) follows.

Corollary 4.11. *If $A = (0, a_1, \ldots, a_n)$ is an abstract vanishing sequence such that g.c.d. of a_j's equals 1, then the arithmetic genus of C^A equals $L_0 + L_\infty$, where L_0 is the number of gaps of the semigroup $\langle a_1, \ldots, a_n\rangle$ and L_∞ is the number of gaps of the semigroup $\langle a_n - a_{n-1}, \ldots, a_n - a_1, a_n\rangle$.*

Proof. Put $r = 0$ in Eq. (4).

5. Equations defining certain monomial curves

To be able to apply Proposition 4.8, one should know degrees of generators of the homogeneous ideal of the curve C^A. In general, it is not clear how these degrees depend on A. In this section we show at first that in the simplest cases J^A is generated by quadrics, and then we use a result from [9] to obtain some information about the general case.

Proposition 5.1. *Let $A \in \mathscr{A}_n$ be an increasing sequence of integers. Suppose that one of the following conditions holds:*

(i) $n \ge 2$ and A is an arithmetic progression;
(ii) $n \ge 3$ and either $a_n - a_{n-1} = 2(a_j - a_{j-1})$ for all $j \in [1; n-1]$ or $a_1 - a_0 = 2(a_j - a_{j-1})$ for all $j \in [2; n]$.

Then $J^A = RJ_2^A$ and $\sum_{j\ge m} J_m^A = RJ_m^A$ for all $m \ge 2$.

Remark 5.2. If $n = 2$ in the case (ii), then C^A is the plane cuspidal cubic, which is not defined by quadratic equations.

Remark 5.3. The curve $C^A \subset \mathbb{P}^n$ is a rational normal curve in the case (i) and a linearly normal curve of arithmetic genus 1 in the case (ii). It is well known that homogeneous ideals of such curves are generated by their components of degree 2 (this is especially true for the case (i)), but it is not easy to give a suitable reference. Our assertion must follow from the algorithm for finding minimal systems of generators of J^A that is proposed in [4], but we prefer to give direct proofs.

Proof. Of course, it suffices to show that $J^A = RJ_2^A$. The result for the case (i) follows, for example, from the fact that rational normal curves are 2-regular. To prove (ii), note first that we may, by virtue of Proposition 3.1, assume that $a_j = j$ for $j > n$ and $a_n = n + 1$. Observe that C^A, being a linearly normal curve of arithmetic genus 1, is 3-regular. Thus, $J_m^A \subset RJ_3^A$ for $m \geq 3$, and it remains to show that $J_3^A = R_1 J_2^A$.

To that end observe that J_3^A is generated, as a linear space, by differences of monomials of equal A-weight. Denote by $J' \subset J_3^A$ the subspace generated by the monomials of the form $x_i x_j x_k - x_l x_m x_p$, where all the indices are less than n and $i + j + k = l + m + p$. Since $J' = J_3^B$, where $B = (0, \ldots, n-1) \in \mathscr{A}_{n-1}$, it follows from case (i) that $J' \subset R_1 J_2$. Suppose that

$$\xi = x_i x_j x_k - x_l x_m x_p \in J_3^A \,. \tag{7}$$

To prove that $\xi \in R_1 J_2^A$, we may assume that $\{i, j, k\} \cap \{l, m, n\} = \emptyset$ (otherwise the two monomials in the right-hand side of Eq. (7) have a common factor) and that at least one of the indices, say k, equals n (otherwise $\xi \in J' \subset R_1 J_2^A$ and there is nothing more to prove). After a suitable reordering of indices, only three cases are possible:

1. $k = n,\ j < n,\ i \leq n - 3$;
2. $k = n,\ i = j = n - 2$;
3. $j = k = n,\ i \leq n - 3$.

In the first case one can write

$$\xi = x_j (x_i x_n - x_{i+2} x_{n-1}) + (x_j x_{i+2} x_{n-1} - x_l x_m x_p)$$

with the first summand in $x_j J_2^A$ and the second in J', whence $\xi \in R_1 J_2^A$. In the second case the only possibility for l, m, and p is that they all equal $n - 1$; hence,

$$\xi = x_{n-2}^2 x_x - x_{n-1}^3 = x_n (x_n^2 - x_{n-3} x_{n-1}) + x_{n-1} (x_{n-3} x_n - x_{n-1}^2) \in R_1 J_2^A$$

(it is here that we use the assumption $n \geq 3$). Finally, in the third case we have

$$\xi = x_n (x_i x_n - x_{i+2} x_{n-1}) + (x_{i+2} x_{n-1} x_n - x_l x_m x_p) \,,$$

where the first summand is in $x_n J_2^A$ and the second summand is in $R_1 J_2^A$ by virtue of what we have proved for the first two cases. This completes the proof.

Our next objective is to prove a result on t-regularity of monomial curves. We derive it from results of the paper [9]. We start with an auxiliary result.

Proposition 5.4. *Let $A = (0 = a_0, a_1, \dots, a_n)$ be an abstract vanishing sequence. Identifying $H^0(\mathscr{O}_{\mathbb{P}^1}(m))$ with the vector space of degree m binary forms in u and v, denote by V the subspace of $H^0(\mathscr{O}_{\mathbb{P}^1}(a_n))$ spanned by $u^{a_n - a_j} v^{a_j}$ for $j \in [0; n]$. If we consider V as linear system on $\mathbb{P}^1$, then $\mathscr{M}_V \cong \sum_{i=1}^n \mathscr{O}_{\mathbb{P}^1}(a_{i-1} - a_i)$.*

Proof. It follows from the definition that $\mathscr{M}_V \subset (\mathscr{O}_{\mathbb{P}^1})^{n+1}$ consists of $(n+1)$-tuples $(f_0, \dots, f_n)$ of local sections of $\mathscr{O}_{\mathbb{P}^1}$ such that

$$u^{a_n - a_0} v^{a_0} f_0 + u^{a_n - a_1} v^{a_1} f_1 + \dots + v^{a_n} f_n = 0 .$$

For each $j \in [1; n]$ define an injection $s_j : \mathscr{O}_{\mathbb{P}^1}(a_{j-1} - a_j) \to (\mathscr{O}_{\mathbb{P}^1})^{n+1}$ as $\sigma \mapsto (\sigma_0, \dots, \sigma_n)$, where $\sigma_{j-1} = v^{a_j - a_{j-1}} \sigma$, $\sigma_j = -u^{a_j - a_{j-1}} \sigma$, and $\sigma_i = 0$ for $i \neq j, j-1$. Put $\mathscr{M}' = \sum \mathscr{O}_{\mathbb{P}^1}(a_{i-1} - a_i)$. It is clear that $\bigoplus s_j$ defines an injection of $\mathscr{M}'$ into $\mathscr{M}_V$. Since $a_0 = 0$, the linear system V has no base points, whence $\deg \mathscr{M}_V = -\deg \mathscr{O}_{\mathbb{P}^1}(a_n) = -a_n$. On the other hand, $\deg \mathscr{M}' = -a_n$, as well. The locally free sheaves $\mathscr{M}' \subseteq \mathscr{M}_V$ on a smooth curve have the same rank and degree. Hence, $\mathscr{M}' = \mathscr{M}_V$.

Now we can state a result on t-regularity of monomial curves.

Proposition 5.5. *Let $A = (0 = a_0, a_1, \dots, a_n)$ be an abstract vanishing sequence such that g.c.d. of a_j's equals 1, and let $C^A \subset \mathbb{P}^n$ be the corresponding monomial curve. Put $\delta = \max_{1 \le i < j \le n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\}$. Then the curve C^A is δ-regular.*

Remark 5.6. The bound provided by this proposition cannot be sharpened uniformly for all vanishing sequences $A = (a_0, \dots, a_n) \in \mathscr{A}_n$. For example, take an integer $l > n$ and put $a_0 = 0$, $a_1 = 1$, and $a_j = l - n + j$ for $j \ge 2$. Then $H^1(\mathscr{I}_{C^A}(l - n)) \neq 0$ (this follows from Proposition 4.10, because $l - n + 1 \notin v_{l-n}(A)$), whence the curve C^A is not $(l - n + 1)$-regular. Since $\delta = l - n + 2$, we see that the bound is sharp for this curve.

On the other hand, there are examples when this bound is very far from being optimal. To wit, suppose that $l \ge 0$ is an integer and consider an $A \in \mathscr{A}_n$ such that $a_j = j$ for $j \in [0; n-1]$ and $a_n = n + l$. Then it can be checked that the curve C^A is t-regular, where $t = \left\lceil \dfrac{l}{n-1} \right\rceil + 2$. For large n this t is much less than $\delta = l + 2$.

Proof of the proposition. Recall a result of Gruson, Lazarsfeld and Peskine:

Proposition 5.7 ([9], Proposition 1.2). *Let $(\mathscr{L}, V)$ be a linear system without base points on a smooth curve $\tilde{C}$. Suppose that this linear system defines a birational morphism of $\tilde{C}$ onto the curve $C \subset \mathbb{P}^n$. If $\mathscr{A}$ is an invertible sheaf on $\tilde{C}$ such that $H^0(\tilde{C}, \wedge^2 \mathscr{M}_V \otimes \mathscr{A}) = 0$, then C is $h^0(\mathscr{A})$-regular.*

To derive our proposition from this result, put $\tilde{C} = \mathbb{P}^1$, $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(a_n)$. If V is as in the proof of Proposition 5.4, then $C = C^A$. Put $\mathscr{A} = \mathscr{O}_{\mathbb{P}^1}(\delta - 1)$. Then Proposition 5.4 implies that $H^0(\wedge^2 \mathscr{M}_V \otimes \mathscr{A}) = 0$. Since $h^0(\mathscr{A}) = \delta$, we are done.

In the corollary that follows, R stands for $k[X_0, \dots, X_n]$ and J^A is as in Section 4.

Corollary 5.8. *Suppose that $A = (0 = a_0, a_1, \dots, a_n) \in \mathscr{A}_n$ and that g.c.d. of a_j's equals 1. If δ is the same as in Proposition 5.5, then $J^A = RJ^A_\delta$.*

Our next corollary strengthens Theorem 4.1 from [1].

Corollary 5.9. *Suppose that $A = (0 = a_0, a_1, \dots, a_n) \in \mathscr{A}_n$. Denote by d the g.c.d. of a_j's and put $\delta = \max_{1 \le i < j \le n}\{((a_i - a_{i-1}) + (a_j - a_{j-1}))/d\}$. Then for all $m \ge \delta - 1$ one has $\dim S^A_m = (a_n/d) \cdot m + 1 - L_0 - L_\infty$, where L_0 is the number of gaps of the semigroup $\langle (a_1 - a_0)/d, \dots, (a_n - a_0)/d \rangle$ and L_∞ is the number of gaps of the semigroup $\langle (a_n - a_{n-1})/d, \dots, (a_n - a_0)/d \rangle$.*

Proof. By virtue of Proposition 3.1, one may assume that $a_0 = 0$ and g.c.d. of a_j's equals 1. If C^A is the monomial curve corresponding to A, then it is δ-regular by Proposition 5.5. Hence, $\dim S^A_m = \chi(\mathscr{O}_{C^A}(m))$ for all $m \ge \delta - 1$. Now the conclusion follows from Proposition 5.1 and Corollary 4.11.

Finally, let us restate Proposition 5.5 in combinatorial terms.

Corollary 5.10. *Suppose that $a_1 < \dots < a_n$ is an increasing sequence of integers such that g.c.d. of a_j's equals 1. Put $a_0 = 0$ and $\delta = \max_{1 \le i < j \le n}\{(a_i - a_{i-1}) + (a_j - a_{j-1})\}$. Then:*

(i) Conductor of the semigroup $\langle a_1, \dots, a_n \rangle$ is at most $(\delta - 2)a_n + 1$.
(ii) Put $b_j = a_n - a_{n-j}$; if $r \ge \delta - 2$ and $x \in [0; ra_n]$ is a gap of the semigroup $\langle a_1, \dots, a_n \rangle$, then $ra_n - x$ is a non-gap of $\langle b_1, \dots, b_n \rangle$.
(iii) If $r \ge \delta - 1$, then any $x \in [0; ra_n]$ such that x is not a gap of A and $ra_n - x$ is not a gap of $\hat{A}$, lies in $v_r(A)$.

Proof. Put $A = (0, a_1, \dots, a_n) \in \mathscr{A}_n$. The curve $X = C^A$ is δ-regular by Proposition 5.5. Now Proposition 4.10 applies: since $H^2(\mathscr{J}_X(\delta - 2)) = H^1(\mathscr{O}_X(\delta - 2)) = 0$, we have (i) and (ii); since $H^1(\mathscr{J}_X(\delta - 1)) = 0$, we have (iii).

6. Proofs of main results

Proof of Theorem 1.4. If $X \subset \mathbb{P}^n$ is an irreducible non-degenerate curve, then X is the image of a birational morphism $f: C \to \mathbb{P}^n$ defined by a linear system $(\mathscr{L}, V)$, where C is a smooth curve. If $p \in C$ is any point with the vanishing sequence $A = (a_0, \dots, a_n)$, then Proposition 3.3(i) shows that $\dim S^A_m \ge mn + 1$, whence $\dim \operatorname{im}(\operatorname{Sym}^m(V) \to H^0(\mathscr{L}^{\otimes m})) \ge mn + 1$ by Proposition 3.2. The first part of the theorem follows immediately from this inequality.

The fact that the bound is attained for rational normal curves is trivial. Hence, we can assume for the sequel that $h^0(\mathscr{J}_X(m)) = \binom{m+n}{n} - mn - 1$, or, equivalently, that $\dim \operatorname{im}(\operatorname{Sym}^m(V) \to H^0(\mathscr{L}^{\otimes m})) = mn + 1$, and we are to prove that X is a rational normal curve.

To that end, pick a point $p \in C$ and denote by $A = (a_0, \dots, a_n)$ its vanishing sequence at p. It follows from Proposition 3.2 that $\dim S_m^A \le mn + 1$. Now Proposition 3.3(i,ii) shows that A is an arithmetic progression and that $(\mathscr{L}, V)$ is m-extremal at p.

Hence, C^A is a normal rational curve of degree n in $\mathbb{P}^n$. Propositions 4.8 and 5.1 show that X has degree n and arithmetic genus 0 as well, whence X is a rational normal curve.

Proof of Theorem 1.7. Denote by $\nu: C \to X$ the normalization of X and consider the linear system $(\mathscr{L}, V)$, where $\mathscr{L} = \nu^*\mathscr{O}_{\mathbb{P}^n}(1)$ and $V = \operatorname{im}(H^0(\mathscr{O}_{\mathbb{P}^n}(1) \to H^0(\mathscr{O}_C(1))$. If this linear system has no inflection points, then X is a normal rational curve by Proposition 2.1, whence $h^0(\mathscr{I}_X(m)) = \binom{m+n}{n} - mn - 1$, which contradicts the hypothesis. Hence, we may assume that there is an inflection point $p \in C$. Denote its vanishing sequence by $A = (a_0, \dots, a_n)$. Lemma 2.2 implies that g.c.d. of a_j's equals 1. Hence, A is not an arithmetic progression and proposition 3.3(iii) implies that $\dim S_m^A \ge m(n+1)$, whence $\dim \operatorname{im}\left(\operatorname{Sym}^m V \to H^0(\mathscr{L}^{\otimes m})\right) \ge m(n+1)$ by virtue of Proposition 3.2. Thus, the first assertion of the theorem follows.

To prove the second assertion, suppose that $h^0(\mathscr{I}_X(m)) = \binom{m+n}{n} - m(n+1)$. Propositions 3.3(iv) and 3.2 imply that A is of the form $(0, 1, \dots, n-1, n+1)$ or $(0, 2, \dots, n+1)$ and the linear system $(\mathscr{L}, V)$ is 2-extremal at p. Corollary 5.1 shows that Proposition 4.8 applies, whence the degree and arithmetic genus of X equal those of C^A. Since C^A is a cuspidal rational curve of degree $n+1$ and arithmetic genus 1, we are done.

Proof of Theorems 1.2 and 1.6. We will proceed by induction on $\dim X$. If $\dim X = 1$, then our theorem are just the $m = 2$ case of Theorems 1.4 and 1.7. If $\dim X > 1$, consider a generic hyperplane section $Y \subset X$. There is an exact sequence

$$0 \to \mathscr{I}_X(1) \to \mathscr{I}_X(2) \to \mathscr{I}_Y(2) \to 0 .$$

Since X is non-degenerate, this sequence yields the inequality $h^0\mathscr{I}_X(2) \le h^0\mathscr{I}_Y(2)$, and the theorems follow from the induction hypothesis.

Proof of Theorem 1.8. Denote by A the vanishing sequence of X at p. It is clear that $a_0 = 0$, and $a_1 = 1$ since p is a smooth point. Hence, the semigroup $\langle a_1, \dots, a_n \rangle$ has no gaps, and $p_a(C^A) = L$ by Corollary 4.11. Using the notation of Corollary 4.11, we see that $L_0 = 0$ and $L_\infty = L$. Corollary 5.9 implies that $\dim S_m^A = a_n \cdot m + 1 - L$. Now the first assertion follows immediately from Proposition 3.2.

To prove the second assertion, assume that $h^0(\mathscr{I}_X(m)) = \binom{m+n}{m} - a_n \cdot m - 1 + L$ for some $m \ge \delta$. If $(\mathscr{L}, V)$ is the linear system defining a birational morphism of a smooth curve onto X, then Corollary 5.9 implies that this linear system is m-extremal. Then Corollary 5.8 together with Proposition 4.8 implies that $\deg X = \deg C^A = a_n$ and $p_a(X) = p_a(C^A) = L$, and the same corollary together with Proposition 4.7 implies that the homogeneous ideal of X is generated by elements of degree $\le m$. This completes the proof.

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