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Introduction

The theory of local aggregation of preferences, which is based on the pairwise comparisons of alternatives, was constructed in the classical work by Arrow [7], and its developments were presented in [2]. The purpose of this paper is to construct a nonlocal model of decision making, in which a noncompensatory nature of aggregation is thoroughly taken into account. The axiomatics of the threshold aggregation was presented in a series of recent papers [3]–[6] and [8]–[9]. This paper is a continuation of these works.

In practice it is quite customary that an alternative is evaluated by means of $n \geqslant 2$ grades $x_1, \ldots, x_n$, each of which taking an integer value from 1 ("bad") to $m \geqslant 3$ ("perfect"). Thus, a problem arises to rank the set X of all n-dimensional vectors x with integer components from 1 to m. Under the assumption that a low grade in the vector $x = (x_1, \ldots, x_n)$ cannot be compensated by (any number of) high grades, in this paper we introduce a notion of the enumerating preference function for the weak order on X, generated by the threshold rule, and evaluate this function explicitly. This permits us also to evaluate all equivalence classes and indifference classes of the weak order. An algorithm of ordering of monotone representatives of indifference classes is given, which correponds to the weak order on X. A dual model to that considered above is presented including an explicit dual enumerating preference function and the dual ordering algorithm of corresponding monotone representatives.

The main results of this paper were presented at the 9th International Meeting of the Society for Social Choice and Welfare (Concordia University, Montreal, Canada, June 2008), seminars of the Institute of Quantitative Social Sciences of Harvard University and in Operations Research Department in MIT (October 2008) and the 1st Russian Economic Congress (Moscow State University, Moscow, Russian Federation, December 2009). Part of the results of the paper were announced in [3, 8] without proofs.

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1. Preliminaries

Let X be a finite set of alternatives of cardinality $|X| \geqslant 2$ and $n \geqslant 1$ and $m \geqslant 2$ be two integers. The set $[1,n] = \{1,2,\dots,n\}$ is interpreted as the set of parameters (or qualities, properties, agents, entities) and the set $[1,m] = \{1,2,\dots,m\}$—as the set of ordered grades (criteria) $1 < 2 < \cdots < m$. By an evaluation procedure for alternatives from X we understand a map E from $X \times [1,n]$ into $[1,m]$, so that to each alternative $x \in X$ and each parameter $i \in [1,n]$ a certain unique grade $x_i = E(x,i) \in [1,m]$ is assigned. In this way each alternative $x \in X$ is characterized by means of n grades $x_1,\dots,x_n$ from $[1,m]$ via the map $x \mapsto \widehat{x} = E(x,\cdot) = (x_1,\dots,x_n) \in [1,m]^n$, where the last set is the Cartesian product of n sets $[1,m]$, i.e., the set of all n-dimensional vectors with components from $[1,m]$. In practice the vector-grades $\widehat{x} = (x_1,\dots,x_n)$ for an alternative $x \in X$ may represent expert grades, questionnaire data, device readings, test data, etc.

We are going to study the problem of ranking the elements from X from the algorithmic point of view using the set $\widehat{X} = \{\widehat{x} : x \in X\}$, an individual (evaluation) profile of X. By a ranking of X we mean a complete and transitive binary relation on X or, equivalently, a weak order on X defined below. Since $\widehat{X} \subset [1,m]^n$ and each alternative $x \in X$ is completely characterized by its profile vector $\widehat{x}$, with no loss of generality we assume throughout the paper that the profile $\widehat{X}$ fills the whole evaluation space, i.e., $X = \widehat{X} = [1,m]^n$. Thus $x \in X = [1,m]^n$ iff $x = \widehat{x} = (x_1,\dots,x_n)$ with $x_i \in [1,m]$, where 'iff' means as usual 'if and only if'. Note that $|X| = m^n$.

Given two nonnegative integers k and l with $k \leqslant l$, we denote by

$$[k,l] = \{i \in \{0\} \cup \mathbb{N} : k \leqslant i \leqslant l\} = \{k, k+1, \dots, l-1, l\}$$

the (natural) *interval* with the endpoints k and l and 'length' $|[k,l]| = l-k+1$ expressing the number of elements in $[k,l]$. We also set $[1,0] = \varnothing$.

Given $j \in [1,m]$ and $x \in X$, we denote by

$$v_j(x) \equiv v_j^{(n)}(x) = |\{i \in [1,n] : x_i = j\}| \tag{1.1}$$

the multiplicity of grade j in the vector $x = (x_1,\dots,x_n)$ and set

$$V_j(x) \equiv V_j^{(n)}(x) = \sum_{k=1}^{j} v_k(x) \quad \text{and} \quad V_0(x) = 0. \tag{1.2}$$

It follows that $0 \leqslant v_j(x) \leqslant n$, $0 \leqslant V_{j-1}(x) \leqslant V_j(x) \leqslant n$ for all $j \in [1,m]$ and

$$\sum_{j=1}^{m} v_j(x) = n \quad \text{or} \quad V_m(x) = n \quad \text{for all} \quad x \in X. \tag{1.3}$$

We say that a binary relation $P = P_{m-1}$ on $X = [1, m]^n$ is *generated by the threshold rule* ([4] if $m = 3$ and [3, 8] in the general case) provided, given $x, y \in X$, we have: if $m = 2$, then $(x, y) \in P = P_1$ iff $v_1(x) < v_1(y)$, and if $m \geqslant 3$, then $(x, y) \in P = P_{m-1}$ iff $v_1(x) < v_1(y)$ or there exists a $k \in [2, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$ and $v_k(x) < v_k(y)$. The inclusion $(x, y) \in P$ can be interpreted in the sense that the alternative x is strictly more preferable than the alternative y. It is known from the references above that P is a *weak order on* X in the sense that it is: (P.1) transitive (given $x, y, z \in X$, $(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$); (P.2) irreflexive ($(x, x) \notin P$ for all $x \in X$); and (P.3) negatively transitive (given $x, y, z \in P$, $(x, y) \notin P$ and $(y, z) \notin P$ imply $(x, z) \notin P$). In addition, if $x, y \in X$, then: (P.4) condition $(x, y) \notin P$ is equivalent to $(y, x) \in P$ or $v_j(x) = v_j(y)$ for all $j \in [1, m]$, and so, (P.5) $X^2 \setminus P$ is complete ($(x, y) \notin P$ or $(y, x) \notin P$ for all $x, y \in X$).

It is known (e.g., [2, 12]) that any weak order P on a finite set X can be characterized by the family of its equivalence classes based on the following construction. Set $X'_1 = \pi(X)$ where, given $\varnothing \neq A \subset X$,

$$\pi(A) = \{x \in A : (y, x) \notin P \text{ for all } y \in A\}$$

is the choice function for P (cf. [1, Section 2.3]). Inductively, if $\ell \geqslant 2$ and nonempty disjoint subsets $X'_1, \dots, X'_{\ell-1}$ of X such that $\bigcup_{k=1}^{\ell-1} X'_k \neq X$ are already defined, we set $X'_\ell = \pi\big(X \setminus (\bigcup_{k=1}^{\ell-1} X'_k)\big)$. Since X is finite, there exists a unique positive integer $s = s(X)$ such that $X = \bigcup_{\ell=1}^{s} X'_\ell$. Setting $X_\ell = X'_{s-\ell+1}$ for $\ell \in [1, s]$, the collection $\{X_\ell\}_{\ell=1}^{s}$ of pairwise disjoint sets is said to be the *family of equivalence classes* of the weak order P, and has the following characteristic property: given $x, y \in X$, $(x, y) \in P$ iff there exist two integers k and ℓ in $[1, s]$ with $k < \ell$ such that $x \in X_\ell$ and $y \in X_k$. Thus, the family of equivalence classes provides the canonical strict ranking of X. The number s of equivalence classes of the weak order P on X generated by the threshold rule is given by formula (1.5) below.

The axioms for the threshold decision making were laid in [4]–[6] for $m = 3$ and extended in [3, 8, 9] for the general case when $m \geqslant 2$ is arbitrary. In order to formulate them, we need a definition. A function $\varphi : X \to \mathbb{R}$ is said to be *coherent* with the family $\{X_\ell\}_{\ell=1}^{s}$ of equivalence classes of a weak order P if, given $x, y \in X$, inequality $\varphi(x) > \varphi(y)$ holds iff there exist integers k and ℓ in $[1, s]$ such that $x \in X_\ell$ and $y \in X_k$. Thus, φ is coherent with $\{X_\ell\}_{\ell=1}^{s}$ iff it is a *preference function* for the relation P in the sense that, given $x, y \in X$, we have: $(x, y) \in P$ iff $\varphi(x) > \varphi(y)$. Such a function φ (nonuniquely determined, in general) plays the role of an aggregation function in the sense that $\varphi(x) > \varphi(y)$ iff x is strictly more preferable than y with respect to criteria, and $\varphi(x) = \varphi(y)$ iff x and y are indifferent. The following result was announced in [3, 8] and proved in [9, Theorem 8.3]:

Theorem A. *A function $\varphi : X \to \mathbb{R}$ is a preference function for the weak order $P = P_{m-1}$ on $X = [1,m]^n$ generated by the threshold rule iff, given $x, y \in X$, it satisfies the following two axioms* $(\mathrm{A.1})_2$ *and* $(\mathrm{A.2})_2$ *if $m = 2$ or three axioms* $(\mathrm{A.1})_m$, $(\mathrm{A.2})_m$ *and* $(\mathrm{A.3})_m$ *if $m \geqslant 3$*:

$(\mathrm{A.1})_m$ *if $v_j(x) = v_j(y)$ for all $j \in [1, m-1]$, then $\varphi(x) = \varphi(y)$;*

$(\mathrm{A.2})_m$ *if $x \succ y$ in X, then $\varphi(x) > \varphi(y)$, where $x \succ y$ means that $x_i \geqslant y_i$ for all $i \in [1,n]$ and there is an $i_0 \in [1,n]$ such that $x_{i_0} > y_{i_0}$;*

$(\mathrm{A.3})_m$ *for each $k \in [3,m]$ the following condition* $(\mathrm{A.3}.k)_m$ *holds: if $v_j(x) = v_j(y)$ for all $j \in [1, m-k]$ (no assumption if $k = m$), $v_{m-k+1}(x)+1 = v_{m-k+1}(y) \neq n - V_{m-k}(y)$, $V_{m-k+2}(x) = n$ and $V_{m-k+1}(y) + v_m(y) = n$, then $\varphi(x) > \varphi(y)$.*

If P is a weak order on X (in particular, generated by the threshold rule), the *indifference* relation I on X is canonically defined as follows: given $x, y \in X$, $(x,y) \in I$ iff $(x,y) \notin P$ and $(y,x) \notin P$. Clearly, I is an equivalence relation on X and, by virtue of property (P.4) above, we have: $(x,y) \in I$ iff $v_j(x) = v_j(y)$ for all $j \in [1,m]$, i. e., vectors x and y can be transformed to each other by certain permutations of their coordinates. Given $x \in X$, we denote by $I_x = \{y \in X : (x,y) \in I\}$ the *indifference class* of x. It was shown in [9, Lemma 4.4(a)] that the family $\{X_\ell\}_{\ell=1}^s$ coincides with the quotient set $X/I = \{I_x : x \in X\}$.

There is a subset X^* of X such that the restriction of P to $X^* \times X^*$, again denoted by P, is a *linear order* on X^* (i. e., P is transitive, irreflexive and, connected, i.e., given $x, y \in X^*$ with $x \neq y$, $(x,y) \in P$ or $(y,x) \in P$). Its construction ([8], [9, Section 4.3]) is recalled below. Given $x \in X = [1,m]^n$, there exists a permutation σ of $[1,n]$ (nonunique, in general) such that the coordinates of the vector

$$x^* = (x_1, \ldots, x_n)^* = (x_1^*, \ldots, x_n^*) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

are ordered in ascending order: $x_1^* = x_{\sigma(1)} \leqslant x_2^* = x_{\sigma(2)} \leqslant \ldots \leqslant x_n^* = x_{\sigma(n)}$. The vector x^*, which is determined uniquely, is called the *monotone representative* of x (or of the indifference class I_x): clearly, $v_j(x^*) = v_j(x)$ for all $j \in [1,m]$ implying $(x^*, x) \in I$ and $I_{x^*} = I_x$, and so, x^* is of the form:

$$\begin{aligned} x^* &= (\underbrace{\overbrace{1,\ldots,1}^{v_1(x)}, \overbrace{2,\ldots,2}^{v_2(x)}, \ldots, \overbrace{m-1,\ldots,m-1}^{v_{m-1}(x)}, \overbrace{m,\ldots,m}^{v_m(x)}}_{n}) = \\ &= (1^{v_1}, 2^{v_2}, \ldots, (m-1)^{v_{m-1}}, m^{v_m}), \end{aligned} \tag{1.4}$$

the number $v_j = v_j(x)$ being the multiplicity of the grade j in x^* and x. In what follows if the multiplicity of some grade j is zero, $v_j(x) = 0$, then the expression j^0 will be omitted in (1.4) (e. g., the vector $(1, \ldots, 1)$ from $[1, m]^n$ is simply (1^n), and so on). Given $A \subset X$, we denote by $A^* = \{x^* : x \in A\}$ the set of all monotone representatives of elements from A. It is known (cf. [4] and [5] if $m = 3$, and [8] and [9, Lemma 4.4(b)] if $m \geqslant 2$ is arbitrary) that

$$s = |X^*| = |[1, m]^{n*}| = C_{n+m-1}^{m-1} = C_{n+m-1}^{n} = \frac{(n+m-1)!}{n!\,(m-1)!}, \tag{1.5}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the usual binomial coefficient, $k \in [0, n]$, and $0! = 1$.

The following two properties will play an important role below (cf. [8] and [9, Lemmas 4.3 and 7.5]): given $x, y \in X$,

$$x \succ y \quad \text{implies} \quad x^* \succ y^*, \quad \text{and} \tag{1.6}$$

$$x^* \succ y^* \quad \text{iff} \quad \begin{array}{l}\text{there exists a } k \in [1, m-1] \text{ such that } v_j(x) = v_j(y) \text{ for} \\ \text{all } j \in [1, k-1] \text{ (no condition if } k = 1\text{)}, v_k(x) < v_k(y) \\ \text{and } V_p(x) \leqslant V_p(y) \text{ for all } p \in [k+1, m-1] \text{ (with no} \\ \text{last condition if } k = m-1\text{).}\end{array} \tag{1.7}$$

Finally, let us point out briefly on the connection of the threshold rule with the lexicographic order and the leximin rule.

Recall that a binary relation $\angle_N$ on the set $\mathbb{R}^N$ of all N-dimensional vectors with real components is said to be the *lexicographic order* if, given $u = (u_1, \ldots, u_N)$ and $v = (v_1, \ldots, v_N)$ from $\mathbb{R}^N$, we have: $u \angle_N v$ iff there exists a $p \in [1, N]$ such that $u_i = v_i$ for all $i \in [1, p-1]$ (no condition if $p = 1$, since $[1, 0] = \varnothing$) and $u_p < v_p$. It is well known that $\angle_N$ is a linear order on $\mathbb{R}^N$; more precisely, $\angle_N$ is transitive (i.e., $u \angle_N v$ and $v \angle_N w$ imply $u \angle_N w$), the negation of $\angle_N$ is of the form: $\neg(u \angle_N v)$ iff $v \angle_N u$ or $v = u$, and $\angle_N$ is trichotomous (i.e., either $u = v$, or $u \angle_N v$, or $v \angle_N u$).

Setting $X = [1, m]^n$ with $n \geqslant 1$ and $m \geqslant 2$ and

$$v(x) = (v_1(x), \ldots, v_{m-1}(x)) \in [0, n]^{m-1} \quad \text{for} \quad x \in X,$$

we find that

$$P_{m-1} = \{(x, y) \in X \times X : v(x) \angle_{m-1} v(y)\},$$

and the following assertion holds.

Lemma 1. *Given x, $y \in X$, we have*: $(x, y) \in P_{m-1}$ *iff* $y^* \angle_n x^*$.

Proof. First, observe that the inclusion $(x, y) \in P_{m-1}$ is equivalent to the existence of a $k \in [1, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$ and $v_k(x) < v_k(y)$, and the relation $y^* \angle_n x^*$ is equivalent to the existence of a $p \in [1, n]$ such that $y_i^* = x_i^*$ for all $i \in [1, p-1]$ and $y_p^* < x_p^*$.

Necessity. If $(x, y) \in P_{m-1}$, then taking into account the observation above and (1.2), we set $p = V_k(x) + 1$ and note that $p \leqslant V_k(y)$, and so, $p \in [1, n]$. Since $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$, then $y_i^* = x_i^*$ for all $i \in [1, p-1]$ and $y_p^* = k < k+1 \leqslant x_p^*$. It follows that $y^* \angle_n x^*$.

Sufficiency. Now, let $y^* \angle_n x^*$. Taking into account the observation at the beginning of the proof, we set $k = y_p^*$ and note that the inequality $k < x_p^*$ implies $k \in [1, m-1]$. By condition $y_i^* = x_i^*$ for all $i \in [1, p-1]$, we find that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$. Let us put $q = |\{i \in [1, p-1] : y_i^* = x_i^* = k\}|$. Then $v_k(y) \geqslant q+1$ and, since $k = y_p^* < x_p^* \leqslant x_{p+1}^* \leqslant \ldots \leqslant x_n^*$, we have $v_k(x) = q$. It follows that $v_k(x) < v_k(y)$, and so, $v(x) \angle_{m-1} v(y)$ implying $(x, y) \in P_{m-1}$. □

The *leximin* rule (an expression invented by Sen [13]) is extensively used in the literature on Social Choice and can be applied to general *social welfare functionals* (cf. [10, Section 2.2.3]). In the context under consideration it is as follows: to every given profile vector x the well ordered vector x^* is associated and x is said to be more preferable than y iff $y^* \angle_n x^*$. In this respect it is to be noted that Theorem A above can be interpreted in terms of principally new axiomatics of aggregation functions φ as compared to the axiomatic approaches usually adopted in the literature on Social Choice. Moreover, the results in the next Section 2 show that the threshold relation P_{m-1} can be effectively algorithmized, which is hardly possible in the framework of general Sen's social welfare functionals.

2. Main results: the enumerating preference function

In order to present our main results, we begin with the following observation.

Lemma 2. *Suppose that $\Phi : X \to \mathbb{R}$ is a preference function for the weak order $P = P_{m-1}$ on $X = [1, m]^n$ generated by the threshold rule. Then*

$$\Phi(A) = \Phi(A^*) \quad \textit{and} \quad |\Phi(A)| = |\Phi(A^*)| = |A^*| \quad \textit{for all} \quad A \subset X, \tag{2.1}$$

where $\Phi(A) = \{\Phi(x) : x \in A\}$ is the image of A under Φ.

Proof. In fact, given $l \in \Phi(A)$, we have $l = \Phi(x)$ for some $x \in A$, and so, $x^* \in A^*$ and, by axiom $(\mathrm{A.1})_m$ from Theorem A, $\Phi(x^*) = \Phi(x) = l$ implying $l \in \Phi(A^*)$. Conversely, if $l \in \Phi(A^*)$, then $l = \Phi(x)$ for some $x \in A^*$, and so, there exists an $a \in A$ such that $a^* = x$, which, again by virtue of axiom $(\mathrm{A.1})_m$, gives $\Phi(a) = \Phi(a^*) = \Phi(x) = l$ and $l \in \Phi(A)$. This proves the first equality in (2.1). In order to establish the third equality in (2.1), it suffices to verify that Φ maps A^* into $\mathbb{R}$ injectively. Given $x^*, y^* \in A^*$ with $x^* \neq y^*$, by virtue of properties (P.5) and (P.4) of P from Section 1, we have $(x^*, y^*) \in P$ or $(y^*, x^*) \in P$ (implying, in particular, that if $A = X$, then the restriction of P to $X^* \times X^*$ is a linear order on X^*), and so, since Φ is a preference function for P, then $\Phi(x^*) > \Phi(y^*)$ or $\Phi(y^*) > \Phi(x^*)$. Thus, Φ maps A^* onto $\Phi(A^*)$ bijectively, and so, $|\Phi(A^*)| = |A^*|$. □

It follows from (2.1) that the number of elements in the image $\Phi(X)$ is equal to $s = |X^*|$ from (1.5), and it is quite natural to look for a preference function Φ for P mapping X onto the interval $[1, |X^*|]$. Fortunately, such a function can be given explicitly in a combinatorial way as the following first main result of this paper asserts:

Theorem 1. *Let two integers $n \geqslant 1$ and $m \geqslant 2$ be given. If $X = [1, m]^n$ and $P = P_{m-1}$ is the weak order on X generated by the threshold rule, then a function Φ maps X onto $[1, |X^*|]$ and it is a preference function for P on X iff it can be represented as*

$$\Phi(x) = \sum_{j=1}^{m} C_{n-V_j(x)+m-j-1}^{m-j} \quad \text{for all} \quad x \in X, \tag{2.2}$$

with $C_k^{k+1} = 0$ for all $k \in [0, m-1]$ and $C_{-1}^{0} = 1$.

It is to be noted that, by virtue of (1.3), the last two terms in (2.2) corresponding to $j = m-1$ and $j = m$ are equal to $C_{v_m(x)}^{1} = v_m(x)$ and $C_{-1}^{0} = 1$, respectively. The function Φ from (2.2) will be called the *enumerating preference function* for P on X.

As a corollary of Theorem 1, we are able to characterize the family of equivalence classes $\{X_\ell\}_{\ell=1}^{s}$ of the weak order P generated by the threshold rule as well as the family of indifference classes $\{I_x\}_{x \in X}$ in Theorem 3 below. For this, we need the following auxiliary result, which is of independent interest and needed in the proof of Theorem 1.

Theorem 2. *Suppose $n \geqslant 1$ and $m \geqslant 2$ are two integers, and set $n_0 = n$. An integer ℓ belongs to the interval $[1, C_{n+m-1}^{m-1}]$ iff there exists a unique collection of $m-2$ integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $0 \leqslant n_j \leqslant n_{j-1}$ for all $j \in [1, m-2]$ such that*

$$\ell \in [L+1, L+1+n_{m-2}], \quad \text{where} \quad L = \sum_{j=1}^{m-2} C_{n_j+m-j-1}^{m-j}. \tag{2.3}$$

Theorem 3. *Given $\ell \in [1, |X^*|]$, we have*:
(a) $X_\ell = \{x \in X : \Phi(x) = \ell\}$; *in other words,* $x \in X_{\Phi(x)}$ *and*

$$I_x = X_{\Phi(x)} = \{y \in X : \Phi(y) = \Phi(x)\} \quad \textit{for all} \quad x \in X;$$

(b) *given* $x \in X$, x *lies in* X_ℓ *iff* (*in the notation of Theorem* 2)

$$v_j(x) = n_{j-1} - n_j \quad \textit{for all} \quad j \in [1, m-2], \tag{2.4}$$

$$v_{m-1}(x) = L + 1 + n_{m-2} - \ell \quad \textit{and} \quad v_m(x) = \ell - L - 1. \tag{2.5}$$

We note that the statement b) of Theorem 3 answers the following question: given ℓ from $[1, s] = [1, |X^*|]$, what are the vectors $x \in X$ satisfying $x \in X_\ell$? Taking into account the statement a) of Theorem 3, this can be reformulated as: find all solutions $x \in X$ of the equation $\Phi(x) = \ell$. In other words, in Theorem 3(b) the equivalence class X_ℓ of the weak order P generated by the threshold rule is restored via its ordinal number ℓ. The number of elements in X_ℓ can be calculated as follows: if the generic vector x from X_ℓ satisfies conditions (2.4) and (2.5), then

$$|X_\ell| = \frac{n!}{v_1(x)! \cdots v_m(x)!} = \frac{n!}{\prod_{j=1}^{m-2}(n_{j-1}-n_j)! \cdot (L+1+n_{m-2}-\ell)! \cdot (\ell-L-1)!}.$$

3. Proofs of the main results

Throughout the proofs we need the following summation over lower indices formula for binomial coefficients (e.g., [11, formulas (5.9) and (5.10)]): if p and q are nonnegative integers, then

$$\sum_{k=0}^{q} C_{p+k}^{p} = C_p^p + C_{p+1}^p + \cdots + C_{p+q}^p = C_{p+q+1}^{p+1} = C_{p+q+1}^{q}. \tag{3.1}$$

Proof of Theorem 2. If there are nonnegative integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $n_j \leqslant n_{j-1}$ for all $j \in [1, m-2]$ such that (2.3) holds, then, by virtue of (3.1),

$$\begin{aligned} 1 \quad &\leqslant \quad L + 1 \leqslant \ell \leqslant L + 1 + n_{m-2} \leqslant \sum_{j=1}^{m-2} C_{n+m-j-1}^{m-j} + 1 + n = \\ &= \quad \sum_{j=1}^{m} C_{n+m-j-1}^{m-j} = \sum_{j=1}^{m} C_{n-1+m-j}^{n-1} = \sum_{k=0}^{m-1} C_{n-1+k}^{n-1} = C_{n+m-1}^{m-1}. \end{aligned} \tag{3.2}$$

Conversely, we apply the induction argument on m for each integer $n \geqslant 1$. If $m = 2$, then $C_{n+m-1}^{m-1} = C_{n+1}^{1} = n + 1$, $n_{m-2} = n_0 = n$ and $L = 0$, and so, the assertion in this case is a tautology. If $m=3$, then $C^{m-1}_{n+m-1}=C^2_{n+2}$ and $[1,C^2_{n+2}]=\bigcup_{k=0}^n[C^2_{k+1}+1,C^2_{k+2}]$ (disjoint union), and so, given $\ell\in[1,C^2_{n+2}]$, there exists a unique $n_1\in[0,n]$ such that

$$\ell\in[C^2_{n_1+1}+1,C^2_{n_1+2}]=[C^2_{n_1+1}+1,C^2_{n_1+1}+1+n_1],$$

and it remains to note that $L=C^2_{n_1+1}=C^{3-1}_{n_1+3-1-1}$. Now, suppose that the necessity part in Theorem 2 holds for some $m\geqslant 3$ and all $n\geqslant 1$, and let $\ell\in[1,C^m_{n+m}]$. Noting that $[1,C^m_{n+m}]=\bigcup_{k=0}^n[C^m_{k+m-1}+1,C^m_{k+m}]$ (disjoint union), we find a unique integer $n_1\in[0,n]$ such that

$$C^m_{n_1+m-1}+1\leqslant\ell\leqslant C^m_{n_1+m}=C^m_{n_1+m-1}+C^{m-1}_{n_1+m-1},$$

and so, $1\leqslant\ell'\equiv\ell-C^m_{n_1+m-1}\leqslant C^{m-1}_{n_1+m-1}$. Applying the induction hypothesis to the integer ℓ' we obtain a unique collection of $m-2$ nonnegative integers $n_1',n_2',\dots,n_{m-2}'$ satisfying $n_j'\leqslant n_{j-1}'$ for all $j\in[1,m-2]$, where $n_0'=n_1$, such that if $L'=\sum_{j=1}^{m-2}C^{m-j}_{n_j'+m-j-1}$, then

$$L'+1\leqslant\ell'=\ell-C^m_{n_1+m-1}\leqslant L'+1+n_{m-2}'.$$

We set $n_j=n_{j-1}'$ for all $j\in[2,m-1]$. It follows that $0\leqslant n_j\leqslant n_{j-1}$ for all $j\in[1,(m+1)-2]$, $n_j'=n_{j+1}$ for all $j\in[0,m-2]$,

$$L'=\sum_{j=1}^{m-2}C^{m-j}_{n_{j+1}+m-j-1}=\sum_{j=2}^{m-1}C^{m+1-j}_{n_j+m-j}=\sum_{j=2}^{(m+1)-2}C^{(m+1)-j}_{n_j+(m+1)-j-1}$$

and $L+1\leqslant\ell\leqslant L+1+n_{(m+1)-2}$, where

$$L=C^m_{n_1+m-1}+L'=\sum_{j=1}^{(m+1)-2}C^{(m+1)-j}_{n_j+(m+1)-j-1},$$

and assertion (2.3) follows with m replaced by $m+1$. □

Proof of Theorem 1. We begin with proving the necessity part. We apply the induction argument on $m\geqslant 2$ for each integer $n\geqslant 1$ and divide the proof into several steps for clarity.

Step 1. Suppose that $m=2$. We have: $X=[1,2]^n$, $(x,y)\in P=P_1$ iff $v_1(x)<v_1(y)$, $v_1(x)+v_2(x)=n$ if $x,y\in X$, $X^*=\{(1^{n-k},2^k):0\leqslant k\leqslant n\}$ (in the notation (1.4)) and $|X^*|=n+1$. Let $\Phi:X\overset{\text{onto}}{\to}[1,n+1]$ be a preference function for P on X, and so, axioms $(\mathrm{A}.1)_2$ and $(\mathrm{A}.2)_2$ from Theorem A are satisfied. Then Φ maps X^* into $[1,n+1]$ bijectively. Noting that the relation

P coincides with $\succ$ on X^* and $(2^n) \succ (1, 2^{n-1}) \succ \ldots \succ (1^{n-1}, 2) \succ (1^n)$ in X, we find from axiom $(A.2)_2$ that $\Phi(2^n) > \Phi(1, 2^{n-1}) > \ldots > \Phi(1^{n-1}, 2) > \Phi(1^n)$. There are $n+1$ different values in this chain of inequalities and, since the image of X^* under Φ is $[1, n+1]$, then $\Phi(1^n) = 1$, $\Phi(1^{n-1}, 2) = 2$, $\ldots$, $\Phi(1, 2^{n-1}) = n$ and $\Phi(2^n) = n + 1$, and so, $\Phi(1^{n-k}, 2^k) = k + 1$ for all $k \in [0, n]$. It follows that if $x \in X$, then $x^* = (1^{n-k}, 2^k)$ with $k = v_2(x^*) = v_2(x)$, and so, by axiom $(A.1)_2$, we get

$$\Phi(x) = \Phi(x^*) = v_2(x) + 1 \quad \text{for all} \quad x \in X = [1, 2]^n. \tag{3.3}$$

Clearly, the function Φ from (3.3), which is of the form (2.2) with $m = 2$, maps X onto $[1, n+1] = [1, |X^*|]$ and, by virtue of (1.3) with $m = 2$, satisfies axioms $(A.1)_2$ and $(A.2)_2$, and so, it is a preference function for $P = P_1$ on X.

Thus, Theorem 1 is established for $m = 2$ and all integer $n \geqslant 1$.

Step 2. Now suppose that the necessity part holds for some $m \geqslant 2$ and all $n \geqslant 1$, and let us show that it remains valid for $m + 1$ and all $n \geqslant 1$, as well.

Let $X = [1, m + 1]^n$. The weak order $P = P_m$ on X generated by the threshold rule is given for $x, y \in X$ by: $(x, y) \in P$ iff $v_1(x) < v_1(y)$ or there exists a $k \in [2, m]$ such that $v_j(x) = v_j(y)$ for all $j \in [1, k-1]$ and $v_k(x) < v_k(y)$. Also, we have:

$$v_1(x) + v_2(x) + \cdots + v_m(x) + v_{m+1}(x) = n \quad \text{for all} \quad x \in X \tag{3.4}$$

and, by virtue of (1.5) with m replaced by $m+1$, $|X^*| = |[1, m+1]^{n*}| = C^m_{n+m}$.

Given $i \in [0, n]$, we set $X(i) = \{x \in X : v_1(x) = v_1^{(n)}(x) = i\}$ and

$$X^*(i) \equiv X(i)^* = X(i) \cap X^* = \{x^* \in X^* : (1^i, 2^{n-i}) \preccurlyeq x^* \preccurlyeq (1^i, (m+1)^{n-i})\},$$

where $x \succcurlyeq y$ or $y \preccurlyeq x$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ means that $x_k \geqslant y_k$ for all $k \in [1, n]$. Note that $X(n) = \{(1^n)\}$. Let us fix $i \in [0, n-1]$ and define a function $\beta_i : X' \equiv [1, m]^{n-i} \to X(i)$ by the rule:

given $x' = (x'_1, \ldots, x'_{n-i}) \in X'$, we set $\beta_i(x') = (1^i, x'_1 + 1, \ldots, x'_{n-i} + 1)$.

Clearly, β_i maps X' into $X(i)$ injectively and X'^* into $X^*(i)$ bijectively, and so, by virtue of (1.5) with n replaced by $n - i$,

$$|X^*(i)| = |X'^*| = |([1, m]^{n-i})^*| = C^{m-1}_{(n-i)+m-1} \quad \forall\, i \in [0, n]. \tag{3.5}$$

Also, note that

$$v_j(x') \equiv v_j^{(n-i)}(x') = v_{j+1}(\beta_i(x')) \quad \text{for all} \quad j \in [1, m]. \tag{3.6}$$

Now, assume that $\Phi : X \overset{\text{onto}}{\to} [1, |X^*|]$ is a preference function for P_m on X.

Step 2a. Let us show that the composed function Φ_i, defined by the rule $\Phi_i(x') = \Phi(\beta_i(x'))$, $x' \in X'$, is a preference function for $P' = P_{m-1}$ on X'. Let $x', y' \in X'$. First, suppose that $m = 2$. By the definition of P_1, we have: $(x', y') \in P' = P_1$ iff $v_1(x') < v_1(y')$, and so, by virtue of (3.6), this is equivalent to $v_1(\beta_i(x')) = i = v_1(\beta_i(y'))$ and $v_2(\beta_i(x')) < v_2(\beta_i(y'))$, that is, $(\beta_i(x'), \beta_i(y')) \in P_2 = P_m$. Now, suppose that $m \geqslant 3$. By the definition of P_{m-1}, we have: $(x', y') \in P'$ iff $v_1(x') < v_1(y')$ or there exists a $k' \in [2, m-1]$ such that $v_j(x') = v_j(y')$ for all $j \in [1, k'-1]$ and $v_{k'}(x') < v_{k'}(y')$, which, by virtue of (3.6), is equivalent to: $v_2(\beta_i(x')) < v_2(\beta_i(y'))$ or there exists a $k' \in [2, m-1]$ such that $v_{j+1}(\beta_i(x')) = v_{j+1}(\beta_i(y'))$ for all $j \in [1, k'-1]$ and $v_{k'+1}(\beta_i(x')) < v_{k'+1}(\beta_i(y'))$. Since $v_1(\beta_i(x')) = i = v_1(\beta_i(y'))$, it follows that $(x', y') \in P'$ iff there exists a $k \in [2, m]$ such that $v_j(\beta_i(x')) = v_j(\beta_i(y'))$ for all $j \in [1, k-1]$ and $v_k(\beta_i(x')) < v_k(\beta_i(y'))$, i.e., $(\beta_i(x'), \beta_i(y')) \in P_m = P$. Thus, given $m \geqslant 2$ and $x', y' \in X'$, $(x', y') \in P'$ iff $(\beta_i(x'), \beta_i(y')) \in P$, and so, since Φ is a preference function for P on X, we find that

$$\begin{aligned} (x', y') \in P' \quad &\text{iff} \quad (\beta_i(x'), \beta_i(y')) \in P \ \text{ iff } \ \Phi(\beta_i(x')) > \Phi(\beta_i(y')) \\ &\text{iff} \quad \Phi_i(x') > \Phi_i(y'), \end{aligned}$$

which proves that $\Phi_i : X' \to \mathbb{R}$ is a preference function for the weak order P' on X' generated by the threshold rule.

Step 2b. Let us show that

$$\Phi(X^*(i)) = [\Phi(1^i, 2^{n-i}), \Phi(1^i, (m+1)^{n-i})] \qquad \forall\, i \in [0, n]. \tag{3.7}$$

If $x^* \in X^*(i)$, then $(1^i, 2^{n-i}) \preccurlyeq x^* \preccurlyeq (1^i, (m+1)^{n-i})$, and so, by axioms $(\mathrm{A.1})_{m+1}$ and $(\mathrm{A.2})_{m+1}$ for the preference function Φ for P on X (Theorem A), we get $\Phi(1^i, 2^{n-i}) \leqslant \Phi(x^*) \leqslant \Phi(1^i, (m+1)^{n-i})$, which establishes the inclusion $\subset$ in (3.7). Conversely, suppose that ℓ lies in the right hand side set of (3.7). Since, by the assumption, $\Phi(X) = [1, |X^*|]$, and, by (2.1), $\Phi(X^*) = \Phi(X)$, we find that $\ell \in [1, |X^*|]$, and so, there exists an $x^* \in X^*$ such that $\ell = \Phi(x^*)$. Noting that X^* is the disjoint union of sets $X^*(k)$ over all $k \in [0, n]$, we find a $k \in [0, n]$ such that $x^* \in X^*(k)$. If we show that $k = i$, then $\ell \in \Phi(X^*(i))$, which completes the proof of (3.7). In fact, if $k < i$, then $v_1(x^*) = k < i = v_1(1^i, (m+1)^{n-i})$, and so, by the definition of P, $(x^*, (1^i, (m+1)^{n-i})) \in P$ implying $\ell = \Phi(x^*) > \Phi(1^i, (m+1)^{n-i})$, which is a contradiction. Similarly, if $k > i$, then $v_1(1^i, 2^{n-i}) = i < k = v_1(x^*)$, whence $((1^i, 2^{n-i}), x^*) \in P$, and so, $\Phi(1^i, 2^{n-i}) > \Phi(x^*) = \ell$, which is a contradiction again. Thus, $k = i$.

Step 2c. Given $i \in [0, n-1]$, we are going to find a preference function $\Psi_i : X' \overset{\text{onto}}{\to} [1, |X'^*|]$ for P' on X' (in the notations of Steps 2 and 2a) and apply the induction hypothesis to it.

Since the function Φ_i from Step 2a is a preference function for P' on X', then applying (3.5) and (2.1), recalling the definition of Φ_i and that $\beta_i(X'^*) = X^*(i)$ and taking into account (3.7), we get:

$$\begin{aligned} |X^*(i)| &= |X'^*| = |\Phi_i(X'^*)| = |\Phi(\beta_i(X'^*))| = |\Phi(X^*(i))| = \\ &= \Phi(1^i, (m+1)^{n-i}) - \Phi(1^i, 2^{n-i}) + 1. \end{aligned} \tag{3.8}$$

Since Φ is a preference function for P on X, then applying (2.1), noting that $(\beta_i(X'))^* = X^*(i)$ and taking into account (3.7) once again and (3.8), we find

$$\begin{aligned} \Phi_i(X') &= \Phi(\beta_i(X')) = \Phi((\beta_i(X'))^*) = \Phi(X^*(i)) = \\ &= \Phi(1^i, 2^{n-i}) - 1 + [1, \Phi(1^i, (m+1)^{n-i}) - \Phi(1^i, 2^{n-i}) + 1] = \\ &= \Phi(1^i, 2^{n-i}) - 1 + [1, |X'^*|]. \end{aligned} \tag{3.9}$$

Given $x' \in X'$, we set $\Psi_i(x') = \Phi_i(x') - \Phi(1^i, 2^{n-i}) + 1$. It follows from Step 2a and (3.9) that $\Psi_i : X' \stackrel{\text{onto}}{\to} [1, |X'^*|]$ is a preference function for P' on X'. Since $X' = [1, m]^{n-i}$ and $P' = P_{m-1}$ is a weak order on X' generated by the threshold rule, by the induction hypothesis, given $x' \in X'$, we get:

$$\Psi_i(x') = \sum_{j=1}^{m} C^{\,m-j}_{(n-i)-V_j(x')+m-j-1},$$

and so, since, as noticed earlier, the last term in the sum above corresponding to $j = m$ is equal to $C^{\,0}_{-1} = 1$, we obtain the following equality:

$$\Phi_i(x') = \Phi(1^i, 2^{n-i}) + \sum_{j=1}^{m-1} C^{\,m-j}_{(n-i)-V_j(x')+m-j-1}, \qquad x' \in X' = [1, m]^{n-i}. \tag{3.10}$$

By virtue of (1.2) and (3.6), we have:

$$\begin{aligned} V_j(x') &= \sum_{k=1}^{j} v_k(x') = \sum_{k=1}^{j} v_{k+1}(\beta_i(x')) = \sum_{k=2}^{j+1} v_k(\beta_i(x')) = \\ &= V_{j+1}(\beta_i(x')) - v_1(\beta_i(x')), \end{aligned}$$

and so, the lower index in the binomial coefficient in (3.10) is equal to

$$(n-i) - V_j(x') + m - j - 1 = (n-i) + v_1(\beta_i(x')) - V_{j+1}(\beta_i(x')) + m - j - 1.$$

Taking into account the definition of Φ_i and changing the summation index $j+1 \mapsto j$ in (3.10), we find that, given $x' \in X'$,

$$\Phi(\beta_i(x')) = \Phi(1^i, 2^{n-i}) + \sum_{j=2}^{m} C^{(m+1)-j}_{(n-i)+v_1(\beta_i(x'))-V_j(\beta_i(x'))+m-j} . \qquad (3.11)$$

Given $x \in X(i)$, we have $x^* \in X^*(i) = \beta_i(X'^*)$ and, since β_i maps X'^* into $X^*(i)$ bijectively, there exists a unique $x'^* \in X'^*$ such that $x^* = \beta_i(x'^*)$. Setting $x' = x'^*$ in (3.11) and noting that $v_j(\beta_i(x'^*)) = v_j(x^*) = v_j(x)$ for all $j \in [1, m]$, and so, by axiom $(A.1)_{m+1}$, $\Phi(\beta_i(x'^*)) = \Phi(x^*) = \Phi(x)$, we arrive at the equality

$$\Phi(x) = \Phi(1^i, 2^{n-i}) + \sum_{j=2}^{m} C^{(m+1)-j}_{(n-i)+v_1(x)-V_j(x)+m-j}, \qquad x \in X(i), \qquad (3.12)$$

where $i \in [0, n-1]$. Note that equality (3.12) holds for $i = n$ as well: in fact, if $i = n$, then $x \in X(i) = X(n)$ iff $v_1(x) = n$ iff $x = (1^n)$, $(1^i, 2^{n-i}) = (1^n, 2^0) = (1^n)$ and $V_j(x) = n$, and so, $C^{(m+1)-j}_{m-j} = 0$ for all $j \in [2, m]$.

It remains to calculate the value $\Phi(1^i, 2^{n-i})$ in (3.12). For this, we need the following equality:

$$\Phi(1^i, 2^{n-i}) = \Phi(1^{i+1}, (m+1)^{n-(i+1)}) + 1 \quad \text{for all} \quad i \in [0, n-1]. \qquad (3.13)$$

Step 2d. Proof of (3.13). First, note that $\Phi(1^n) = 1$ and $\Phi((m+1)^n) = |X^*|$. In fact, given $x \in X$, we have $(m+1)^n \succcurlyeq x \succcurlyeq (1^n)$, and so, by axioms $(A.1)_{m+1}$ and $(A.2)_{m+1}$, we have $\Phi((m+1)^n) \geqslant \Phi(x) \geqslant \Phi(1^n)$, and since $\Phi : X \overset{\text{onto}}{\to} [1, |X^*|]$, the desired equalities follow.

In order to prove (3.13), let us fix $i \in [0, n-1]$. For the sake of brevity, we set $z^* = (1^i, 2^{n-i})$ and $w^* = (1^{i+1}, (m+1)^{n-(i+1)})$. Note that, since $v_1(z^*) = i < i+1 = v_1(w^*)$, then $(z^*, w^*) \in P$, and so, $\Phi(z^*) > \Phi(w^*)$. For any $x^* \in \bigcup_{k=0}^{i} X^*(k)$ (disjoint union) we have $\Phi(z^*) \leqslant \Phi(x^*) \leqslant \Phi((m+1)^n) = |X^*|$: in fact, if $x^* \in X^*(k)$ with $k \in [0, i-1]$, then $v_1(x^*) = k < i = v_1(z^*)$, and so, $(x^*, z^*) \in P$ implying $\Phi(x^*) > \Phi(z^*)$, and if $x^* \in X^*(i)$, then $x^* \succcurlyeq z^*$, and so, by axiom $(A.2)_{m+1}$, $\Phi(x^*) \geqslant \Phi(z^*)$. Similarly, if $y^* \in \bigcup_{k=i+1}^{n} X^*(k)$ (disjoint union), then $1 = \Phi(1^n) \leqslant \Phi(y^*) \leqslant \Phi(w^*)$: in fact, if $y^* \in X^*(i+1)$, then $y^* \preccurlyeq w^*$, and so, by axiom $(A.2)_{m+1}$, $\Phi(y^*) \leqslant \Phi(w^*)$, and if $y^* \in X^*(k)$ with $i+1 < k \leqslant n$, then $v_1(w^*) = i+1 < k = v_1(y^*)$, and so, $(w^*, y^*) \in P$ implying $\Phi(w^*) > \Phi(y^*)$. It follows that

$$\begin{aligned} \Phi\big(\textstyle\bigcup_{k=0}^{i} X^*(k)\big) &\subset [\Phi(z^*), |X^*|] \quad \text{and} \\ \Phi\big(\textstyle\bigcup_{k=i+1}^{n} X^*(k)\big) &\subset [1, \Phi(w^*)], \end{aligned}$$

and, since $X^* = \bigcup_{k=0}^n X^*(k) = \big(\bigcup_{k=0}^i X^*(k)\big) \cup \big(\bigcup_{k=i+1}^n X^*(k)\big)$ (disjoint union), we find

$$[1, |X^*|] = \Phi(X^*) = \Phi\big(\bigcup_{k=0}^n X^*(k)\big) \subset [1, \Phi(w^*)] \cup [\Phi(z^*), |X^*|],$$

where $\Phi(w^*) < \Phi(z^*)$. Since the intervals in this inclusion are natural, we get $\Phi(w^*) + 1 = \Phi(z^*)$, and equality (3.13) follows.

Step 2e. In order to establish equality (2.2) for $m+1$ from (3.12), let $i \in [0, n]$ and let us calculate the value $\Phi(1^i, 2^{n-i})$. By virtue of (3.13), (3.8), (3.5) and (3.1), we have:

$$\begin{aligned}
\Phi(1^i, 2^{n-i}) &= \Phi(1^{i+1}, (m+1)^{n-(i+1)}) - \Phi(1^{i+1}, 2^{n-(i+1)}) + 1 + \Phi(1^{i+1}, 2^{n-(i+1)}) \\
&= |X^*(i+1)| + \Phi(1^{i+1}, 2^{n-(i+1)}) = \\
&= |X^*(i+1)| + |X^*(i+2)| + \cdots + |X^*(n)| + \Phi(1^n, 2^0) = \\
&= \sum_{l=i+1}^{n} C_{(n-l)+m-1}^{m-1} + \Phi(1^n) = \sum_{k=0}^{n-i-1} C_{(m-1)+k}^{m-1} + 1 = \\
&= C_{(n-i)+m-1}^{m} + 1.
\end{aligned}$$

It follows from (3.12) that, given $i \in [0, n]$ and $x \in X(i)$,

$$\Phi(x) = C_{(n-i)+(m+1)-1-1}^{(m+1)-1} + \sum_{j=2}^{m} C_{(n-i)+v_1(x)-V_j(x)+(m+1)-j-1}^{(m+1)-j} + 1. \quad (3.14)$$

Now, given $x \in X$, we find that $x \in X(i)$ with $i = v_1(x)$, and so, applying (3.14) and noting that, by virtue of (3.4),

$$1 = C_{-1}^{0} = C_{(n-v_1(x))+v_1(x)-V_{m+1}(x)+(m+1)-(m+1)-1}^{(m+1)-(m+1)},$$

we conclude that

$$\Phi(x) = \sum_{j=1}^{m+1} C_{n-V_j(x)+(m+1)-j-1}^{(m+1)-j} \quad \text{for all} \quad x \in X = [1, m+1]^n,$$

as asserted in (2.2) for $m+1$ in place of m.

This completes the proof of the necessity part of Theorem 1. Now we turn to the proof of the sufficiency part.

Step 3. First, we prove that the function Φ given by (2.2) is a preference function for $P = P_{m-1}$ on $X = [1, m]^n$ with $m \geqslant 3$. For this, it suffices to verify that Φ satisfies the three axioms from Theorem A. Let $x, y \in X$.

Axiom $(\mathrm{A.1})_m$. If $v_j(x) = v_j(y)$ for all $j \in [1, m-1]$, then, by virtue of (1.2) and (1.3), $V_j(x) = V_j(y)$ for all $j \in [1, m]$, and so, formula (2.2) implies $\Phi(x) = \Phi(y)$.

Axiom $(\mathrm{A.2})_m$. Suppose that $x \succ y$ in X. Then, by (1.6), $x^* \succ y^*$, and so, condition at the right in (1.7) is satisfied. It follows from (1.2) that $V_j(x) = V_j(y)$ for all $j \in [1, k-1]$, $V_k(x) < V_k(y)$, $V_p(x) \leqslant V_p(y)$ for all $p \in [k+1, m-1]$ and $V_m(x) = V_m(y) = n$. Therefore,

$$\begin{aligned} C^{m-j}_{n-V_j(x)+m-j-1} &= C^{m-j}_{n-V_j(y)+m-j-1} \quad \text{for all} \quad j \in [1, k-1], \\ C^{m-k}_{n-V_k(x)+m-k-1} &> C^{m-k}_{n-V_k(y)+m-k-1}, \quad \text{and} \\ C^{m-j}_{n-V_j(x)+m-j-1} &\geqslant C^{m-j}_{n-V_j(y)+m-j-1} \quad \text{for all} \quad j \in [k+1, m], \end{aligned}$$

and so, summing these (in)equalities over all $j \in [1, m]$ and taking into account equality (2.2), we get $\Phi(x) > \Phi(y)$.

Axiom $(\mathrm{A.3})_m$. Given $k \in [3, m]$, suppose that condition $(\mathrm{A.3}.k)_m$ in Theorem A is satisfied. Since $v_j(x) = v_j(y)$ for all $j \in [1, m-k]$, we have:

$$\sum_{j=1}^{m-k} C^{m-j}_{n-V_j(x)+m-j-1} = \sum_{j=1}^{m-k} C^{m-j}_{n-V_j(y)+m-j-1}. \tag{3.15}$$

Set $\nu = v_{m-k+2}(x)$. Then condition $V_{m-k+2}(x) = n$ implies $V_{m-k+1}(x) = n - \nu$ and $V_j(x) = n$ for all $j \in [m-k+2, m]$, and condition $v_{m-k+1}(x) + 1 = v_{m-k+1}(y)$ implies $V_{m-k+1}(x) + 1 = V_{m-k+1}(y)$, and so, $V_{m-k+1}(y) = n - \nu + 1$ (in particular, it follows that $\nu \in [2, n]$). Finally, condition $V_{m-k+1}(y) + v_m(y) = n$ implies $V_j(y) = V_{m-k+1}(y) = n - \nu + 1$ for all $j \in [m-k+1, m-1]$, $V_m(y) = n$ and $v_m(y) = \nu - 1$. It follows that

$$\begin{aligned} \sum_{j=m-k+1}^{m} C^{m-j}_{n-V_j(x)+m-j-1} &= C^{m-(m-k+1)}_{n-V_{m-k+1}(x)+m-(m-k+1)-1} + \\ &\qquad + \sum_{j=m-k+2}^{m-1} C^{m-j}_{n-n+m-j-1} + 1 = \\ &= C^{k-1}_{\nu+k-2} + 1, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\sum_{j=m-k+1}^{m} C^{m-j}_{n-V_j(y)+m-j-1} &= \sum_{j=m-k+1}^{m-1} C^{m-j}_{n-(n-\nu+1)+m-j-1} + 1 = \\
&= \sum_{j=m-k+1}^{m-1} C^{m-j}_{\nu-2+m-j} + 1 = \sum_{j=m-k+1}^{m} C^{\nu-2}_{\nu-2+m-j} = \\
&= \sum_{j=0}^{k-1} C^{\nu-2}_{\nu-2+j} = C^{k-1}_{\nu-2+k}, \qquad (3.17)
\end{aligned}$$

where equality (3.17) follows from (3.1). Now, (3.15)–(3.17) and (2.2) imply that $\Phi(x) = \Phi(y) + 1 > \Phi(y)$, as asserted.

Step 4. Finally, we show that $\Phi : X \overset{\text{onto}}{\to} [1, |X^*|]$, that is, $\Phi(X) = [1, |X^*|]$.

Given $x \in X$, we have $(1^n) \preccurlyeq x \preccurlyeq (m^n)$, and so, by axioms $(\mathrm{A.1})_m$ and $(\mathrm{A.2})_m$, we find $\Phi(1^n) \leqslant \Phi(x) \leqslant \Phi(m^n)$. Since $V_j(1^n) = n$ and $V_j(m^n) = 0$ for all $j \in [1, m-1]$, we get: $\Phi(1^n) = \sum_{j=1}^{m-1} C^{m-j}_{m-j-1} + 1 = 1$ and, by virtue of (3.2) and (1.5), $\Phi(m^n) = \sum_{j=1}^{m} C^{m-j}_{n+m-j-1} = |X^*|$, and so, $\Phi(x)$ is in $[\Phi(1^n), \Phi(m^n)] = [1, |X^*|]$ implying $\Phi(X) \subset [1, |X^*|]$.

In order to prove the reverse inclusion $[1, |X^*|] \subset \Phi(X)$, we let ℓ be in $[1, |X^*|] = [1, C^{m-1}_{n+m-1}]$ and apply Theorem 2: there is a unique collection of nonnegative integers $n_1, n_2, \dots, n_{m-2}$ satisfying $n_j \leqslant n_{j-1}$ for all $j \in [1, m-2]$ such that (2.3) holds. Consider a vector $x \in X = [1, m]^n$ (well) defined by equalities (2.4) and (2.5). Then, given $j \in [1, m-2]$, we have:

$$V_j(x) = \sum_{k=1}^{j} v_k(x) = \sum_{k=1}^{j} (n_{k-1} - n_k) = n_0 - n_j = n - n_j$$

and $n - V_j(x) = n_j$, and so, by virtue of (2.2) and (2.3), we get:

$$\begin{aligned}
\Phi(x) &= \sum_{j=1}^{m-2} C^{m-j}_{n-V_j(x)+m-j-1} + v_m(x) + 1 = \\
&= \sum_{j=1}^{m-2} C^{m-j}_{n_j+m-j-1} + (\ell - L - 1) + 1 = \ell.
\end{aligned}$$

It follows that $\ell \in \Phi(X)$, and so, $[1, |X^*|] \subset \Phi(X)$.

This completes the proof of Theorem 1. □

Proof of Theorem 3. (a) To begin with, we note that, by virtue of property (P.4) from p. 6, given $x, y \in X$, we have: $(y, x) \notin P$ iff $(x, y) \in P$ or $v_j(x) = v_j(y)$ for all $j \in [1, m-1]$. Since, by Theorem 1, Φ is a preference function for P on X (cf. also property (P.5) and axiom $(\mathrm{A.1})_m$), we get:

$$v_j(x) = v_j(y) \ \text{ for all } \ j \in [1, m] \quad \text{iff} \quad \Phi(x) = \Phi(y). \tag{3.18}$$

It follows that $(y, x) \notin P$ iff $\Phi(x) > \Phi(y)$ or $\Phi(x) = \Phi(y)$, i.e., $\Phi(x) \geqslant \Phi(y)$. As in (1.5), we set $s = |X^*|$.

By the definition of X_s (Section 1), we find

$$\begin{aligned} X_s &= X_1' = \pi(X) = \{x \in X : (y, x) \notin P \ \text{ for all } \ y \in X\} = \\ &= \{x \in X : \Phi(x) \geqslant \Phi(y) \ \text{ for all } \ y \in X\}. \end{aligned}$$

Let us show that the last set is equal to $\{x \in X : \Phi(x) = s\}$. In fact, let $x \in X$. If $\Phi(x) = s$, then since, by Theorem 1, $\Phi(y) \in [1, s]$ for all $y \in X$, we get $\Phi(x) = s \geqslant \Phi(y)$ for all $y \in X$. Now, if $\Phi(x) \geqslant \Phi(y)$ for all $y \in X$, then setting $y = (m^n)$ we find $s \geqslant \Phi(x) \geqslant \Phi(y) = \Phi(m^n) = s$, and so, $\Phi(x) = s$. Thus, $X_s = \{x \in X : \Phi(x) = s\} = \{(m^n)\}$.

Now, suppose that for some $\ell \in [2, s]$ we have already shown that X_k is equal to $\{x \in X : \Phi(x) = k\}$ for all $k \in [\ell, s]$, and let us show that $X_{\ell-1} = \{x \in X : \Phi(x) = \ell - 1\}$. By the definition,

$$X_{\ell-1} = X'_{s-(\ell-1)+1} = X'_{s-\ell+2} = \pi\big(X \setminus \big(\textstyle\bigcup_{k=1}^{s-\ell+1} X_k'\big)\big)$$

is the set of all $x \in X \setminus (\bigcup_{k=1}^{s-\ell+1} X_k')$ such that $(y, x) \notin P$ for all $y \in X$ which lie outside of $\bigcup_{k=1}^{s-\ell+1} X_k'$. Since, again by the definition, $X_k = X'_{s-k+1}$ for all $k \in [\ell, s]$ or, equivalently, $X_k' = X_{s-k+1}$ for all $k \in [1, s-\ell+1]$, by the hypothesis above, we find

$$\bigcup_{k=1}^{s-\ell+1} X_k' = \bigcup_{k=\ell}^{s} X_k = \{x \in X : \Phi(x) \in [\ell, s]\},$$

and so, Theorem 1 implies

$$\begin{aligned} X_{\ell-1} = \{x \in X : \quad & \Phi(x) \in [1, \ell-1] \text{ and } \Phi(x) \geqslant \Phi(y) \text{ for all } y \in X \\ & \text{such that } \Phi(y) \in [1, \ell-1]\}. \end{aligned}$$

We claim that $X_{\ell-1} = \{x \in X : \Phi(x) = \ell - 1\}$; in fact, given $x \in X$, we have: clearly, if $\Phi(x) = \ell - 1$, then $x \in X_{\ell-1}$, and if $x \in X_{\ell-1}$, then, by virtue of Theorem 1 and equality $\Phi(X) = [1, s]$, we can choose a $y \in X$ such that $\Phi(y) = \ell - 1$, and so, $\ell - 1 \geqslant \Phi(x) \geqslant \Phi(y) = \ell - 1$ implying $\Phi(x) = \ell - 1$. In this way we have proved that $X_\ell = \{x \in X : \Phi(x) = \ell\}$ for all $\ell \in [1, |X^*|]$.

Now, given $x \in X$, there is an $\ell \in [1, s]$ such that $x \in X_\ell = \{x \in X : \Phi(x) = \ell\}$, and so, $x \in X_{\Phi(x)}$, i.e., $X_{\Phi(x)}$ is (as noticed on p. 7) the indifference class of x, which establishes the equality $X_{\Phi(x)} = I_x$.

(b) It was shown in Step 4 of the proof of Theorem 1 that if $x \in X$ satisfies (2.4) and (2.5), then $\Phi(x) = \ell$, and so, by item (a), $x \in X_\ell$.

Now, suppose that $x \in X_\ell$, so that $\Phi(x) = \ell$. By Theorem 2, there exists a unique collection of $m - 2$ nonnegative integers $n_1, n_2, \ldots, n_{m-2}$ satisfying $n_j \leqslant n_{j-1}$ for all $j \in [1, m-2]$ such that the inclusion in (2.3) holds. Consider a vector $x' \in X$ having the properties (2.4) and (2.5). Then $\Phi(x') = \ell$, and so, $\Phi(x) = \Phi(x')$. Taking into account (3.18), we find that $v_j(x) = v_j(x')$ for all $j \in [1, m]$, and so, x satisfies conditions (2.4) and (2.5) as well. □

4. Algorithmic order on X^*

Recall that, given $x, y \in X$, we have: $(x, y) \in P$ iff $(x^*, y^*) \in P^*$, where P^* is the restriction of the relation P to $X^* \times X^*$, and that P^* is a linear order on X^*. Moreover, $I_x = I_{x^*}$ for all $x \in X$. It follows that if we are interested in more properties of the relation P on X, then it suffices to study them for P^* on X^*. Recall also that the restriction of the function Φ from (2.2) to X^* is a bijection between X^* and $[1, |X^*|]$, so that the pairs (X^*, P^*) and $([1, |X^*|], >)$ are order isomorphic in the sense that, given $x^*, y^* \in X^*$, $(x^*, y^*) \in P$ iff $\Phi(x^*) > \Phi(y^*)$.

Let $x \in X = [1, m]^n$. Since $X = \bigcup_{\ell=1}^{s} X_\ell$ (disjoint union) with $s = |X^*|$, there exists a unique $\ell \in [1, |X^*|]$ such that $x \in X_\ell$. By Theorem 2, the number ℓ determines uniquely a collection of $m - 2$ nonnegative integers $n_1, n_2, \ldots, n_{m-2}$ with appropriate properties, so that, in particular, equalities (2.4) and (2.5) hold. Setting $n_{m-1} = v_m(x) = \ell - L - 1$ and $n_m = 0$, we find that $v_{m-1}(x) = n_{m-2} - n_{m-1}$ and $v_m(x) = n_{m-1}$, and so, $0 \leqslant n_{m-1} \leqslant n_{m-2}$. Thus, we have shown that, given $x \in X$, there exists a unique collection of m integers $n_1, n_2, \ldots, n_{m-1}$ and $n_m = 0$ satisfying $0 \leqslant n_j \leqslant n_{j-1}$ for all $j \in [1, m]$ such that

$$v_j(x) = n_{j-1} - n_j \quad \text{for all} \quad j \in [1, m-1] \quad \text{and} \quad v_m(x) = n_{m-1}. \tag{4.1}$$

Moreover, Theorem 3(a), (2.3) and definitions of n_{m-1} and n_m imply

$$
\begin{aligned}
\Phi(x) &= \ell = L + n_{m-1} + 1 = \\
&= \sum_{j=1}^{m-2} C_{n_j+m-j-1}^{m-j} + C_{n_{m-1}+m-(m-1)-1}^{m-(m-1)} + C_{n_m+m-m-1}^{m-m} = \\
&= \sum_{j=1}^{m} C_{n_j+m-j-1}^{m-j}. \qquad (4.2)
\end{aligned}
$$

On the other hand, due to the uniqueness of collection $\{n_j\}_{j=0}^{m}$, it is clear that, given $x \in X$, we have:

$$n_j \equiv n_j(x) = n - V_j(x) \quad \text{for all} \quad j \in [1, m] \qquad (4.3)$$

and, in particular, numbers (4.3) satisfy conditions (4.1), and so, the monotone representative x^* of x is of the form:

$$x^*(\widetilde{n}) = \left(1^{n-n_1}, 2^{n_1-n_2}, 3^{n_2-n_3}, \ldots, (m-1)^{n_{m-2}-n_{m-1}}, m^{n_{m-1}}\right), \qquad (4.4)$$

where $\widetilde{n} = (n_1, n_2, \ldots, n_{m-1})$, $n_0 = n$ and $n_j \in [0, n_{j-1}]$ for all $j \in [1, m-1]$. Denote by $\widetilde{N}$ the set of all such vectors $\widetilde{n}$. In this way we have shown that the set $\widetilde{N}$ is bijective to X^* via the map (4.4) (cf. also (4.3)). Moreover, $\widetilde{N}$ and X^* are order isomorphic in the following sense: given $\widetilde{n} = (n_1, n_2, \ldots, n_{m-1})$, $\widetilde{n'} = (n'_1, n'_2, \ldots, n'_{m-1}) \in \widetilde{N}$, we have: $(x^*(\widetilde{n}), x^*(\widetilde{n'})) \in P^*$ iff $n'_1 < n_1$ or there exists a $k \in [2, m-1]$ such that $n'_j = n_j$ for all $j \in [1, k-1]$ and $n'_k < n_k$. In fact, in order to see this, it suffices to note only that $v_j(x^*(\widetilde{n})) = n_{j-1} - n_j$ and $v_j(x^*(\widetilde{n'})) = n'_{j-1} - n'_j$ for all $j \in [1, m-1]$ and $n_0 = n'_0 = n$.

Thus, the linear order on $\widetilde{N}$, exposed in the previous paragraph, defines the *algorithmic order on* X^* via (4.4) corresponding to the more greater P-preferability, which can be described by the following rule: write out one by one a string of vectors $x^*(\widetilde{n})$ of the form (4.4) in such a way that n_1 assumes successively the values $0, 1, \ldots, n$, and if n_1 is fixed, then the number n_2 assumes successively the values $0, 1, \ldots, n_1$, and if n_1 and n_2 are fixed in the ranges $0 \leqslant n_1 \leqslant n$ and $0 \leqslant n_2 \leqslant n_1$, then the number n_3 assumes successively the values $0, 1, \ldots, n_2$, and so on, and finally, if $n_1, n_2, \ldots, n_{m-2}$ are fixed in their respective ranges ($0 \leqslant n_1 \leqslant n$, $0 \leqslant n_2 \leqslant n_1$, $\ldots$, $0 \leqslant n_{m-2} \leqslant n_{m-3}$), then the number n_{m-1} assumes successively the values $0, 1, \ldots, n_{m-2}$. According to the algorithmic order on X^*, to each $x^* \in X^*$ there corresponds a unique natural number, which is the ordinal number of x^* and, if x^* is of the form (4.4) for some collection $\widetilde{n} = (n_1, n_2, \ldots, n_{m-1}) \in \widetilde{N}$, then this ordinal number of x^* is given by formula (4.2).

Examples of the algorithmic orderings of the class of monotone representatives X^* of elements from $X = [1, m]^n$ for $m = 3, 4, 5$ and $n = 2, 3, 4, 5\,(6, 7)$ are presented in Section 6.

5. Dual threshold aggregation axiomatics and algorithms

If the utmost perfection (quality) of alternatives is of main concern, we can apply the threshold rule $P = P_{m-1}$ from Section 1 to rank the set of alternatives $X = [1, m^n]$. However, if we are interested in at least one good feature of alternatives, we should employ a different, but related, aggregation (decision making) procedure, which will be called the *dual threshold aggregation*. The possibility to develop such a dual model for three-graded rankings had already been mentioned in [4]. In this Section we develop an axiomatic theory as well as algorithms of the dual threshold aggregation in the general case when $m \geqslant 2$ is arbitrary.

The dual procedure is based on the following notion. A binary relation $P^d = P^d_{m-1}$ on $X = [1, m]^n$ is said to be *generated by the dual threshold rule* provided, given $x, y \in X$, we have: if $m = 2$, then $(x, y) \in P^d = P^d_1$ iff $v_2(y) < v_2(x)$, and if $m \geqslant 3$, then $(x, y) \in P^d = P^d_{m-1}$ iff $v_m(y) < v_m(x)$ or there exists a $k \in [2, m-1]$ such that $v_j(x) = v_j(y)$ for all $j \in [k+1, m]$ and $v_k(y) < v_k(x)$. As it will be seen later, this notion gives exactly the dual model to that considered above with all advantages of the dual model including the construction of the dual enumerating preference function Φ^d and the dual algorithmic order on X^*.

5.1. Dual axiomatics. Making use of the lexicographic order (cf. Section 1) and notation (1.1), given an alternative $x \in X = [1, m]^n$, we set

$$v^d(x) = (v_m(x), v_{m-1}(x), \ldots, v_2(x)) \in [0, n]^{m-1}$$

and note that

$$P^d \equiv P^d_{m-1} = \{(x, y) \in X \times X : v^d(y) \angle_{m-1} v^d(x)\}.$$

We are going to reduce the dual threshold aggregation axiomatics to Theorem A. In order to do this, we introduce a permutation r of $[1, m]$ as follows:

$$r(j) = m - j + 1 \qquad \text{for all} \qquad j \in [1, m].$$

Note that r is a bijection between the sets $[1, m-1] = \{1, 2, \ldots, m-1\}$ and $[m, 2] \equiv \{m, m-1, \ldots, 2\}$, reversing the order of the numbers, and so, its self composition $r^2 = r \circ r$ is the identity on $[1, m-1]$ and on $[m, 2]$: $r(r(j)) = j$ for all appropriate j. Given $x = (x_1, \ldots, x_n) \in X = [1, m]^n$, we set

$$\mathbf{r}(x) = (r(x_1), r(x_2), \ldots, r(x_n)) = (m - x_1 + 1, m - x_2 + 1, \ldots, m - x_n + 1),$$

and note that $\mathbf{r}(\mathbf{r}(x)) = x$, i.e., $\mathbf{r}(x') = x$ iff $x' = \mathbf{r}(x)$.

The following two properties (5.1) and (5.2) of $\mathbf{r}$ will be of significance:

$$v_j(\mathbf{r}(x)) = v_{r(j)}(x) \quad \text{for all} \quad x \in X \quad \text{and} \quad j \in [1, m]. \tag{5.1}$$

In fact, we have:

$$\begin{aligned} v_j(\mathbf{r}(x)) &= |\{i \in [1, n] : r(x_i) = j\}| = |\{i \in [1, n] : m - x_i + 1 = j\}| = \\ &= |\{i \in [1, n] : x_i = m - j + 1\}| = |\{i \in [1, n] : x_i = r(j)\}| = \\ &= v_{r(j)}(x). \end{aligned}$$

It follows that $v_j(x) = v_{r(j)}(\mathbf{r}(x))$ and

$$v^d(x) = v(\mathbf{r}(x)) \quad \text{and} \quad v^d(\mathbf{r}(x)) = v(x) \quad \text{for all} \quad x \in X, \tag{5.2}$$

because

$$\begin{aligned} v^d(x) &= (v_m(x), v_{m-1}(x), \ldots, v_2(x)) = (v_{r(1)}(x), v_{r(2)}(x), \ldots, v_{r(m-1)}(x)) = \\ &= (v_1(\mathbf{r}(x)), v_2(\mathbf{r}(x)), \ldots, v_{m-1}(\mathbf{r}(x))) = v(\mathbf{r}(x)). \end{aligned}$$

Now, given $x, y \in X$, we have:

$$\begin{aligned} (x, y) \in P^d \quad &\text{iff} \quad v^d(y) \angle_{m-1} v^d(x) \quad \text{iff} \quad v(\mathbf{r}(y)) \angle_{m-1} v(\mathbf{r}(x)) \\ &\text{iff} \quad (\mathbf{r}(y), \mathbf{r}(x)) \in P \end{aligned} \tag{5.3}$$

or, equivalently, $(x, y) \in P$ iff $(\mathbf{r}(y), \mathbf{r}(x)) \in P^d$. It follows from Lemma 1 and (5.3) that $(x, y) \in P^d$ iff $(\mathbf{r}(x))^* \angle_n (\mathbf{r}(y))^*$.

By virtue of (5.3), the relation P^d on X satisfies the properties (P.1)–(P.5) (if we replace P in these properties on p. 6 by P^d), and so, P^d is a weak order on X. For instance, the negation of P^d is of the form: given $x, y \in X$, $(x, y) \notin P^d$ iff $(y, x) \in P^d$ or $v(y) = v(x)$; in fact, it follows from (5.3) that

$$\begin{aligned} (x, y) \notin P^d \quad &\text{iff} \quad (\mathbf{r}(y), \mathbf{r}(x)) \notin P \quad \text{iff} \quad [(\mathbf{r}(x), \mathbf{r}(y)) \in P \text{ or } v(\mathbf{r}(x)) = v(\mathbf{r}(y))] \\ &\text{iff} \quad [(y, x) \in P^d \text{ or } v^d(x) = v^d(y)], \end{aligned}$$

and it remains to note that, in view of (1.3), the condition "$v_j(x) = v_j(y)$ for all $j \in [2, m]$" is equivalent to the condition "$v_j(x) = v_j(y)$ for all $j \in [1, m-1]$".

This observation also shows that the indifference relation I^d on X generated by P^d coincides with the indifference relation I:

$$I^d = \{(x,y) : (x,y) \notin P^d \text{ and } (y,x) \notin P^d\} = \{(x,y) : v(x) = v(y)\} = I.$$

In order to treat the axiomatics of preference functions for the relation P^d, we note that if φ is a preference function for P and ψ is a preference function for P^d, then, given $x, y \in X$, we have:

$$\begin{aligned} \psi(x) > \psi(y) \quad \text{iff} \quad (x,y) \in P^d \quad \text{iff} \quad (\mathbf{r}(y), \mathbf{r}(x)) \in P \quad \text{iff} \quad \varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x)) \\ \text{iff} \quad [-\varphi(\mathbf{r}(x)) > -\varphi(\mathbf{r}(y))]. \end{aligned} \tag{5.4}$$

We conclude that φ is a preference function for P iff the function φ^d, defined by $\varphi^d(x) = -\varphi(\mathbf{r}(x))$ for all $x \in X$, is a preference function for P^d, and vice versa: φ^d is a preference function for P^d iff the function φ, defined for $x \in X$ by $\varphi(x) = -\varphi^d(\mathbf{r}(x))$, is a preference function for P. It follows from Theorem A that a function $\varphi^d : X \to \mathbb{R}$ is a preference function for P^d iff the function $\varphi(x) = -\varphi^d(\mathbf{r}(x))$ satisfies axioms $(\mathrm{A.1})_m$–$(\mathrm{A.3})_m$, and by virtue of (5.4) with ψ replaced by φ^d, given $x, y \in X$, we have:

$$\varphi^d(x) > \varphi^d(y) \quad \text{iff} \quad \varphi(x') > \varphi(y'), \quad \text{where } x' = \mathbf{r}(y) \text{ and } y' = \mathbf{r}(x).$$

So, replacing x by $\mathbf{r}(y)$ and y by $\mathbf{r}(x)$ in axioms $(\mathrm{A.1})_m$–$(\mathrm{A.3})_m$ and taking into account equalities (5.1) and (5.2), we obtain the following (dual) axioms for function φ^d. Axioms $(\mathrm{A.1})_m$ and $(\mathrm{A.2})_m$ remain the same, because conditions "$v^d(x) = v^d(y)$" and "$v(x) = v(y)$" are equivalent, and if $x \succ y$, then $\mathbf{r}(y) \succ \mathbf{r}(x)$, and so, $\varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x))$ implying $\varphi^d(x) > \varphi^d(y)$. The third dual axiom assumes the following form:

Axiom $(\mathrm{A.3})_m^d$: for each $k \in [3, m]$ the following condition $(\mathrm{A.3}.k)_m^d$ holds: given $x, y \in X$, if $v_j(x) = v_j(y)$ for all $j \in [k+1, m]$ (if $k = m$, this condition is absent), $v_k(y) + 1 = v_k(x) \neq V_k(x)$, $V_{k-2}(y) = 0$ and $V_{k-1}(x) = v_1(x)$, then $\varphi^d(x) > \varphi^d(y)$.

The observations above lead to the following corollary of Theorem A.

Theorem B. *A function $\varphi^d : X \to \mathbb{R}$ is a preference function for the weak order $P^d = P_{m-1}^d$ on $X = [1, m]^n$ generated by the dual threshold rule iff it satisfies the two axioms $(\mathrm{A.1})_2$ and $(\mathrm{A.2})_2$ if $m = 2$ or three axioms $(\mathrm{A.1})_m$, $(\mathrm{A.2})_m$ and $(\mathrm{A.3})_m^d$ if $m \geqslant 3$.*

5.2. The dual enumerating preference function. Recall from the above that $\varphi : X \to \mathbb{R}$ is a preference function for the relation P on X iff the function $\varphi^d : X \to \mathbb{R}$, defined by $\varphi^d(x) = -\varphi(\mathbf{r}(x))$ for all $x \in X$, is a preference function for P^d on X. Taking into account Theorem 1, we shall look for the dual enumerating preference function for P^d on X in the form $\Phi^d(x) = c - \Phi(\mathbf{r}(x))$, $x \in X$, where c is an appropriate constant to be found below. Given $j \in [1, m]$, equalities (1.2) and (5.1) imply

$$V_j(\mathbf{r}(x)) = \sum_{k=1}^{j} v_k(\mathbf{r}(x)) = \sum_{k=1}^{j} v_{r(k)}(x) = \sum_{k=1}^{j} v_{m-k+1}(x) = \sum_{i=m-j+1}^{m} v_i(x),$$

and so, $n - V_j(\mathbf{r}(x)) = V_{m-j}(x)$, and equality (2.2) gives

$$\Phi^d(x) = c - \Phi(\mathbf{r}(x)) = c - \sum_{j=1}^{m} C^{m-j}_{V_{m-j}(x)+m-j-1} = c - \sum_{i=0}^{m-1} C^{i}_{V_i(x)+i-1}.$$

If we want to have the property of Φ^d that Φ^d maps X onto $[1, |X^*|]$, then we should have $\Phi^d(1^n) = 1$. Since $V_i(1^n) = n$ for all $i \in [1, m-1]$, then, by virtue of (1.2) and (3.1), we get:

$$\begin{aligned} 1 &= \Phi^d(1^n) = c - C^0_{-1} - \sum_{i=1}^{m-1} C^i_{n+i-1} = c - 1 - \sum_{i=0}^{m-1} C^i_{(n-1)+i} + C^0_{n-1} = \\ &= c - \sum_{i=0}^{m-1} C^{n-1}_{(n-1)+i} = c - C^{m-1}_{(n-1)+(m-1)+1} = c - C^{m-1}_{n+m-1}, \end{aligned}$$

and so, according to (1.5), $c = 1 + C^{m-1}_{n+m-1} = 1 + |X^*|$. Taking into account that $C^0_{-1} = 1$, we conclude that

$$\Phi^d(x) = C^{m-1}_{n+m-1} - \sum_{i=1}^{m-1} C^i_{V_i(x)+i-1} \quad \text{for all} \quad x \in X. \tag{5.5}$$

Note that $V_i(m^n) = 0$ for all $i \in [1, m-1]$, and so, $\Phi^d(m^n) = |X^*|$. Thus, as a corollary of Theorem 1, we get the following

Theorem 4. *Given two integers $n \geqslant 1$ and $m \geqslant 2$, if $X = [1, m]^n$ and $P^d = P^d_{m-1}$ is the weak order on X generated by the dual threshold rule, then we have: a function Φ^d maps X onto $[1, |X^*|]$ and is a preference function for P^d on X iff it is of the form (5.5).*

Note, in particular, that function (5.5) satisfies the axioms from Theorem B.

5.3. Dual algorithmic order on X^*. In order to present the dual algorithmic order on X^* corresponding to the weak order P^d, following (4.3) we set $n_i = n - V_i(x)$ for all $x \in X$ and $i \in [0, m]$. It follows that $n_0 = n$, $n_m = 0$ and $0 \leqslant n_i \leqslant n_{i-1}$ and $v_i(x) = n_{i-1} - n_i$ for all $i \in [1, m]$. Therefore, the monotone representative x^* of $x \in X$ is of the form (4.4) where $\widetilde{n} = (n_{m-1}, n_{m-2}, \dots, n_2, n_1)$ is such that $n_i \in [0, n_{i-1}]$ for all $i \in [1, m-1]$. If $\widetilde{n'} = (n'_{m-1}, n'_{m-2}, \dots, n'_2, n'_1)$ is such that $n'_i \in [0, n'_{i-1}]$ for all $i \in [1, m-1]$, then we have: $(x^*(\widetilde{n}), x^*(\widetilde{n'})) \in P^d$ iff $n'_{m-1} < n_{m-1}$ or there exists a number $k \in [1, m-2]$ such that $n'_i = n_i$ for all $i \in [k+1, m-1]$ and $n'_k < n_k$. It follows that the *dual algorithmic order on* X^* via (4.4), corresponding to the more greater P^d-preferability, can be described by the following rule: write out one by one a string of vectors $x^*(\widetilde{n})$ of the form (4.4) in such a way that n_{m-1} assumes successively the values $0, 1, \dots, n$, and if n_{m-1} is fixed, then the number n_{m-2} assumes successively the values $n_{m-1}, n_{m-1}+1, \dots, n$, and if n_{m-1} and n_{m-2} are fixed in the ranges $0 \leqslant n_{m-1} \leqslant n$ and $n_{m-1} \leqslant n_{m-2} \leqslant n$, then the number n_{m-3} assumes successively the values $n_{m-2}, n_{m-2}+1, \dots, n$, and so on, and finally, if $n_{m-1}, n_{m-2}, \dots, n_2$ are fixed and such that $n_i \leqslant n_{i-1} \leqslant n$ for all $i \in [3, m-1]$, then the number n_1 assumes successively the values $n_2, n_2+1, \dots, n$. According to the dual algorithmic order on X^*, to each $x^* \in X^*$ there corresponds a unique natural number, which is the ordinal number of x^* and, if x^* is of the form (4.4) for some collection $\widetilde{n} = (n_{m-1}, n_{m-2}, \dots, n_2, n_1)$ as above, then, by virtue of (5.5), this ordinal number of x^* is equal to

$$\Phi^d(x^*) = C^{m-1}_{n+m-1} - \sum_{i=1}^{m-1} C^{i}_{n-n_i+i-1}. \tag{5.6}$$

Examples of the dual algorithmic ordering of $X^* = [1, m]^{n*}$ for $m = 3, 4, 5$ and $n = 2, 3, 4, 5\,(6, 7)$, as compared to the algorithmic ordering of X^* in Section 6, are presented in Section 7.

6. Appendix 1. Tables of threshold orderings of X^*

Here we present examples of the algorithmic orderings of X^* for $X = [1, m]^n$ with $m = 3, 4, 5$ and $n = 2, 3, 4, 5\,(6, 7)$. The lower index at the right of a vector denotes its ordinal number with respect to the algorithmic order on X^*, i. e., the value (4.2) or (2.2) at the vector x^* from (4.4) (and all vectors $x \in I_{x^*}$). The greater this lower index is the more P-preferable is the corresponding vector-alternative. The number of elements in X^* is given by (1.5), which at the same time is equal to the maximal value of Φ on X^* and the ordinal number of the most P-preferable vector (m^n).

If $m = 2$, then $X = [1,2]^n$ and, according to (1.5), the value $s = |X^*|$ is equal to $n+1$, and the function (2.2) assumes the form: $\Phi(x) = v_2(x) + 1$. For example, if $n = 5$, we have the following natural ordering of X^*:

$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,2,2)_3$, $(1,1,2,2,2)_4$, $(1,2,2,2,2)_5$, $(2,2,2,2,2)_6$.

Suppose $m = 3$ or $X = [1,3]^n$. According to (1.5), the value $s = |X^*|$ is equal to $(n+2)(n+1)/2$, i.e., $6, 10, 15, 21, 28, 36$ for $n = 2,3,4,5,6,7$, respectively. The slightly transformed function (2.2) assumes the form:

$$\begin{aligned}\Phi(x) &= \frac{(v_2(x)+v_3(x)+1)(v_2(x)+v_3(x)+2)}{2} - v_2(x) = \\ &= \frac{(n+1-v_1(x))(n+2-v_1(x))}{2} - v_2(x),\end{aligned}$$

where $n = v_1(x) + v_2(x) + v_3(x)$. The set X^* is algorithmically ordered as follows.

<u>$n = 2$:</u>
$(1,1)_1$, $(1,2)_2$, $(1,3)_3$,
$(2,2)_4$, $(2,3)_5$, $(3,3)_6$.

<u>$n = 3$:</u>
$(1,1,1)_1$, $(1,1,2)_2$, $(1,1,3)_3$,
$(1,2,2)_4$, $(1,2,3)_5$, $(1,3,3)_6$,
$(2,2,2)_7$, $(2,2,3)_8$, $(2,3,3)_9$, $(3,3,3)_{10}$.

<u>$n = 4$:</u>
$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,1,3)_3$,
$(1,1,2,2)_4$, $(1,1,2,3)_5$, $(1,1,3,3)_6$,
$(1,2,2,2)_7$, $(1,2,2,3)_8$, $(1,2,3,3)_9$, $(1,3,3,3)_{10}$,
$(2,2,2,2)_{11}$, $(2,2,2,3)_{12}$, $(2,2,3,3)_{13}$, $(2,3,3,3)_{14}$, $(3,3,3,3)_{15}$.

<u>$n = 5$:</u>
$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$,
$(1,1,1,2,2)_4$, $(1,1,1,2,3)_5$, $(1,1,1,3,3)_6$,
$(1,1,2,2,2)_7$, $(1,1,2,2,3)_8$, $(1,1,2,3,3)_9$, $(1,1,3,3,3)_{10}$,
$(1,2,2,2,2)_{11}$, $(1,2,2,2,3)_{12}$, $(1,2,2,3,3)_{13}$, $(1,2,3,3,3)_{14}$, $(1,3,3,3,3)_{15}$,
$(2,2,2,2,2)_{16}$, $(2,2,2,2,3)_{17}$, $(2,2,2,3,3)_{18}$, $(2,2,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$,
$(3,3,3,3,3)_{21}$.

<u>$n = 6$:</u>
$(1,1,1,1,1,1)_1$, $(1,1,1,1,1,2)_2$, $(1,1,1,1,1,3)_3$,
$(1,1,1,1,2,2)_4$, $(1,1,1,1,2,3)_5$, $(1,1,1,1,3,3)_6$,
$(1,1,1,2,2,2)_7$, $(1,1,1,2,2,3)_8$, $(1,1,1,2,3,3)_9$, $(1,1,1,3,3,3)_{10}$,
$(1,1,2,2,2,2)_{11}$, $(1,1,2,2,2,3)_{12}$, $(1,1,2,2,3,3)_{13}$, $(1,1,2,3,3,3)_{14}$,
$(1,1,3,3,3,3)_{15}$,
$(1,2,2,2,2,2)_{16}$, $(1,2,2,2,2,3)_{17}$, $(1,2,2,2,3,3)_{18}$, $(1,2,2,3,3,3)_{19}$,
$(1,2,3,3,3,3)_{20}$, $(1,3,3,3,3,3)_{21}$,
$(2,2,2,2,2,2)_{22}$, $(2,2,2,2,2,3)_{23}$, $(2,2,2,2,3,3)_{24}$, $(2,2,2,3,3,3)_{25}$,
$(2,2,3,3,3,3)_{26}$, $(2,3,3,3,3,3)_{27}$, $(3,3,3,3,3,3)_{28}$.

$\underline{n=7:}$

$(1,1,1,1,1,1,1)_1$, $(1,1,1,1,1,1,2)_2$, $(1,1,1,1,1,1,3)_3$,
$(1,1,1,1,1,2,2)_4$, $(1,1,1,1,1,2,3)_5$, $(1,1,1,1,1,3,3)_6$,
$(1,1,1,1,2,2,2)_7$, $(1,1,1,1,2,2,3)_8$, $(1,1,1,1,2,3,3)_9$, $(1,1,1,1,3,3,3)_{10}$,
$(1,1,1,2,2,2,2)_{11}$, $(1,1,1,2,2,2,3)_{12}$, $(1,1,1,2,2,3,3)_{13}$, $(1,1,1,2,3,3,3)_{14}$,
$(1,1,1,3,3,3,3)_{15}$,
$(1,1,2,2,2,2,2)_{16}$, $(1,1,2,2,2,2,3)_{17}$, $(1,1,2,2,2,3,3)_{18}$, $(1,1,2,2,3,3,3)_{19}$,
$(1,1,2,3,3,3,3)_{20}$, $(1,1,3,3,3,3,3)_{21}$,
$(1,2,2,2,2,2,2)_{22}$, $(1,2,2,2,2,2,3)_{23}$, $(1,2,2,2,2,3,3)_{24}$, $(1,2,2,2,3,3,3)_{25}$,
$(1,2,2,3,3,3,3)_{26}$, $(1,2,3,3,3,3,3)_{27}$, $(1,3,3,3,3,3,3)_{28}$,
$(2,2,2,2,2,2,2)_{29}$, $(2,2,2,2,2,2,3)_{30}$, $(2,2,2,2,2,3,3)_{31}$, $(2,2,2,2,3,3,3)_{32}$,
$(2,2,2,3,3,3,3)_{33}$, $(2,2,3,3,3,3,3)_{34}$, $(2,3,3,3,3,3,3)_{35}$,
$(3,3,3,3,3,3,3)_{36}$.

Suppose $m=4$ or $X=[1,4]^n$. According to (1.5), the value $s=|X^*|$ is equal to $(n+3)(n+2)(n+1)/6$, i.e., $10, 20, 35, 56, 84$ for $n=2,3,4,5,6$, respectively. The slightly transformed function (2.2) assumes the form:

$$\Phi(x) = \frac{(n-v_1(x))(n+1-v_1(x))(n+2-v_1(x))}{6} + \\ + \frac{(v_3(x)+v_4(x)+1)(v_3(x)+v_4(x)+2)}{2} - v_3(x),$$

where $n=v_1(x)+v_2(x)+v_3(x)+v_4(x)$. We have the following algorithmic ordering of X^*.

$\underline{n=2:}$

$(1,1)_1$, $(1,2)_2$, $(1,3)_3$, $(1,4)_4$,
$(2,2)_5$, $(2,3)_6$, $(2,4)_7$,
$(3,3)_8$, $(3,4)_9$, $(4,4)_{10}$.

$\underline{n=3:}$

$(1,1,1)_1$, $(1,1,2)_2$, $(1,1,3)_3$, $(1,1,4)_4$,
$(1,2,2)_5$, $(1,2,3)_6$, $(1,2,4)_7$,
$(1,3,3)_8$, $(1,3,4)_9$, $(1,4,4)_{10}$,
$(2,2,2)_{11}$, $(2,2,3)_{12}$, $(2,2,4)_{13}$,
$(2,3,3)_{14}$, $(2,3,4)_{15}$, $(2,4,4)_{16}$,
$(3,3,3)_{17}$, $(3,3,4)_{18}$, $(3,4,4)_{19}$, $(4,4,4)_{20}$.

$\underline{n=4:}$

$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,1,3)_3$, $(1,1,1,4)_4$,
$(1,1,2,2)_5$, $(1,1,2,3)_6$, $(1,1,2,4)_7$,
$(1,1,3,3)_8$, $(1,1,3,4)_9$, $(1,1,4,4)_{10}$,
$(1,2,2,2)_{11}$, $(1,2,2,3)_{12}$, $(1,2,2,4)_{13}$,
$(1,2,3,3)_{14}$, $(1,2,3,4)_{15}$, $(1,2,4,4)_{16}$,
$(1,3,3,3)_{17}$, $(1,3,3,4)_{18}$, $(1,3,4,4)_{19}$, $(1,4,4,4)_{20}$,
$(2,2,2,2)_{21}$, $(2,2,2,3)_{22}$, $(2,2,2,4)_{23}$,

$(2,2,3,3)_{24}$, $(2,2,3,4)_{25}$, $(2,2,4,4)_{26}$,
$(2,3,3,3)_{27}$, $(2,3,3,4)_{28}$, $(2,3,4,4)_{29}$, $(2,4,4,4)_{30}$,
$(3,3,3,3)_{31}$, $(3,3,3,4)_{32}$, $(3,3,4,4)_{33}$, $(3,4,4,4)_{34}$, $(4,4,4,4)_{35}$.

$\underline{n=5}$:
$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$, $(1,1,1,1,4)_4$,
$(1,1,1,2,2)_5$, $(1,1,1,2,3)_6$, $(1,1,1,2,4)_7$,
$(1,1,1,3,3)_8$, $(1,1,1,3,4)_9$, $(1,1,1,4,4)_{10}$,
$(1,1,2,2,2)_{11}$, $(1,1,2,2,3)_{12}$, $(1,1,2,2,4)_{13}$,
$(1,1,2,3,3)_{14}$, $(1,1,2,3,4)_{15}$, $(1,1,2,4,4)_{16}$,
$(1,1,3,3,3)_{17}$, $(1,1,3,3,4)_{18}$, $(1,1,3,4,4)_{19}$, $(1,1,4,4,4)_{20}$,
$(1,2,2,2,2)_{21}$, $(1,2,2,2,3)_{22}$, $(1,2,2,2,4)_{23}$,
$(1,2,2,3,3)_{24}$, $(1,2,2,3,4)_{25}$, $(1,2,2,4,4)_{26}$,
$(1,2,3,3,3)_{27}$, $(1,2,3,3,4)_{28}$, $(1,2,3,4,4)_{29}$, $(1,2,4,4,4)_{30}$,
$(1,3,3,3,3)_{31}$, $(1,3,3,3,4)_{32}$, $(1,3,3,4,4)_{33}$, $(1,3,4,4,4)_{34}$, $(1,4,4,4,4)_{35}$,
$(2,2,2,2,2)_{36}$, $(2,2,2,2,3)_{37}$, $(2,2,2,2,4)_{38}$,
$(2,2,2,3,3)_{39}$, $(2,2,2,3,4)_{40}$, $(2,2,2,4,4)_{41}$,
$(2,2,3,3,3)_{42}$, $(2,2,3,3,4)_{43}$, $(2,2,3,4,4)_{44}$, $(2,2,4,4,4)_{45}$,
$(2,3,3,3,3)_{46}$, $(2,3,3,3,4)_{47}$, $(2,3,3,4,4)_{48}$, $(2,3,4,4,4)_{49}$, $(2,4,4,4,4)_{50}$,
$(3,3,3,3,3)_{51}$, $(3,3,3,3,4)_{52}$, $(3,3,3,4,4)_{53}$, $(3,3,4,4,4)_{54}$, $(3,4,4,4,4)_{55}$,
$(4,4,4,4,4)_{56}$.

$\underline{n=6}$:
$(1,1,1,1,1,1)_1$, $(1,1,1,1,1,2)_2$, $(1,1,1,1,1,3)_3$, $(1,1,1,1,1,4)_4$,
$(1,1,1,1,2,2)_5$, $(1,1,1,1,2,3)_6$, $(1,1,1,1,2,4)_7$,
$(1,1,1,1,3,3)_8$, $(1,1,1,1,3,4)_9$, $(1,1,1,1,4,4)_{10}$,
$(1,1,1,2,2,2)_{11}$, $(1,1,1,2,2,3)_{12}$, $(1,1,1,2,2,4)_{13}$,
$(1,1,1,2,3,3)_{14}$, $(1,1,1,2,3,4)_{15}$, $(1,1,1,2,4,4)_{16}$,
$(1,1,1,3,3,3)_{17}$, $(1,1,1,3,3,4)_{18}$, $(1,1,1,3,4,4)_{19}$, $(1,1,1,4,4,4)_{20}$,
$(1,1,2,2,2,2)_{21}$, $(1,1,2,2,2,3)_{22}$, $(1,1,2,2,2,4)_{23}$,
$(1,1,2,2,3,3)_{24}$, $(1,1,2,2,3,4)_{25}$, $(1,1,2,2,4,4)_{26}$,
$(1,1,2,3,3,3)_{27}$, $(1,1,2,3,3,4)_{28}$, $(1,1,2,3,4,4)_{29}$, $(1,1,2,4,4,4)_{30}$,
$(1,1,3,3,3,3)_{31}$, $(1,1,3,3,3,4)_{32}$, $(1,1,3,3,4,4)_{33}$, $(1,1,3,4,4,4)_{34}$,
$(1,1,4,4,4,4)_{35}$,
$(1,2,2,2,2,2)_{36}$, $(1,2,2,2,2,3)_{37}$, $(1,2,2,2,2,4)_{38}$,
$(1,2,2,2,3,3)_{39}$, $(1,2,2,2,3,4)_{40}$, $(1,2,2,2,4,4)_{41}$,
$(1,2,2,3,3,3)_{42}$, $(1,2,2,3,3,4)_{43}$, $(1,2,2,3,4,4)_{44}$, $(1,2,2,4,4,4)_{45}$,
$(1,2,3,3,3,3)_{46}$, $(1,2,3,3,3,4)_{47}$, $(1,2,3,3,4,4)_{48}$, $(1,2,3,4,4,4)_{49}$,
$(1,2,4,4,4,4)_{50}$,
$(1,3,3,3,3,3)_{51}$, $(1,3,3,3,3,4)_{52}$, $(1,3,3,3,4,4)_{53}$, $(1,3,3,4,4,4)_{54}$,
$(1,3,4,4,4,4)_{55}$, $(1,4,4,4,4,4)_{56}$,
$(2,2,2,2,2,2)_{57}$, $(2,2,2,2,2,3)_{58}$, $(2,2,2,2,2,4)_{59}$,
$(2,2,2,2,3,3)_{60}$, $(2,2,2,2,3,4)_{61}$, $(2,2,2,2,4,4)_{62}$,
$(2,2,2,3,3,3)_{63}$, $(2,2,2,3,3,4)_{64}$, $(2,2,2,3,4,4)_{65}$, $(2,2,2,4,4,4)_{66}$,
$(2,2,3,3,3,3)_{67}$, $(2,2,3,3,3,4)_{68}$, $(2,2,3,3,4,4)_{69}$, $(2,2,3,4,4,4)_{70}$,
$(2,2,4,4,4,4)_{71}$,

$(2,3,3,3,3,3)_{72}$, $(2,3,3,3,3,4)_{73}$, $(2,3,3,3,4,4)_{74}$, $(2,3,3,4,4,4)_{75}$,
$(2,3,4,4,4,4)_{76}$, $(2,4,4,4,4,4)_{77}$,
$(3,3,3,3,3,3)_{78}$, $(3,3,3,3,3,4)_{79}$, $(3,3,3,3,4,4)_{80}$, $(3,3,3,4,4,4)_{81}$,
$(3,3,4,4,4,4)_{82}$, $(3,4,4,4,4,4)_{83}$, $(4,4,4,4,4,4)_{84}$.

Suppose $m = 5$ or $X = [1,5]^n$. According to (1.5), the value $s = |X^*|$ is equal to $(n+4)(n+3)(n+2)(n+1)/(24)$, i.e., $15, 35, 70, 126$ for $n = 2,3,4,5$, respectively. The set X^* is algorithmically ordered as follows.

$\underline{n = 2}$:
$(1,1)_1$, $(1,2)_2$, $(1,3)_3$, $(1,4)_4$, $(1,5)_5$,
$(2,2)_6$, $(2,3)_7$, $(2,4)_8$, $(2,5)_9$,
$(3,3)_{10}$, $(3,4)_{11}$, $(3,5)_{12}$,
$(4,4)_{13}$, $(4,5)_{14}$, $(5,5)_{15}$.

$\underline{n = 3}$:
$(1,1,1)_1$, $(1,1,2)_2$, $(1,1,3)_3$, $(1,1,4)_4$, $(1,1,5)_5$,
$(1,2,2)_6$, $(1,2,3)_7$, $(1,2,4)_8$, $(1,2,5)_9$,
$(1,3,3)_{10}$, $(1,3,4)_{11}$, $(1,3,5)_{12}$,
$(1,4,4)_{13}$, $(1,4,5)_{14}$, $(1,5,5)_{15}$,
$(2,2,2)_{16}$, $(2,2,3)_{17}$, $(2,2,4)_{18}$, $(2,2,5)_{19}$,
$(2,3,3)_{20}$, $(2,3,4)_{21}$, $(2,3,5)_{22}$,
$(2,4,4)_{23}$, $(2,4,5)_{24}$, $(2,5,5)_{25}$,
$(3,3,3)_{26}$, $(3,3,4)_{27}$, $(3,3,5)_{28}$,
$(3,4,4)_{29}$, $(3,4,5)_{30}$, $(3,5,5)_{31}$,
$(4,4,4)_{32}$, $(4,4,5)_{33}$, $(4,5,5)_{34}$, $(5,5,5)_{35}$.

$\underline{n = 4}$:
$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,1,3)_3$, $(1,1,1,4)_4$, $(1,1,1,5)_5$,
$(1,1,2,2)_6$, $(1,1,2,3)_7$, $(1,1,2,4)_8$, $(1,1,2,5)_9$,
$(1,1,3,3)_{10}$, $(1,1,3,4)_{11}$, $(1,1,3,5)_{12}$,
$(1,1,4,4)_{13}$, $(1,1,4,5)_{14}$, $(1,1,5,5)_{15}$,
$(1,2,2,2)_{16}$, $(1,2,2,3)_{17}$, $(1,2,2,4)_{18}$, $(1,2,2,5)_{19}$,
$(1,2,3,3)_{20}$, $(1,2,3,4)_{21}$, $(1,2,3,5)_{22}$,
$(1,2,4,4)_{23}$, $(1,2,4,5)_{24}$, $(1,2,5,5)_{25}$,
$(1,3,3,3)_{26}$, $(1,3,3,4)_{27}$, $(1,3,3,5)_{28}$,
$(1,3,4,4)_{29}$, $(1,3,4,5)_{30}$, $(1,3,5,5)_{31}$,
$(1,4,4,4)_{32}$, $(1,4,4,5)_{33}$, $(1,4,5,5)_{34}$, $(1,5,5,5)_{35}$,
$(2,2,2,2)_{36}$, $(2,2,2,3)_{37}$, $(2,2,2,4)_{38}$, $(2,2,2,5)_{39}$,
$(2,2,3,3)_{40}$, $(2,2,3,4)_{41}$, $(2,2,3,5)_{42}$,
$(2,2,4,4)_{43}$, $(2,2,4,5)_{44}$, $(2,2,5,5)_{45}$,
$(2,3,3,3)_{46}$, $(2,3,3,4)_{47}$, $(2,3,3,5)_{48}$,
$(2,3,4,4)_{49}$, $(2,3,4,5)_{50}$, $(2,3,5,5)_{51}$,
$(2,4,4,4)_{52}$, $(2,4,4,5)_{53}$, $(2,4,5,5)_{54}$, $(2,5,5,5)_{55}$,
$(3,3,3,3)_{56}$, $(3,3,3,4)_{57}$, $(3,3,3,5)_{58}$,
$(3,3,4,4)_{59}$, $(3,3,4,5)_{60}$, $(3,3,5,5)_{61}$,

$(3,4,4,4)_{62}$, $(3,4,4,5)_{63}$, $(3,4,5,5)_{64}$, $(3,5,5,5)_{65}$,
$(4,4,4,4)_{66}$, $(4,4,4,5)_{67}$, $(4,4,5,5)_{68}$, $(4,5,5,5)_{69}$, $(5,5,5,5)_{70}$.

$\underline{n=5:}$
$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,1,3)_3$, $(1,1,1,1,4)_4$, $(1,1,1,1,5)_5$,
$(1,1,1,2,2)_6$, $(1,1,1,2,3)_7$, $(1,1,1,2,4)_8$, $(1,1,1,2,5)_9$,
$(1,1,1,3,3)_{10}$, $(1,1,1,3,4)_{11}$, $(1,1,1,3,5)_{12}$,
$(1,1,1,4,4)_{13}$, $(1,1,1,4,5)_{14}$, $(1,1,1,5,5)_{15}$,
$(1,1,2,2,2)_{16}$, $(1,1,2,2,3)_{17}$, $(1,1,2,2,4)_{18}$, $(1,1,2,2,5)_{19}$,
$(1,1,2,3,3)_{20}$, $(1,1,2,3,4)_{21}$, $(1,1,2,3,5)_{22}$,
$(1,1,2,4,4)_{23}$, $(1,1,2,4,5)_{24}$, $(1,1,2,5,5)_{25}$,
$(1,1,3,3,3)_{26}$, $(1,1,3,3,4)_{27}$, $(1,1,3,3,5)_{28}$,
$(1,1,3,4,4)_{29}$, $(1,1,3,4,5)_{30}$, $(1,1,3,5,5)_{31}$,
$(1,1,4,4,4)_{32}$, $(1,1,4,4,5)_{33}$, $(1,1,4,5,5)_{34}$, $(1,1,5,5,5)_{35}$,
$(1,2,2,2,2)_{36}$, $(1,2,2,2,3)_{37}$, $(1,2,2,2,4)_{38}$, $(1,2,2,2,5)_{39}$,
$(1,2,2,3,3)_{40}$, $(1,2,2,3,4)_{41}$, $(1,2,2,3,5)_{42}$,
$(1,2,2,4,4)_{43}$, $(1,2,2,4,5)_{44}$, $(1,2,2,5,5)_{45}$,
$(1,2,3,3,3)_{46}$, $(1,2,3,3,4)_{47}$, $(1,2,3,3,5)_{48}$,
$(1,2,3,4,4)_{49}$, $(1,2,3,4,5)_{50}$, $(1,2,3,5,5)_{51}$,
$(1,2,4,4,4)_{52}$, $(1,2,4,4,5)_{53}$, $(1,2,4,5,5)_{54}$, $(1,2,5,5,5)_{55}$,
$(1,3,3,3,3)_{56}$, $(1,3,3,3,4)_{57}$, $(1,3,3,3,5)_{58}$,
$(1,3,3,4,4)_{59}$, $(1,3,3,4,5)_{60}$, $(1,3,3,5,5)_{61}$,
$(1,3,4,4,4)_{62}$, $(1,3,4,4,5)_{63}$, $(1,3,4,5,5)_{64}$, $(1,3,5,5,5)_{65}$,
$(1,4,4,4,4)_{66}$, $(1,4,4,4,5)_{67}$, $(1,4,4,5,5)_{68}$, $(1,4,5,5,5)_{69}$, $(1,5,5,5,5)_{70}$,
$(2,2,2,2,2)_{71}$, $(2,2,2,2,3)_{72}$, $(2,2,2,2,4)_{73}$, $(2,2,2,2,5)_{74}$,
$(2,2,2,3,3)_{75}$, $(2,2,2,3,4)_{76}$, $(2,2,2,3,5)_{77}$,
$(2,2,2,4,4)_{78}$, $(2,2,2,4,5)_{79}$, $(2,2,2,5,5)_{80}$,
$(2,2,3,3,3)_{81}$, $(2,2,3,3,4)_{82}$, $(2,2,3,3,5)_{83}$,
$(2,2,3,4,4)_{84}$, $(2,2,3,4,5)_{85}$, $(2,2,3,5,5)_{86}$,
$(2,2,4,4,4)_{87}$, $(2,2,4,4,5)_{88}$, $(2,2,4,5,5)_{89}$, $(2,2,5,5,5)_{90}$,
$(2,3,3,3,3)_{91}$, $(2,3,3,3,4)_{92}$, $(2,3,3,3,5)_{93}$,
$(2,3,3,4,4)_{94}$, $(2,3,3,4,5)_{95}$, $(2,3,3,5,5)_{96}$,
$(2,3,4,4,4)_{97}$, $(2,3,4,4,5)_{98}$, $(2,3,4,5,5)_{99}$, $(2,3,5,5,5)_{100}$,
$(2,4,4,4,4)_{101}$, $(2,4,4,4,5)_{102}$, $(2,4,4,5,5)_{103}$, $(2,4,5,5,5)_{104}$, $(2,5,5,5,5)_{105}$,
$(3,3,3,3,3)_{106}$, $(3,3,3,3,4)_{107}$, $(3,3,3,3,5)_{108}$,
$(3,3,3,4,4)_{109}$, $(3,3,3,4,5)_{110}$, $(3,3,3,5,5)_{111}$,
$(3,3,4,4,4)_{112}$, $(3,3,4,4,5)_{113}$, $(3,3,4,5,5)_{114}$, $(3,3,5,5,5)_{115}$,
$(3,4,4,4,4)_{116}$, $(3,4,4,4,5)_{117}$, $(3,4,4,5,5)_{118}$, $(3,4,5,5,5)_{119}$, $(3,5,5,5,5)_{120}$,
$(4,4,4,4,4)_{121}$, $(4,4,4,4,5)_{122}$, $(4,4,4,5,5)_{123}$, $(4,4,5,5,5)_{124}$, $(4,5,5,5,5)_{125}$,
$(5,5,5,5,5)_{126}$.

7. Appendix 2. Tables of dual threshold orderings of X^*

In this Section we present examples of the dual algorithmic orderings of X^* for $X=[1,m]^n$ with $m=3,4,5$ and $n=2,3,4,5\,(6,7)$. The lower index at the right of a vector is the ordinal number of the vector with respect to the dual algorithmic order on X^*, i. e., the value (5.6) or (5.5) at the vector x^* from (4.4) (and all vectors $x \in I_{x^*}$). The greater this lower index is the more P^d-preferable is the corresponding vector-alternative. The number of elements in X^* is given by (1.5), which at the same time is equal to the maximal value of Φ^d on X^* and the ordinal number of the most P^d-preferable vector (m^n).

If $m = 2$, then $X = [1, 2]^n$ and, according to (1.5), the value $s = |X^*|$ is equal to $n + 1$, and the function (5.5) assumes the form: $\Phi^d(x) = v_2(x) + 1$.

Suppose $m = 3$ or $X = [1, 3]^n$. According to (1.5), the value $s = |X^*|$ is equal to $(n + 2)(n + 1)/2$, i.e., $6, 10, 15, 21, 28, 36$ for $n = 2, 3, 4, 5, 6, 7$, respectively. The function (5.5) assumes the form:

$$\Phi^d(x) = \frac{(n+1)(n+2)}{2} - v_1(x) - \frac{(v_1(x) + v_2(x))(v_1(x) + v_2(x) + 1)}{2}.$$

where $n = v_1(x) + v_2(x) + v_3(x)$. The set X^* is ordered with respect to the dual algorithmic order as follows.

<u>$n = 2$:</u>
$(1,1)_1$, $(1,2)_2$, $(2,2)_3$,
$(1,3)_4$, $(2,3)_5$, $(3,3)_6$.

<u>$n = 3$:</u>
$(1,1,1)_1$, $(1,1,2)_2$, $(1,2,2)_3$, $(2,2,2)_4$,
$(1,1,3)_5$, $(1,2,3)_6$, $(2,2,3)_7$,
$(1,3,3)_8$, $(2,3,3)_9$, $(3,3,3)_{10}$.

<u>$n = 4$:</u>
$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,2,2)_3$, $(1,2,2,2)_4$, $(2,2,2,2)_5$,
$(1,1,1,3)_6$, $(1,1,2,3)_7$, $(1,2,2,3)_8$, $(2,2,2,3)_9$,
$(1,1,3,3)_{10}$, $(1,2,3,3)_{11}$, $(2,2,3,3)_{12}$,
$(1,3,3,3)_{13}$, $(2,3,3,3)_{14}$, $(3,3,3,3)_{15}$.

<u>$n = 5$:</u>
$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,2,2)_3$, $(1,1,2,2,2)_4$, $(1,2,2,2,2)_5$,
$(2,2,2,2,2)_6$,
$(1,1,1,1,3)_7$, $(1,1,1,2,3)_8$, $(1,1,2,2,3)_9$, $(1,2,2,2,3)_{10}$, $(2,2,2,2,3)_{11}$,
$(1,1,1,3,3)_{12}$, $(1,1,2,3,3)_{13}$, $(1,2,2,3,3)_{14}$, $(2,2,2,3,3)_{15}$,
$(1,1,3,3,3)_{16}$, $(1,2,3,3,3)_{17}$, $(2,2,3,3,3)_{18}$,
$(1,3,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$, $(3,3,3,3,3)_{21}$.

<u>$n = 6$:</u>
$(1,1,1,1,1,1)_1$, $(1,1,1,1,1,2)_2$, $(1,1,1,1,2,2)_3$, $(1,1,1,2,2,2)_4$,
$(1,1,2,2,2,2)_5$, $(1,2,2,2,2,2)_6$, $(2,2,2,2,2,2)_7$,
$(1,1,1,1,1,3)_8$, $(1,1,1,1,2,3)_9$, $(1,1,1,2,2,3)_{10}$, $(1,1,2,2,2,3)_{11}$,
$(1,2,2,2,2,3)_{12}$, $(2,2,2,2,2,3)_{13}$,
$(1,1,1,1,3,3)_{14}$, $(1,1,1,2,3,3)_{15}$, $(1,1,2,2,3,3)_{16}$, $(1,2,2,2,3,3)_{17}$,

$(2,2,2,2,3,3)_{18}$,
$(1,1,1,3,3,3)_{19}$, $(1,1,2,3,3,3)_{20}$, $(1,2,2,3,3,3)_{21}$, $(2,2,2,3,3,3)_{22}$,
$(1,1,3,3,3,3)_{23}$, $(1,2,3,3,3,3)_{24}$, $(2,2,3,3,3,3)_{25}$,
$(1,3,3,3,3,3)_{26}$, $(2,3,3,3,3,3)_{27}$, $(3,3,3,3,3,3)_{28}$.

$\underline{n=7}$:
$(1,1,1,1,1,1,1)_1$, $(1,1,1,1,1,1,2)_2$, $(1,1,1,1,1,2,2)_3$, $(1,1,1,1,2,2,2)_4$,
$(1,1,1,2,2,2,2)_5$, $(1,1,2,2,2,2,2)_6$, $(1,2,2,2,2,2,2)_7$,
$(2,2,2,2,2,2,2)_8$,
$(1,1,1,1,1,1,3)_9$, $(1,1,1,1,1,2,3)_{10}$, $(1,1,1,1,2,2,3)_{11}$, $(1,1,1,2,2,2,3)_{12}$,
$(1,1,2,2,2,2,3)_{13}$, $(1,2,2,2,2,2,3)_{14}$, $(2,2,2,2,2,2,3)_{15}$,
$(1,1,1,1,1,3,3)_{16}$, $(1,1,1,1,2,3,3)_{17}$, $(1,1,1,2,2,3,3)_{18}$, $(1,1,2,2,2,3,3)_{19}$,
$(1,2,2,2,2,3,3)_{20}$, $(2,2,2,2,2,3,3)_{21}$,
$(1,1,1,1,3,3,3)_{22}$, $(1,1,1,2,3,3,3)_{23}$, $(1,1,2,2,3,3,3)_{24}$, $(1,2,2,2,3,3,3)_{25}$,
$(2,2,2,2,3,3,3)_{26}$,
$(1,1,1,3,3,3,3)_{27}$, $(1,1,2,3,3,3,3)_{28}$, $(1,2,2,3,3,3,3)_{29}$, $(2,2,2,3,3,3,3)_{30}$,
$(1,1,3,3,3,3,3)_{31}$, $(1,2,3,3,3,3,3)_{32}$, $(2,2,3,3,3,3,3)_{33}$,
$(1,3,3,3,3,3,3)_{34}$, $(2,3,3,3,3,3,3)_{35}$, $(3,3,3,3,3,3,3)_{36}$.

Suppose $m=4$ or $X=[1,4]^n$. According to (1.5), the value $s=|X^*|$ is equal to $(n+3)(n+2)(n+1)/6$, i.e., $10, 20, 35, 56, 84$ for $n=2,3,4,5,6$, respectively. The function (5.5) is of the form:

$$\Phi^d(x) = \frac{(n+1)(n+2)(n+3)}{6} - v_1(x) - \frac{(v_1(x)+v_2(x))(v_1(x)+v_2(x)+1)}{2} - \\ - \frac{(n-v_4(x))(n+1-v_4(x))(n+2-v_4(x))}{6}\,.$$

where $n = v_1(x)+v_2(x)+v_3(x)+v_4(x)$. We have the following dual algorithmic ordering of X^*.

$\underline{n=2}$:
$(1,1)_1$, $(1,2)_2$, $(2,2)_3$,
$(1,3)_4$, $(2,3)_5$, $(3,3)_6$,
$(1,4)_7$, $(2,4)_8$, $(3,4)_9$, $(4,4)_{10}$.

$\underline{n=3}$:
$(1,1,1)_1$, $(1,1,2)_2$, $(1,2,2)_3$, $(2,2,2)_4$,
$(1,1,3)_5$, $(1,2,3)_6$, $(2,2,3)_7$,
$(1,3,3)_8$, $(2,3,3)_9$, $(3,3,3)_{10}$,
$(1,1,4)_{11}$, $(1,2,4)_{12}$, $(2,2,4)_{13}$,
$(1,3,4)_{14}$, $(2,3,4)_{15}$, $(3,3,4)_{16}$,
$(1,4,4)_{17}$, $(2,4,4)_{18}$, $(3,4,4)_{19}$, $(4,4,4)_{20}$.

$\underline{n=4}$:
$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,2,2)_3$, $(1,2,2,2)_4$, $(2,2,2,2)_5$,
$(1,1,1,3)_6$, $(1,1,2,3)_7$, $(1,2,2,3)_8$, $(2,2,2,3)_9$,

$(1,1,3,3)_{10}$, $(1,2,3,3)_{11}$, $(2,2,3,3)_{12}$,
$(1,3,3,3)_{13}$, $(2,3,3,3)_{14}$, $(3,3,3,3)_{15}$,
$(1,1,1,4)_{16}$, $(1,1,2,4)_{17}$, $(1,2,2,4)_{18}$, $(2,2,2,4)_{19}$,
$(1,1,3,4)_{20}$, $(1,2,3,4)_{21}$, $(2,2,3,4)_{22}$,
$(1,3,3,4)_{23}$, $(2,3,3,4)_{24}$, $(3,3,3,4)_{25}$,
$(1,1,4,4)_{26}$, $(1,2,4,4)_{27}$, $(2,2,4,4)_{28}$,
$(1,3,4,4)_{29}$, $(2,3,4,4)_{30}$, $(3,3,4,4)_{31}$,
$(1,4,4,4)_{32}$, $(2,4,4,4)_{33}$, $(3,4,4,4)_{34}$, $(4,4,4,4)_{35}$.

$n=5$:
$(1,1,1,1,1)_1$, $(1,1,1,1,2)_2$, $(1,1,1,2,2)_3$, $(1,1,2,2,2)_4$, $(1,2,2,2,2)_5$, $(2,2,2,2,2)_6$,
$(1,1,1,1,3)_7$, $(1,1,1,2,3)_8$, $(1,1,2,2,3)_9$, $(1,2,2,2,3)_{10}$, $(2,2,2,2,3)_{11}$,
$(1,1,1,3,3)_{12}$, $(1,1,2,3,3)_{13}$, $(1,2,2,3,3)_{14}$, $(2,2,2,3,3)_{15}$,
$(1,1,3,3,3)_{16}$, $(1,2,3,3,3)_{17}$, $(2,2,3,3,3)_{18}$,
$(1,3,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$, $(3,3,3,3,3)_{21}$,
$(1,1,1,1,4)_{22}$, $(1,1,1,2,4)_{23}$, $(1,1,2,2,4)_{24}$, $(1,2,2,2,4)_{25}$, $(2,2,2,2,4)_{26}$,
$(1,1,1,3,4)_{27}$, $(1,1,2,3,4)_{28}$, $(1,2,2,3,4)_{29}$, $(2,2,2,3,4)_{30}$,
$(1,1,3,3,4)_{31}$, $(1,2,3,3,4)_{32}$, $(2,2,3,3,4)_{33}$,
$(1,3,3,3,4)_{34}$, $(2,3,3,3,4)_{35}$, $(3,3,3,3,4)_{36}$,
$(1,1,1,4,4)_{37}$, $(1,1,2,4,4)_{38}$, $(1,2,2,4,4)_{39}$, $(2,2,2,4,4)_{40}$,
$(1,1,3,4,4)_{41}$, $(1,2,3,4,4)_{42}$, $(2,2,3,4,4)_{43}$,
$(1,3,3,4,4)_{44}$, $(2,3,3,4,4)_{45}$, $(3,3,3,4,4)_{46}$,
$(1,1,4,4,4)_{47}$, $(1,2,4,4,4)_{48}$, $(2,2,4,4,4)_{49}$,
$(1,3,4,4,4)_{50}$, $(2,3,4,4,4)_{51}$, $(3,3,4,4,4)_{52}$,
$(1,4,4,4,4)_{53}$, $(2,4,4,4,4)_{54}$, $(3,4,4,4,4)_{55}$, $(4,4,4,4,4)_{56}$.

$n=6$:
$(1,1,1,1,1,1)_1$, $(1,1,1,1,1,2)_2$, $(1,1,1,1,2,2)_3$, $(1,1,1,2,2,2)_4$, $(1,1,2,2,2,2)_5$, $(1,2,2,2,2,2)_6$, $(2,2,2,2,2,2)_7$,
$(1,1,1,1,1,3)_8$, $(1,1,1,1,2,3)_9$, $(1,1,1,2,2,3)_{10}$, $(1,1,2,2,2,3)_{11}$, $(1,2,2,2,2,3)_{12}$, $(2,2,2,2,2,3)_{13}$,
$(1,1,1,1,3,3)_{14}$, $(1,1,1,2,3,3)_{15}$, $(1,1,2,2,3,3)_{16}$, $(1,2,2,2,3,3)_{17}$, $(2,2,2,2,3,3)_{18}$,
$(1,1,1,3,3,3)_{19}$, $(1,1,2,3,3,3)_{20}$, $(1,2,2,3,3,3)_{21}$, $(2,2,2,3,3,3)_{22}$,
$(1,1,3,3,3,3)_{23}$, $(1,2,3,3,3,3)_{24}$, $(2,2,3,3,3,3)_{25}$,
$(1,3,3,3,3,3)_{26}$, $(2,3,3,3,3,3)_{27}$, $(3,3,3,3,3,3)_{28}$,
$(1,1,1,1,1,4)_{29}$, $(1,1,1,1,2,4)_{30}$, $(1,1,1,2,2,4)_{31}$, $(1,1,2,2,2,4)_{32}$, $(1,2,2,2,2,4)_{33}$, $(2,2,2,2,2,4)_{34}$,
$(1,1,1,1,3,4)_{35}$, $(1,1,1,2,3,4)_{36}$, $(1,1,2,2,3,4)_{37}$, $(1,2,2,2,3,4)_{38}$, $(2,2,2,2,3,4)_{39}$,
$(1,1,1,3,3,4)_{40}$, $(1,1,2,3,3,4)_{41}$, $(1,2,2,3,3,4)_{42}$, $(2,2,2,3,3,4)_{43}$,
$(1,1,3,3,3,4)_{44}$, $(1,2,3,3,3,4)_{45}$, $(2,2,3,3,3,4)_{46}$,
$(1,3,3,3,3,4)_{47}$, $(2,3,3,3,3,4)_{48}$, $(3,3,3,3,3,4)_{49}$,
$(1,1,1,1,4,4)_{50}$, $(1,1,1,2,4,4)_{51}$, $(1,1,2,2,4,4)_{52}$, $(1,2,2,2,4,4)_{53}$, $(2,2,2,2,4,4)_{54}$,

$(1,1,1,3,4,4)_{55}$, $(1,1,2,3,4,4)_{56}$, $(1,2,2,3,4,4)_{57}$, $(2,2,2,3,4,4)_{58}$,
$(1,1,3,3,4,4)_{59}$, $(1,2,3,3,4,4)_{60}$, $(2,2,3,3,4,4)_{61}$,
$(1,3,3,3,4,4)_{62}$, $(2,3,3,3,4,4)_{63}$, $(3,3,3,3,4,4)_{64}$,
$(1,1,1,4,4,4)_{65}$, $(1,1,2,4,4,4)_{66}$, $(1,2,2,4,4,4)_{67}$, $(2,2,2,4,4,4)_{68}$,
$(1,1,3,4,4,4)_{69}$, $(1,2,3,4,4,4)_{70}$, $(2,2,3,4,4,4)_{71}$,
$(1,3,3,4,4,4)_{72}$, $(2,3,3,4,4,4)_{73}$, $(3,3,3,4,4,4)_{74}$,
$(1,1,4,4,4,4)_{75}$, $(1,2,4,4,4,4)_{76}$, $(2,2,4,4,4,4)_{77}$,
$(1,3,4,4,4,4)_{78}$, $(2,3,4,4,4,4)_{79}$, $(3,3,4,4,4,4)_{80}$,
$(1,4,4,4,4,4)_{81}$, $(2,4,4,4,4,4)_{82}$, $(3,4,4,4,4,4)_{83}$, $(4,4,4,4,4,4)_{84}$.

Suppose $m = 5$ or $X = [1,5]^n$. According to (1.5), the value $s = |X^*|$ is equal to $(n+4)(n+3)(n+2)(n+1)/(24)$, i.e., $15, 35, 70, 126$ for $n = 2,3,4,5$, respectively. The set X^* is dually algorithmically ordered as follows.

$\underline{n = 2}$:
$(1,1)_1$, $(1,2)_2$, $(2,2)_3$,
$(1,3)_4$, $(2,3)_5$, $(3,3)_6$,
$(1,4)_7$, $(2,4)_8$, $(3,4)_9$, $(4,4)_{10}$,
$(1,5)_{11}$, $(2,5)_{12}$, $(3,5)_{13}$, $(4,5)_{14}$, $(5,5)_{15}$.

$\underline{n = 3}$:
$(1,1,1)_1$, $(1,1,2)_2$, $(1,2,2)_3$, $(2,2,2)_4$,
$(1,1,3)_5$, $(1,2,3)_6$, $(2,2,3)_7$,
$(1,3,3)_8$, $(2,3,3)_9$, $(3,3,3)_{10}$,
$(1,1,4)_{11}$, $(1,2,4)_{12}$, $(2,2,4)_{13}$,
$(1,3,4)_{14}$, $(2,3,4)_{15}$, $(3,3,4)_{16}$,
$(1,4,4)_{17}$, $(2,4,4)_{18}$, $(3,4,4)_{19}$, $(4,4,4)_{20}$,
$(1,1,5)_{21}$, $(1,2,5)_{22}$, $(2,2,5)_{23}$,
$(1,3,5)_{24}$, $(2,3,5)_{25}$, $(3,3,5)_{26}$,
$(1,4,5)_{27}$, $(2,4,5)_{28}$, $(3,4,5)_{29}$, $(4,4,5)_{30}$,
$(1,5,5)_{31}$, $(2,5,5)_{32}$, $(3,5,5)_{33}$, $(4,5,5)_{34}$, $(5,5,5)_{35}$.

$\underline{n = 4}$:
$(1,1,1,1)_1$, $(1,1,1,2)_2$, $(1,1,2,2)_3$, $(1,2,2,2)_4$, $(2,2,2,2)_5$,
$(1,1,1,3)_6$, $(1,1,2,3)_7$, $(1,2,2,3)_8$, $(2,2,2,3)_9$,
$(1,1,3,3)_{10}$, $(1,2,3,3)_{11}$, $(2,2,3,3)_{12}$,
$(1,3,3,3)_{13}$, $(2,3,3,3)_{14}$, $(3,3,3,3)_{15}$,
$(1,1,1,4)_{16}$, $(1,1,2,4)_{17}$, $(1,2,2,4)_{18}$, $(2,2,2,4)_{19}$,
$(1,1,3,4)_{20}$, $(1,2,3,4)_{21}$, $(2,2,3,4)_{22}$,
$(1,3,3,4)_{23}$, $(2,3,3,4)_{24}$, $(3,3,3,4)_{25}$,
$(1,1,4,4)_{26}$, $(1,2,4,4)_{27}$, $(2,2,4,4)_{28}$,
$(1,3,4,4)_{29}$, $(2,3,4,4)_{30}$, $(3,3,4,4)_{31}$,
$(1,4,4,4)_{32}$, $(2,4,4,4)_{33}$, $(3,4,4,4)_{34}$, $(4,4,4,4)_{35}$,
$(1,1,1,5)_{36}$, $(1,1,2,5)_{37}$, $(1,2,2,5)_{38}$, $(2,2,2,5)_{39}$,
$(1,1,3,5)_{40}$, $(1,2,3,5)_{41}$, $(2,2,3,5)_{42}$,
$(1,3,3,5)_{43}$, $(2,3,3,5)_{44}$, $(3,3,3,5)_{45}$,

$(1,1,4,5)_{46}$, $(1,2,4,5)_{47}$, $(2,2,4,5)_{48}$,
$(1,3,4,5)_{49}$, $(2,3,4,5)_{50}$, $(3,3,4,5)_{51}$,
$(1,4,4,5)_{52}$, $(2,4,4,5)_{53}$, $(3,4,4,5)_{54}$, $(4,4,4,5)_{55}$,
$(1,1,5,5)_{56}$, $(1,2,5,5)_{57}$, $(2,2,5,5)_{58}$,
$(1,3,5,5)_{59}$, $(2,3,5,5)_{60}$, $(3,3,5,5)_{61}$,
$(1,4,5,5)_{62}$, $(2,4,5,5)_{63}$, $(3,4,5,5)_{64}$, $(4,4,5,5)_{65}$,
$(1,5,5,5)_{66}$, $(2,5,5,5)_{67}$, $(3,5,5,5)_{68}$, $(4,5,5,5)_{69}$, $(5,5,5,5)_{70}$.

$\underline{n=5}$:
$(1,1,1,1,1)_{1}$, $(1,1,1,1,2)_{2}$, $(1,1,1,2,2)_{3}$, $(1,1,2,2,2)_{4}$, $(1,2,2,2,2)_{5}$,
$(2,2,2,2,2)_{6}$,
$(1,1,1,1,3)_{7}$, $(1,1,1,2,3)_{8}$, $(1,1,2,2,3)_{9}$, $(1,2,2,2,3)_{10}$, $(2,2,2,2,3)_{11}$,
$(1,1,1,3,3)_{12}$, $(1,1,2,3,3)_{13}$, $(1,2,2,3,3)_{14}$, $(2,2,2,3,3)_{15}$,
$(1,1,3,3,3)_{16}$, $(1,2,3,3,3)_{17}$, $(2,2,3,3,3)_{18}$,
$(1,3,3,3,3)_{19}$, $(2,3,3,3,3)_{20}$, $(3,3,3,3,3)_{21}$,
$(1,1,1,1,4)_{22}$, $(1,1,1,2,4)_{23}$, $(1,1,2,2,4)_{24}$, $(1,2,2,2,4)_{25}$, $(2,2,2,2,4)_{26}$,
$(1,1,1,3,4)_{27}$, $(1,1,2,3,4)_{28}$, $(1,2,2,3,4)_{29}$, $(2,2,2,3,4)_{30}$,
$(1,1,3,3,4)_{31}$, $(1,2,3,3,4)_{32}$, $(2,2,3,3,4)_{33}$,
$(1,3,3,3,4)_{34}$, $(2,3,3,3,4)_{35}$, $(3,3,3,3,4)_{36}$,
$(1,1,1,4,4)_{37}$, $(1,1,2,4,4)_{38}$, $(1,2,2,4,4)_{39}$, $(2,2,2,4,4)_{40}$,
$(1,1,3,4,4)_{41}$, $(1,2,3,4,4)_{42}$, $(2,2,3,4,4)_{43}$,
$(1,3,3,4,4)_{44}$, $(2,3,3,4,4)_{45}$, $(3,3,3,4,4)_{46}$,
$(1,1,4,4,4)_{47}$, $(1,2,4,4,4)_{48}$, $(2,2,4,4,4)_{49}$,
$(1,3,4,4,4)_{50}$, $(2,3,4,4,4)_{51}$, $(3,3,4,4,4)_{52}$,
$(1,4,4,4,4)_{53}$, $(2,4,4,4,4)_{54}$, $(3,4,4,4,4)_{55}$, $(4,4,4,4,4)_{56}$,
$(1,1,1,1,5)_{57}$, $(1,1,1,2,5)_{58}$, $(1,1,2,2,5)_{59}$, $(1,2,2,2,5)_{60}$, $(2,2,2,2,5)_{61}$,
$(1,1,1,3,5)_{62}$, $(1,1,2,3,5)_{63}$, $(1,2,2,3,5)_{64}$, $(2,2,2,3,5)_{65}$,
$(1,1,3,3,5)_{66}$, $(1,2,3,3,5)_{67}$, $(2,2,3,3,5)_{68}$,
$(1,3,3,3,5)_{69}$, $(2,3,3,3,5)_{70}$, $(3,3,3,3,5)_{71}$,
$(1,1,1,4,5)_{72}$, $(1,1,2,4,5)_{73}$, $(1,2,2,4,5)_{74}$, $(2,2,2,4,5)_{75}$,
$(1,1,3,4,4)_{76}$, $(1,2,3,4,5)_{77}$, $(2,2,3,4,5)_{78}$,
$(1,3,3,4,5)_{79}$, $(2,3,3,4,5)_{80}$, $(3,3,3,4,5)_{81}$,
$(1,1,4,4,5)_{82}$, $(1,2,4,4,5)_{83}$, $(2,2,4,4,5)_{84}$,
$(1,3,4,4,5)_{85}$, $(2,3,4,4,5)_{86}$, $(3,3,4,4,5)_{87}$,
$(1,4,4,4,5)_{88}$, $(2,4,4,4,5)_{89}$, $(3,4,4,4,5)_{90}$, $(4,4,4,4,5)_{91}$,
$(1,1,1,5,5)_{92}$, $(1,1,2,5,5)_{93}$, $(1,2,2,5,5)_{94}$, $(2,2,2,5,5)_{95}$,
$(1,1,3,5,5)_{96}$, $(1,2,3,5,5)_{97}$, $(2,2,3,5,5)_{98}$,
$(1,3,3,5,5)_{99}$, $(2,3,3,5,5)_{100}$, $(3,3,3,5,5)_{101}$,
$(1,1,4,5,5)_{102}$, $(1,2,4,5,5)_{103}$, $(2,2,4,5,5)_{104}$,
$(1,3,4,5,5)_{105}$, $(2,3,4,5,5)_{106}$, $(3,3,4,5,5)_{107}$,
$(1,4,4,5,5)_{108}$, $(2,4,4,5,5)_{109}$, $(3,4,4,5,5)_{110}$, $(4,4,4,5,5)_{111}$,
$(1,1,5,5,5)_{112}$, $(1,2,5,5,5)_{113}$, $(2,2,5,5,5)_{114}$,
$(1,3,5,5,5)_{115}$, $(2,3,5,5,5)_{116}$, $(3,3,5,5,5)_{117}$,
$(1,4,5,5,5)_{118}$, $(2,4,5,5,5)_{119}$, $(3,4,5,5,5)_{120}$, $(4,4,5,5,5)_{121}$,
$(1,5,5,5,5)_{122}$, $(2,5,5,5,5)_{123}$, $(3,5,5,5,5)_{124}$, $(4,5,5,5,5)_{125}$, $(5,5,5,5,5)_{126}$.

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Алескеров Ф.Т., Чистяков В.В., Калягин В.А.

Многокритериальные пороговые алгоритмы принятия решений: Препринт WP7/2010/02. – М.: Издательский дом Государственного университета – Высшей школы экономики, 2010. – 40 с. (на англ. яз.).

Достаточно часто на практике альтернативы оцениваются $n \geqslant 2$ оценками $x_1, \ldots, x_n$, каждая их которых может принимать целое значение от 1 («плохо») до $m \geqslant 3$ («отлично»). Поэтому возникает проблема ранжировать элементы множества X, состоящего из всех n-мерных векторов с целочисленными компонентами от 1 до m. В предположении, что низкая оценка не может быть компенсирована никаким числом высоких оценок, мы вводим понятие перечислительной функции предпочтений для слабого порядка на X, порожденного пороговым правилом, и находим явное значение этой функции. Это позволяет описать все классы эквивалентности и классы безразличия этого слабого порядка. Дается алгоритм упорядочения монотонных представителей классов эквивалентности. Построена модель, двойственная к рассматриваемой, и для нее приведен явный вид двойственной перечислительной функции предпочтений и алгоритм упорядочения соответствующих монотонных представителей.

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