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THE EXACT BIAS OF THE BANZHAF MEASURE OF POWER WHEN VOTES ARE NEITHER EQUIPROBABLE NOR INDEPENDENT

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This paper discusses a numerical scheme for computing the Banzhaf swing probability when votes are neither equiproba-ble nor independent. Examples indicate a substantial bias in the Banzhaf measure of voting power if neither assumption is met. The analytical part derives the exact magnitude of the bias due to the common probability of an affirmative vote deviating from one half and due to common correlation in unweighted simple-majority games. The former bias is polynomial, the latter is linear. A modified square-root rule for two-tier voting systems that takes into account both the homogeneity and the size of constituencies is also provided.

JEL-Codes: D72 Key Words: voting power, Banzhaf measure, correlated votes

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В статье обсуждается численная схема для вычисления вероятности перемены при расчете индекса Банцафа, когда голосование не удовлетворяет ни условию равновероятности, ни условию независимости. Примеры показывают, что в этом случае имеет место достаточно значительное отклонение значения индекса Банцафа. Аналитически выводятся точные величины отклонений, зависящих от отклонения вероятности голосования «за» от ½ и от корреляции в невзвешенной игре простого большинства. В первом случае оценка величины отклонения положительна, во втором – линейна. Для двухступенчатой системы голосования предлагается модифицированное правило «квадратного корня», которое учитывает однородность и размер избирательного округа.

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1 Introduction

Despite their respectable age, the power indices proposed by [24] and [1], or [25], henceforth Bz and SSI, remain a popular choice in empirical work. Both indices measure the distribution of *a priori* voting power that follows from the constitution and rules of a voting body alone. However, voting situations, both hypothetical and real, exist in which the two indices yield markedly different results. Which index to use therefore becomes a question of practical importance in the empirical work.

To answer this question, [27] derives probabilistic models consistent with each of the two indices. He shows that, depending on the distribution of the voting poll, the expected individual effect of each member of a voting body on the outcome of voting numerically coincides with either the SSI or Bz measure. Straffin's prescription for empirical work is as follows: "If we believe that voters in a certain body have such common standards, the Shapley-Shubik index might be most appropriate; if we believe voters behave independently, the Banzhaf index is the instrument of choice" [28, p. 1137]. The question explored in this paper is: What is the error of an empirical researcher who, following Straffin's prescription, applies the Bz measure to a voting body in which Straffin's Independence Assumption is not met?

To answer this question, I compute the bias of the Bz absolute measure of power in reflecting a voter's probability of being decisive when the votes are neither equiprobable nor independent. I use a numerical scheme to construct a joint probability distribution on the set of voting outcomes (coalitions) for given probabilities and correlation coefficients, and compare the Bz measure for this distribution to its equivalent in the case of equiprobable and independent votes.[1] Section 2 argues that the pairwise correlation as a model of stochastic dependence is sufficiently general for most empirical ap-

[1] The numerical scheme was introduced in [16] for modeling financial default risk. In this paper I provide an analytical solution to a slightly less general

plications including voting by blocs. Section 3 discusses a numerical scheme for computing the Bz swing probability when the votes are not equally probable and correlated, and shows how to estimate the probabilities and correlation coefficients from ballot data. Section 4 presents an analytical derivation of the exact magnitude of the bias due to the common probability of a YES vote deviating from one half and due to common correlation in unweighted simple-majority games. Section 5 derives a modified Penrose's square-root rule in the case of correlated votes. The last section concludes.

2 Straffin's probabilistic voting models

Let p_i be the probability that member i votes YES. [27] introduces two probabilistic assumptions: "*Independence Assumption*: The p_i's are selected independently from the uniform distribution on $[0,1]$. or: *Homogeneity Assumption*: A number p is selected from the uniform distribution on $[0,1]$, and $p_i = p$ for all i" (p. 112). He then proceeds to prove two well-known characterization theorems. Theorem 1 states that under the Independence Assumption the probability of the i-th member's vote being decisive, or the i-th expected individual effect on the outcome of voting, coincides with the Banzhaf measure of voting power for i

$$\beta_i = \frac{\eta_i}{2^{n-1}}. \tag{1}$$

Here η_i is the number of coalitions in which i is decisive, and n the total number of members. The Banzhaf index is obtained by normalizing β_i's to add up to unity, which unfortunately destroys its probabilistic meaning. Theorem 2 makes a similar statement for the Homogeneity Assumption and the SSI.

The crucial assumption in both models is that each member votes independently. This is evident from the proofs, both of which

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rely on multilinear extensions of a game introduced by [22]. A multilinear extension of a game played by $N = \{1, 2, \ldots, n\}$ members is

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq N} \prod_{j \in S} x_j \prod_{j \in N \setminus S} (1 - x_j) w(S), \quad \text{where } 0 \leq x_j \leq 1 \text{ for all } j. \tag{2}$$

The characteristic function, $w(S)$, takes the value of 1 if S is a winning coalition and the value of 0 if it is not. It is completely defined by the voting rule (quota) and the weights assigned to each member. The increment in the multilinear extension incurred by the addition of the i-th member's vote to the voting poll gives the effect of the i-th member on the outcome

$$\Delta_i f(x_1, \ldots, x_n) = \sum_{S \subseteq W_i} \prod_{j \in S \setminus \{i\}} x_j \prod_{j \in N \setminus S} (1 - x_j), \tag{3}$$

where W_i is the set of winning coalitions in which member i is decisive (critical).

Let x_i be the probability that member i votes YES. The assumption of independent votes endows an increment in the multilinear extension with a unique probabilistic interpretation. Then and only then does $\Delta_i f(x_1, \ldots, x_n)$ become the probability that the i-th vote is decisive. Taking this fact as a point of departure, Straffin shows that the Independence Assumption leads to the Bz measure, whereas the Homogeneity Assumption leads to the SSI. In the general case of possibly dependent votes this probability takes the form $P_i = \sum_{S \subseteq W_i} \pi_S$, where π_S is the probability of the occurrence of coalition S. It is given by a joint probability distribution on the set of all coalitions. While summation remains valid due to the coalitions being mutually exclusive, the products in (2) and (3) only apply to independent votes.

It is important to note that while assigning different weights to different members of a voting body, or changing the quota, may change the characteristic function of the game, *stochastic properties of the votes have no effect on the characteristic function.* Coalitions that have been winning under equally probable and independent votes continue to do so when the votes lose either property – what changes are the probabilities of their occurrence. Straffin's Independence Assumption implies that all voting outcomes have an equal probability of occurrence. Computing the probabilities if one departs from this assumption is the focus of the present paper.

For all empirical purposes Straffin's Independence Assumption is equivalent to the "equiprobability of each member voting either way; and independence between members" [9, p. 37]. Note that "equiprobability either way" means two things: First, all members vote YES with equal probability and, second, this probability equals one half. The Independence Assumption thus leads to a binomial distribution with one half as the probability of success. The consequences of relaxing the equiprobability assumption have been previously studied in [13], [7] and [14]. These studies show that the probability of being decisive changes considerably when the votes are not equiprobable.

As argued in [9], Straffin's Independence Assumption can be defended by the Principle of Insufficient Reason. As an assumption it is rational in the absence of prior knowledge about the future issues on the ballot and how divided over these issues the voting body will be. It suits the intended purpose of measuring the *a priori* distribution of voting power, the distribution that follows from the constitution and rules of the voting body, provided that all coalitions are equiprobable.

In Straffin's Homogeneity Assumption, equal probability of acceptance may be interpreted as reflecting the fact that members of a voting body have common standards when evaluating a proposal on ballot. The Homogeneity Assumption thus seems to abandon the *a priori* approach in favor of a more realistic model. The implied individual voting behavior is nevertheless very rigid. In the words of Felsenthal and Machover: "the model ... is appropriate if we assume that all the voters are identical clones, with the same interests and identical [probabilistic] propensities, formalized by the common random variable P, which in each division produces the same probability p for all of them" (p. 201). To an external observer who does not know the true value of a common p, decisions by voting bodies with p close to zero or one would appear highly correlated, as near unanimous outcomes would be frequent in either case.

One possibility is to combine the two models [30]. As [17] define it: "the voters are said to be 'partially homogeneous' when they can be partitioned into groups within which voters are homogeneous, whereas the groups vote independently of each other" (p. 430). However, partial homogeneity suffers from all the limitations of both probabilistic models. In the next section I argue that working directly with correlated votes is a more satisfactory way of modeling truly heterogeneous voting bodies.

2.1 Correlated votes

The crucial assumption is that each member votes independently of all other members is untenable in most voting situations. First, as noted by many authors, including Straffin, members of a voting body may follow common standards when evaluating a proposal on ballot, to the effect that the votes in favor any one such proposal will correlate positively. One example of a common standard is common information. The more the members communicate with each other, the less their votes are likely to be independent. Second, voting may be strategic. Strategic voting is contingent on how other members are expected to vote and is thus, by definition, not independent. Third, and closely related, there may be tacit collusion between certain members of a voting body, so that an outsider to the group will in effect be facing a voting bloc. The existence and behavior of secret or tacit voting blocs may appear probabilistic to an outsider.

Fourth, members may have similar or different preferences, which could lead to correlated voting patterns. All of the above factors suggest that dependent voting must be the norm rather than the exception, and that correlations may either be positive, reflecting a degree of commonality or conformism, or negative, reflecting a degree of rivalry. It is therefore only natural to expect a member's *a priori* power to differ from her actual ability to change the outcome of the voting at any point after the constitutional stage. This expectation is all the more applicable when one considers that the former does not change as long as the rules stay the same, while the latter may change from one issue to another. A realistic model of a voting body should therefore be able to accommodate varying probabilities and correlations between votes.

Correlation between votes provides a general way of taking voters' preferences into account, and the need to do so has been repeatedly stressed in the literature.[2] It is common to represent voter's preferences as points in Euclidean space.[3] Whereas spatial representations typically are deterministic, correlations suggest only a probabilistic tendency of a member toward certain positions and correlation coefficients can easily be estimated from ballot data.

As a simple model of probabilistic dependence, I shall assume that the votes of $n-1$ ($n \geq 3$) members of a voting body are correlated, whereas the i-th member votes independently of all others. Member i is independent because she has already made her choice. Her vote is assumed to be deterministic. I then compute the i-th swing probability and the bias resulting from the application of the Bz measure to i. Alternatively, one could compute the conditional probability of i being decisive, conditioned on her voting YES. Since

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[2] For a recent debate, see [21] and a critique of Napel and Widgrén in [6], as well as a reply and a rejoinder in the same issue of the *Journal of Theoretical Politics*. For a critique of preference-free measures of voting power in the context of the European Union, see [10].

[3] As in [26], [21], the veto player theory of [29], and in a general theory of voting by [20], among others.

the two probabilities differ by the factor $1/p_i$, where p_i is the probability of i voting YES, I will compute the former probability.

The assumption of pairwise correlation implies the existence of a degree of commonality (positive correlation) or a degree of rivalry (negative correlation) between $n-1$ members of a voting body, including their mutual independence as a special case.[4] Note that pairwise correlations cannot capture correlations between an individual member and a bloc of members, but this entails no loss of generality if voting blocs are deterministic, in the sense that each insider votes in unison with all other insiders with a probability of one. In this case, pairwise correlation between an outsider and a bloc is equivalent to pairwise correlation between the outsider and a hypothetical member holding the total weight of the bloc in votes. The voting blocs typically discussed in the literature are deterministic e.g., [2, 18].

However, the above is not the only way to model probabilistic dependence between votes. Several alternatives have been proposed in the literature, including the urn model by [3] and the branching process model by [12]. The urn approach has been most extensively developed in the generalizations of Condorcet's Jury Theorem in [5] and [4]. The approach proposed in this paper has the advantage of extending the probabilistic setting of Straffin's theorem to correlated votes without making explicit or implicit assumptions about the dynamics of a voting procedure or the nature of probabilistic dependence. In contrast, by virtue of an urn process the voting in Berg's model is sequential. This follows by construction of an urn scheme, in which colored balls are drawn one at a time and are then replaced by one or several balls of a given color. A model based on an urn process implicitly assumes that the probability of being correct changes every time a vote is cast. Such a model would imply state-dependence in the process of reaching a decision, with the

[4]With some abuse of terminology, as zero correlation does not imply stochastic independence in general.

possibility of a lock-in on an alternative [23]. [12] approach is based on the Ising model from statistical mechanics. In this model correlations are implicitly defined by a spatial proximity parameter, but the votes are equiprobable.

I show that positive correlation between some members of a voting body is likely to reduce the voting power of an independent member, while negative correlation due to contrarian strategies applied by some members is likely to increase her power. By increasing the probability of ties or near-ties, negative correlation increases probabilities of those voting outcomes in which the independent member is decisive, while positive correlation decreases these probabilities.

3 A numerical scheme for computing the swing probability

Suppose that member i votes YES with probability $p_i \in [0,1]$, member j votes YES with probability $p_j \in [0,1]$, and the two YES votes are correlated with a coefficient of correlation $c_{i,j} \in [-1,1]$. Define the probabilities of the four possible voting outcomes as: $\pi_1 = P\{v_i = 1, v_j = 1\}$, $\pi_2 = P\{v_i = 1, v_j = 0\}$, $\pi_3 = P\{v_i = 0, v_j = 1\}$, $\pi_4 = P\{v_i = 0, v_j = 0\}$, where 1 and 0 indicate the YES and NO vote. We have: $\pi_1 + \pi_2 = p_i$, $\pi_1 + \pi_3 = p_j$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. As the covariance $cov[v_i, v_j]$ between the two Bernoulli random variables v_i and v_j is $E[v_i v_j] - E[v_i]E[v_j] = \pi_1 - p_i p_j$, the coefficient of correlation $c_{i,j} = cov[v_i, v_j]/\sqrt{var[v_i]var[v_j]}$ must satisfy $\pi_1 = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}$. Two uncorrelated Bernoulli random variables are independent. Plugging p_i, p_j and $c_{i,j}$ into the four equations recovers the distribution $(\pi_1, \pi_2, \pi_3, \pi_4)$, provided $\pi_1 \geq 0$. As we shall see below, an analogous system of equations in the general case of more than three members may not have a unique solution. Before proceeding to the general case, it is necessary to introduce notation.

With m members, a voting outcome can be represented by binary vector $\mathbf{s} = (v_1, v_2, \ldots, v_m)$, whose i-th coordinate $v_i = 1$ if member i votes YES, and $v_i = 0$ otherwise. Define the following sets: $\mathbf{S}$ the set of all voting outcomes; $\mathbf{S}(i)$ the set of voting outcomes in which member i votes YES, that is the set of all binary vectors $\mathbf{s}$ such that $v_i = 1$; $\mathbf{S}(i,j) = \mathbf{S}(i) \cap \mathbf{S}(j)$ the set of voting outcomes in which members i and j both vote YES, that is the set of all binary vectors $\mathbf{s}$ such that $v_i = v_j = 1$. Sets $\mathbf{S}$, $\mathbf{S}(i)$ and $\mathbf{S}(i,j)$ respectively contain 2^m, 2^{m-1} and 2^{m-2} elements. For example, for $m = 3$ there will be eight voting outcomes A:(1,1,1), B:(1,1,0), C:(1,0,1), D:(1,0,0), E:(0,1,1), F:(0,1,0), G:(0,0,1), and H:(0,0,0). The set $\mathbf{S}$ contains all eight vectors. The set $\mathbf{S}(2)$ contains vectors A, B, E, and F, as only they have 1 in the second coordinate. The set $\mathbf{S}(2,3)$ contains vectors A and E, as only they have 1 in the second and third coordinates.

For $m \geq 3$, we have

$$\pi_{\mathbf{s}} \in [0,1] \quad \forall \mathbf{s} \in \mathbf{S}; \tag{4}$$

$$\sum_{\mathbf{s}\in\mathbf{S}} \pi_{\mathbf{s}} = 1; \tag{5}$$

$$\sum_{\mathbf{s}\in\mathbf{S}(i)} \pi_{\mathbf{s}} = p_i \text{ for } 1, 2, \ldots, m; \tag{6}$$

$$\sum_{\mathbf{s}\in\mathbf{S}(i,j)} \pi_{\mathbf{s}} = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)} \tag{7}$$

$$\text{for } 1 \leq i < j \leq m,$$

provided the correlation matrix constructed from $c_{i,j}$'s is positive semi-definite.

Given m probabilities and $\binom{m}{2}$ coefficients of correlation, the above system comprises $1+m+\binom{m}{2}$ equations with 2^m unknowns and hence may not have a unique solution for $m \geq 3$. For a particular solution choose the one which is closest in the sense of least squares to the probability distribution in the case of independent votes. This solution can be obtained by solving the following quadratic optimization problem

$$\min_{\pi_{\mathbf{s}}} \frac{1}{2} \sum_{\pi_{\mathbf{s}}} \left[\pi_{\mathbf{s}} - \prod_{i=1}^{m} p_i^{v_i} (1-p_i)^{(1-v_i)} \right]^2 \quad \text{for } \mathbf{s} \in \mathbf{S}, \tag{8}$$

subject to constraints (4)-(7).

The strict convexity of the objective function implies that a solution, if one exists, is unique. Any probability vector of length 2^m can be used as a criterion for computing the smallest sum of squared deviations. This vector is chosen because the resulting optimization problem can be used to compute the bias in the vicinity of the probability vector corresponding to the Bz ideal case of equiprobable and independent votes, as well as in the vicinity of the probability vector corresponding to the more general case of independent but not equiprobable votes.

The formulation of the numerical scheme is essentially independent of the assignment of probabilities in the sense that defining, for example, p_i as the probability of i voting YES and p_j as the probability of j voting NO, leads to a similar system of equations. This is clear with respect to constraints involving the probabilities, while the following simple Lemma shows it also to be true with respect to constraints involving the correlation coefficients.

Lemma 1. *Let v_i and v_j be two Bernoulli random variables with $Ev_i = p_i$ and $Ev_j = p_j$. Further, let*

$$P\{v_i = 1, v_j = 1\} = p_i p_j + c_{i,j} \sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{9}$$

$$P\{v_i = 0, v_j = 1\} = (1-p_i)p_j + \bar{c}_{i,j} \sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{10}$$

$$P\{v_i = 1, v_j = 0\} = p_i(1-p_j) + \hat{c}_{i,j} \sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{11}$$

$$\begin{aligned} P\{v_i = 0, v_j = 0\} = {} & (1-p_i)(1-p_j) + \\ & + \ \tilde{c}_{i,j} \sqrt{p_i(1-p_i)p_j(1-p_j)}. \end{aligned} \tag{12}$$

Then, $c_{i,j} = -\bar{c}_{i,j}$, $c_{i,j} = -\hat{c}_{i,j}$, $c_{i,j} = \tilde{c}_{i,j}$.

Proof. To prove the first equality, substitute (9) and (10) into $P\{v_i = 1, v_j = 1\} + P\{v_i = 0, v_j = 1\} = p_j$. The second equality is obtained by substituting (9) and (11) into $P\{v_i = 1, v_j = 1\} + P\{v_i = 1, v_j = 0\} = p_i$. Finally, the third equality is obtained by substituting (9) and (12) into $P\{v_i = 1, v_j = 1\} - P\{v_i = 0, v_j = 0\} = p_i + p_j - 1$. $\square$

Consequently, each of the four alternative assignments of probabilities leads to systems of equations identical except, perhaps, for the sign of the correlation coefficient. I will use this fact in estimating the probabilities and correlation coefficients from ballot data (Section 3.1).

A numerical solution of the general problem is feasible but can be computationally intensive for a large m. In Appendix A, I analytically solve a slightly less general problem, in which all the probabilities are identical but the correlation coefficients may vary.

Proposition 1. *Let $p_i = p \in [0,1]$ for all $i = 1, 2, \ldots, m$ be the probability of i-th member voting YES and $c_{i,j} \in [-1,1]$, $1 \le i < j \le m$, the correlation coefficient between any two such votes. Setting $q = 1-p$, the probability of occurrence of a voting outcome is given by*

$$\pi_s^* = p^{\sum_{i=1}^m v_i} q^{m-\sum_{i=1}^m v_i} + 2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{i,j} - \\ - 2^{3-m} pq \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{i-1} c_{j,i} + \sum_{j=i+1}^{m} c_{i,j} \right) + 2^{4-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{i,j} v_i v_j, \tag{13}$$

provided $\pi_s^* \ge 0$.

When $c_{i,j} = c$, $\sum\limits_{i=1}^{m-1} \sum\limits_{j=i+1}^{m} c = c\frac{m(m-1)}{2}$ and $\sum\limits_{j=1}^{i-1} c + \sum\limits_{j=i+1}^{m} c = c(m-1)$, so that

$$\pi^*_{\mathbf{s}} = p^{\sum_{i=1}^{m} v_i} q^{m-\sum_{i=1}^{m} v_i} + 2^{2-m} pqc \Bigg(\frac{m(m-1)}{2} - \\ - 2(m-1)\sum_{i=1}^{m} v_i + 4 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j \Bigg). \quad (14)$$

The above optimization problem is completely general in that it admits varying probabilities and correlation coefficients. If distributions that satisfy the probabilities and correlation coefficients exist, it will find one such distribution. If all p's are equal, the analytical solution (13) will yield the same distribution, unless one or more of the voting outcomes occurs with probability zero, or if the complementary slackness condition (4) is binding. The requirement $\pi^*_{\mathbf{s}} \geq 0$ puts an upper bound on $c_{i,j}$'s for given m and p, and constraint (5) ensures $\pi^*_{\mathbf{s}} \leq 1$. This restriction applies to the analytical solution only, as in numerical optimization the full set of constraints (4)-(7) is imposed.

3.1 Estimating the probabilities and correlation coefficients

The proposed methodology allows calibrating an accurate model of the voting body given one's prior beliefs about the preferences of the members and the degree of commonality or rivalry among them. Expressed in terms of the probabilities and correlation coefficients, these beliefs can be used to forecast the probabilities of different voting outcomes. Alternatively, one can estimate probabilities and correlation coefficients based on ballot data.

Since $p_i = \sum\limits_{\mathbf{s} \in \mathbf{S}(i)} \pi_{\mathbf{s}}$, its estimate $\hat{p}_i$ equals the frequency of YES votes in the total number of votes cast by i. For two Bernoulli random variables v_i and v_j with $Ev_i = p_i$ and $Ev_j = p_j$

$$P\{v_i = 1, v_j = 1\} = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{15}$$

$$P\{v_i = 1, v_j = 0\} = p_i(1-p_j) - c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{16}$$

$$P\{v_i = 0, v_j = 1\} = (1-p_i)p_j - c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \tag{17}$$

$$\begin{aligned} P\{v_i = 0, v_j = 0\} &= (1-p_i)(1-p_j) + \\ &+ \quad c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}. \end{aligned} \tag{18}$$

Substituting the frequencies of the four arrangements of votes $f_1^{i,j}$, $f_2^{i,j}$, $f_3^{i,j}$, $f_4^{i,j}$ for the probabilities on the left-hand side, and the estimates $\hat{p}_i$, $\hat{p}_j$ in the right-hand side equations yields a system of four equation with one unknown $c_{i,j}$. An estimate of $c_{i,j}$ can be obtained by minimizing the goodness of fit statistic

$$\min_{c_{i,j}} GF(c_{i,j}) = \sum_{k=1}^{4} \frac{(f_k^{i,j} - h_k(c_{i,j}))^2}{f_k^{i,j}}, \tag{19}$$

where

$$h_1(c_{i,j}) = \hat{p}_i\hat{p}_j + c_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)}; \tag{20}$$

$$h_2(c_{i,j}) = \hat{p}_i(1-\hat{p}_j) - c_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)}; \tag{21}$$

$$h_3(c_{i,j}) = (1-\hat{p}_i)\hat{p}_j - c_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)}; \tag{22}$$

$$h_4(c_{i,j}) = (1-\hat{p}_i)(1-\hat{p}_j) + c_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)}. \tag{23}$$

The value that minimizes $GF(c_{i,j})$ is Neyman and Pearson's minimum χ^2 estimator [15, ch. 1.2]. In a voting body of m members there will be $\binom{m}{2}$ distinct pairs of members and hence that many minimization problems to solve. The independence assumption can be tested using Fisher's exact test based on a hypogeometric distribution [8, chs. 2.4 and 3.6.1].

3.2 Examples

With $n \geq 3$ members, the aim is to compute the i-th member swing probability and the bias resulting from the application of the Bz measure to i, assuming that i votes independently but the remaining $m = n - 1$ votes correlate. The generalized Bz measure, i.e. the probability of casting a decisive vote, for member i can be written as $Bz_i(n, \mathbf{p}, \mathbf{c})$, where $\mathbf{p}$ is the vector of m probabilities and $\mathbf{c}$ the vector of $\binom{m}{2}$ correlation coefficients. If $p_i = p$ and $c_{i,j} = c$, we would write $Bz_i(n, p, c)$. This case will be studied analytically. In the above notation, $Bz_i(n, 0.5, 0) = \beta_i$ is the original Bz measure. The following three examples illustrate the effect of the probabilities and correlation coefficients on the Bz measure, assuming the independent member votes YES.

Example 1 (Table 1): Consider an unweighted simple-majority game with four members, or $\{3; 1, 1, 1, 1\}$. Let the *first* member be independent. If all other members also vote independently, each coalition among the remaining three members would occur with the probability $0.5^3 = 0.125$. The first member is decisive in 3 of the 8 coalitions; her Bz measure is equal to $3 \cdot 0.5^3 = 0.375$ (Case A).

Let any two of the remaining three votes correlate with $c = 0.2$ (Case B). Positive correlation makes broad coalitions more probable, tight coalitions less probable. The opposite is true of negative correlation (Case C). Increasing p shifts the probabilities toward coalitions with a high percentage of 1's (Case D). Introducing positive correlation negates some of this shift due to an increase in the probability of occurrence of all broad coalitions, including those with a high percentage of 0's (Case E).

Case D suggests that a departure from equiprobability increases the voting power of the independent member, but in the next section I show that the opposite can also occur. Cases B and C show that positive correlation between members of a voting body will reduce the voting power of the independent member; negative correlation will have the opposite effect. By increasing the probability of ties or near-ties, negative correlation increases probabilities of those voting outcomes in which the independent member is decisive, while positive correlation decreases these probabilities.

The above examples show that the distribution of voting power in an unweighted simple majority game ceases to be trivial when the votes are neither equiprobable nor independent, and that even small departures from either assumption may generate a substantial discrepancy between the Bz measure and the probability of casting a decisive vote. Application of the Bz measure to these voting situations will result in substantial biases. The absolute and the relative biases for i are computed as:

$$Bz_i(n, 0.5, 0) - Bz_i(n, p, c) \quad \text{and} \quad \frac{Bz_i(n, 0.5, 0) - Bz_i(n, p, c)}{Bz_i(n, 0.5, 0)}. \tag{24}$$

The following example of a weighted voting game shows the versatility of the numerical scheme.

Table 1. Game: $\{2.5; 1, 1, 1, 1\}$

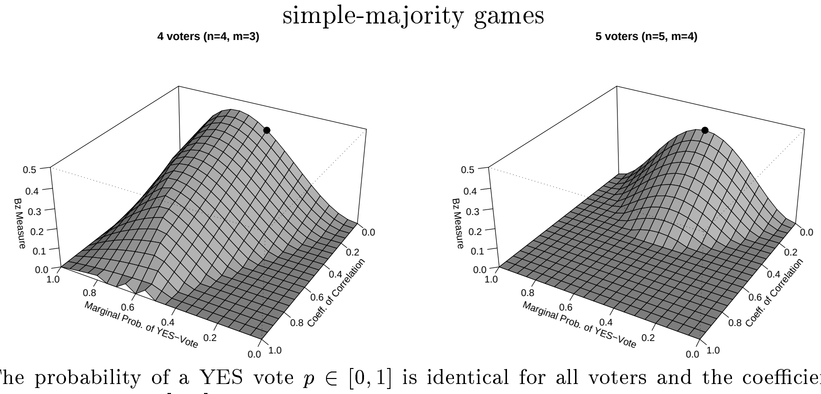
Case						A	B	C	D	E
Coalitions						$p = 0.5$	$p = 0.5$	$p = 0.5$	$p = 0.75$	$p = 0.75$
v_1	v_2	v_3	v_4	Winning	Decisive	$c = 0$	$c = 0.2$	$c = -0.2$	$c = 0$	$c = 0.2$
1	1	1	1	$\surd$	-	0.125	0.200	0.050	0.422	0.478
1	1	1	0	$\surd$	$\surd$	0.125	0.100	0.150	0.141	0.122
1	1	0	1	$\surd$	$\surd$	0.125	0.100	0.150	0.141	0.122
1	1	0	0	-	-	0.125	0.100	0.150	0.047	0.028
1	0	1	1	$\surd$	$\surd$	0.125	0.100	0.150	0.141	0.122
1	0	1	0	-	-	0.125	0.100	0.150	0.047	0.028
1	0	0	1	-	-	0.125	0.100	0.150	0.047	0.028
1	0	0	0	-	-	0.125	0.200	0.050	0.016	0.072
$Bz_1(4, p, c)$						0.375	0.300	0.450	0.422	0.366
Absolute bias						0.000	0.075	-0.075	-0.047	0.009
Relative bias						-	0.200	-0.200	-0.125	0.025

Example 2 (Table 2): Consider the weighted game $\{6;4,2,2,1\}$. When all members vote independently, the Bz vector reads (0.75,0.25,0.25,0.00).

Let $c_{1,2} = c_{1,3} = c_{1,4} = 0.1$, $c_{2,3} = 0.2$, $c_{2,4} = c_{3,4} = 0.5$. This is a situation in which small members are more likely to cooperate with each other than with the large member. Now the Bz vector reads (0.700,0.225,0.225,0.000), allocating respectively 6.7 and 10 percent less power to the large member and medium members (whose powers are equal). The smallest member is a dummy regardless of the stochastic properties of the votes, as the characteristic function is independent of them.

Example 3 (Figure 1): The final example illustrates the effect of a change in p and c on the Bz measure in an unweighted simple-majority game with $n = 4$ and $n = 5$. Figure 1 shows that the bias incurred by p deviating from 0.5 is larger than that incurred by c deviating from 0, which appears to vary linearly with the magnitude of the correlation coefficient. This is established rigorously in the next section.

Figure 1. The absolute Bz measure of voting power in unweighted simple-majority games



The probability of a YES vote $p \in [0,1]$ is identical for all voters and the coefficient of correlation $c \in [0,1]$ between two YES votes is identical for all pairs of voters. The Bz measure is unbiased when $p = 0.5$ and $c = 0$ (filled points). The probability bias incurred by p deviating from 0.5 is polynomial, whereas the correlation bias incurred by c deviating from 0 is linear.

Table 2. Game: $\{6; 4, 2, 2, 1\}$

Coalitions						$p = 0.5, c = 0$	$p = 0.5, \mathbf{c}^*$
v_1	v_2	v_3	v_4	Winning	Decisive	VOTER 1	
1	1	1	1	$\surd$	$\surd$	0.125	0.275
1	1	1	0	$\surd$	$\surd$	0.125	0.025
1	1	0	1	$\surd$	$\surd$	0.125	0.100
1	1	0	0	$\surd$	$\surd$	0.125	0.100
1	0	1	1	$\surd$	$\surd$	0.125	0.100
1	0	1	0	$\surd$	$\surd$	0.125	0.100
1	0	0	1	-	-	0.125	0.025
1	0	0	0	-	-	0.125	0.275
$Bz_1(4, p, \mathbf{c})$						0.750	0.700
Absolute bias						0.000	-0.050
Relative bias						-	-0.067
v_1	v_2	v_3	v_4	Winning	Decisive	VOTER 2	
1	1	1	1	$\surd$	-	0.125	0.2125
1	1	1	0	$\surd$	-	0.125	0.0625
1	1	0	1	$\surd$	$\surd$	0.125	0.0625
1	1	0	0	$\surd$	$\surd$	0.125	0.1625
0	1	1	1	-	-	0.125	0.1625
0	1	1	0	-	-	0.125	0.0625
0	1	0	1	-	-	0.125	0.0625
0	1	0	0	-	-	0.125	0.2125
$Bz_2(4, p, \mathbf{c})$						0.250	0.225
Absolute bias						0.000	-0.025
Relative bias						-	-0.100
v_1	v_2	v_3	v_4	Winning	Decisive	VOTER 4	
1	1	1	1	$\surd$	-	0.125	0.175
1	1	0	1	$\surd$	-	0.125	0.100
1	0	1	1	$\surd$	-	0.125	0.100
1	0	0	1	-	-	0.125	0.125
0	1	1	1	-	-	0.125	0.125
0	1	0	1	-	-	0.125	0.100
0	0	1	1	-	-	0.125	0.100
0	0	0	1	-	-	0.125	0.175
$Bz_4(4, p, \mathbf{c})$						0.000	0.000
Absolute bias						0.000	0.000
Relative bias						-	-

$^*c_{1,2} = c_{1,3} = c_{1,4} = 0.1,\ c_{2,3} = 0.2,\ c_{2,4} = c_{3,4} = 0.5$

4 Assessing the bias of the Bz measure

The examples of the previous section show the Bz measure to be biased when the votes are neither equiprobable nor independent.

This section presents a proposition and a corollary on the magnitude of the probability and correlation biases in unweighted simple-majority games. The model studied will be that of a *homogeneous* voting body in which each vote has an equal probability of being affirmative, and each pair of such votes is correlated with the same coefficient of correlation.

In an unweighted simple majority voting game, the Bz swing probability for the independent member i, assuming i votes YES, is given by

$$Bz_i(n,p,c) = \sum_{\mathbf{s}\text{ s.t. } \sum_{i=1}^{m} v_i = \frac{m+1}{2}} \pi_{\mathbf{s}} \text{ for } \mathbf{s} \in \mathbf{S} \text{ when } n \text{ is even, } m = n-1, \quad (25)$$

or

$$Bz_i(n,p,c) = \sum_{\mathbf{s}\text{ s.t. } \sum_{i=1}^{m} v_i = \frac{m}{2}} \pi_{\mathbf{s}} \text{ for } \mathbf{s} \in \mathbf{S} \text{ when } n \text{ is odd, } m = n-1. \quad (26)$$

Proposition 2. *In a simple-majority game with n members, in which: (1) the probabilities of a YES vote equal p for all members, $q = 1 - p$, and (2) the correlation coefficients equal c for any pair of members, the Banzhaf absolute measure of voting power for an independent member i is given by*

$$Bz_i(n,p,c) = \binom{m}{\frac{m+1}{2}} \left[p^{\frac{m+1}{2}} q^{\frac{m-1}{2}} - 2^{1-m} pqc(m-1) \right] \quad (27)$$
$$\textit{when } n \textit{ is even, } m = n-1;$$

$$Bz_i(n,p,c) = \binom{m}{\frac{m}{2}} \left[(pq)^{\frac{m}{2}} - 2^{1-m} pqcm \right] \quad (28)$$
$$\textit{when } n \textit{ is odd, } m = n-1.$$

Proof. The expression for $Bz_i(n, p, c)$ is obtained by adding the probabilities of the relevant voting outcomes given by (14), which have equal probabilities of occurrence. When n is even (m is odd), there are $\binom{m}{\frac{m+1}{2}}$ voting outcomes in which i is decisive by voting YES, and $\sum_{i=1}^{m} v_i = \frac{(m-1)}{2}$, $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j = \binom{\frac{m-1}{2}}{2}$. Similarly, when n is odd (m is even), there are $\binom{m}{\frac{m}{2}}$ voting outcomes in which i is decisive by voting YES, and $\sum_{i=1}^{m} v_i = \frac{m}{2}$, $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j = \binom{\frac{m}{2}}{2}$. $\square$

Proposition 2 can be adapted to fit any weighted supermajority game by replacing the above combinatorial analysis with a listing of coalitions in which the independent member is decisive, such as the one in Table 2. The number of such coalitions may differ from $\binom{m}{\frac{m+1}{2}}$ and $\binom{m}{\frac{m}{2}}$. The following corollary furnishes the relative bias due to p deviating from 0.5 when $c = 0$, and due to c deviating from 0 when $p = 0.5$.

Corollary 1. *In a simple-majority game with n members, in which: (1) the probabilities of a YES vote equal p for all members, and (2) the votes are uncorrelated, the relative bias equals*

$$\frac{Bz_i(n, 0.5, 0) - Bz_i(n, p, 0)}{Bz_i(n, 0.5, 0)} = 1 - 2^m p^{\frac{m+1}{2}} q^{\frac{m-1}{2}} \tag{29}$$

$$\text{when } n \text{ is even, } m = n - 1;$$

$$\frac{Bz_i(n, 0.5, 0) - Bz_i(n, p, 0)}{Bz_i(n, 0.5, 0)} = 1 - 2^m p^{\frac{m}{2}} q^{\frac{m}{2}} \tag{30}$$

$$\text{when } n \text{ is odd, } m = n - 1.$$

In a simple-majority game with n members, in which: (1) the probabilities of a YES vote equal $p = 0.5$ for all members, and (2) the correlation coefficients equal c for any pair of members, the relative bias equals

$$
\begin{aligned}
\frac{Bz_i(n,0.5,0)-Bz_i(n,0.5,c)}{Bz_i(n,0.5,0)} &= \frac{c(m-1)}{2} \qquad (31)\\
& \quad \textit{when } n \textit{ is even}, \; m=n-1; \\
\frac{Bz_i(n,0.5,0)-Bz_i(n,0.5,c)}{Bz_i(n,0.5,0)} &= c\frac{m}{2} \qquad (32)\\
& \quad \textit{when } n \textit{ is odd}, \; m=n-1.
\end{aligned}
$$

The sign of the probability bias depends on p and the parity of n. When n is even (m is odd), the bias is polynomial. It can have either sign, as $2^m p^{\frac{m+1}{2}} q^{\frac{m-1}{2}}$ can be smaller or larger than 1 for $p \neq 0.5$. For $p \in [0,1]$ and $m = 2k$, $k = 1, 2, \ldots$, the function $f(p) = 2^m p^{\frac{m+1}{2}} (1-p)^{\frac{m-1}{2}}$ attains a unique maximum at $p^* = \frac{m+1}{2m}$. Since $p^* > 0.5$, and $p^* \to 0.5$, $f(p^*) \to 1$ from above as $m \to \infty$, the bias is positive for all $p < 0.5$ and negative for some $p > 0.5$.

The member is the more powerful, the more frequently her vote is decisive. But this will depend on circumstances created by others casting their votes so that she has opportunities to be decisive. The asymmetry of the probability bias about the point $p = 0.5$ for an even n owes to the criterion (25), in which $Bz_i(n,p,0)$ is the highest when voting outcomes satisfying $\sum_{i=1}^{m} v_i = \frac{m+1}{2}$ are highly probable, or when YES votes are slightly more probable than NO votes.

When n is odd (m is even), the inequality $2^2 p(1-p) \leq 1$ for $p \in [0,1]$ implies $2^m p^{\frac{m}{2}} q^{\frac{m}{2}} < 1$ for all $p \neq 0.5$. The probability bias is polynomial and positive.

In any case, positive correlation will bias the Bz measure upwards, negative correlation will have the opposite effect. The absolute and relative correlation biases increase linearly in c. The relative bias increases linearly in m.

5 A modified Penrose's square-root rule

Let N be the number of constituencies, each having n_i citizens. Let i and d_i denote a citizen and the delegate of the i-th constituency. The square-root rule (SRR) gives an approximate answer to the following question: How should voting power be distributed in a council of elected delegates so that each citizen – regardless of the size of her constituency – has an equal *a priori* power in the sense of Banzhaf? The following assumptions lead to a two-stage binomial model: (i) each citizen has one vote, (ii) all citizens' and all delegates' votes are equiprobable and independent, and (iii) the universal voting rule is simple majority. The probability of a citizen being decisive in bringing about her preferred outcome in the council equals the probability that the delegate is decisive in bringing about this outcome, multiplied by the probability that the citizen is decisive in electing the delegate. Formally, $\hat{\beta}_i = \beta_{d_i}(N)\beta_i(n_i)$, where $\beta_i(n_i)$ is the voting power of the citizen i in her constituency, $\beta_{d_i}(N)$ is the voting power of the delegate d_i in the council, and $\hat{\beta}_i$ is the indirect voting power of the citizen i.

To find the ratio of delegate powers that will equilibrate the citizens' indirect powers, set $\hat{\beta}_i/\hat{\beta}_j = 1$ and apply Stirling's approximation to $\beta_i(n_i)$ and $\beta_j(n_j)$. This leads to the well-known result that the citizens' indirect powers are approximately equal if the powers of the delegates in the council are proportional to the square root of the size of their constituencies, or $\beta_{d_i}(N)/\beta_{d_j}(N) \approx \sqrt{n_i/n_j} = 1/\sqrt{h}$. The last step assumes, without any loss of generality, that the constituencies differ in size by the fraction $h > 0$ so that $n_j = hn_i$.

Suppose that in each constituency i the votes are *equiprobable but correlated*, with the coefficient of correlation c_i. A high positive c_i implies a more homogeneous constituency. The larger and the more homogeneous a constituency is, the less power do its citizens have. In contrast, differences of opinion with respect to the candidates on the ballot should lead to closer outcomes, thus increasing the efficacy of a vote.

Proposition 2 can be used to construct the ratio of Bz measures for citizens i and j of two different constituencies. Setting $p_i = p_j = 0.5$, $n_j = hn_i$ and dropping the subscript on n_i leads to

$$\frac{2^{-1-n}\binom{n}{\frac{n}{2}}(2-c_i n)}{2^{-1-[hn]}\binom{[hn]}{\frac{[hn]}{2}}(2-c_j[hn])}. \tag{33}$$

where $[x]$ denotes the integer part of x.[5] By Stirling's approximation

$$\frac{\beta_{d_i}}{\beta_{d_j}} \approx \frac{1}{\sqrt{h}} \frac{2-c_j hn}{2-c_i n}. \tag{34}$$

The above SRR takes into account both the homogeneity and the size of constituencies.[6] All other things being equal, the more homogeneous the constituency is, the lower the voting power of its citizens will be, and the higher the voting power of their delegate ought to be if all citizens are to have equal powers. Setting $c_i = c_j = 0$ leads to the original SRR in [24]. For β_{d_i} and β_{d_j} to remain probabilities, $c_i n$ and $c_j hn$ must be small.

[12] and [11] offer a critical discussion and an empirical test of Penrose's SRR in U.S. presidential elections. Their evidence refutes the binomial model of voting and hence also the SRR as a rule of fair representation. Equation (34) shows the magnitude of the bias in the SRR due to correlation between votes.[7]

[5] Proposition 2 allows relaxing both assumptions. The consequences of relaxing the equiprobability assumption have been discussed in the literature, so I focus on correlation. Since the parity of n does not qualitatively alter the result, the equation for an odd n is used.

[6] [2] derive a rule of fair representation and a voting rule that maximize the sum of the utilities of citizens in all constituencies. In their model "a country's population can be partitioned into blocks: citizens within a block have perfectly correlated preferences, whereas citizens across blocks have independent preferences" (p. 319). This model of heterogeneity reduces the problem to that of blocks' sizes alone.

[7] Other probabilistic voting models may well cause more serious distortions. On an optimistic note, the simulation study by [19] shows Penrose's SRR to be

6 Summary and Conclusions

The crucial probabilistic assumption underlying the classical measures of voting power is that each member of the voting body votes independently of all other members. In the case of the Banzhaf measure this assumption is supplemented by that of equal probabilities of YES and NO votes for each member.

By means of a numerical scheme for computing the Banzhaf swing probability when the votes are neither equiprobable nor independent, this paper studies the magnitude of numerical error or bias in the Banzhaf absolute measure that occurs if neither assumption is met. The numerical scheme admits varying probabilities and correlation coefficients, which makes it suitable for empirical implementation, such as the calibration of an accurate model of a voting body based on beliefs about the preferences of individual members and the degree of commonality or rivalry between them, or the estimation of such a model from ballot data. An analytical solution is provided for a model in which probabilities are identical, but correlation coefficients vary.

The analytical part derives the exact magnitude of the bias for an unweighted simple-majority game in which the probability of an affirmative vote is the same for all members and the correlation coefficients are the same for any pair of members. The bias incurred by the common probability deviating from one half can be positive or negative depending on the probability and the size of the voting body, although it is always positive when the number of members is odd. The probability bias is more serious than that incurred by the common coefficient of correlation deviating from zero. The former is a polynomial function, whereas the latter is a linear function of the deviation. Positive correlation between members of a voting body will reduce the voting power of the independent member, negative

robust for a particular family of distributions. But they assume independent votes.

correlation will have the opposite effect. The magnitude of either bias increases with the size of the voting body.

The magnitude of the bias in a weighted voting game cannot be studied analytically due to the characteristic function of such a game not being amenable to combinatorial methods, despite it being independent of the stochastic properties of the votes. The approach to general weighted voting games has to remain that of listing all voting outcomes in which the independent voter is decisive and summing their probabilities of occurrence. However, the proposed method allows the bias in any weighted voting game to be computed numerically.

As a further result I derive a modified square-root rule for the representation in two-tier voting systems that takes into account the sizes of the constituencies and the heterogeneity of their electorates. Since in a homogeneous electorate the votes are positively correlated, the larger and the more homogeneous the electorate, the less power a vote has.

The main conclusion of this paper is that, despite the Banzhaf measure being a valid measure of *a priori* voting power and thus useful for evaluating the rules at the constitutional stage of a voting body, it is a poor measure of the actual probability of being decisive at any time past that stage. The Banzhaf measure cannot be used to forecast how frequent a voter will be decisive.

A Appendix: Solution to the optimization problem

Write the Lagrangian $\mathcal{L}(\mathbf{x})$ as

$$\Phi(\mathbf{x}) + \lambda \left[\sum_{\mathbf{s} \in \mathbf{S}} x_{V(\mathbf{s})} - 1 \right] + \sum_{i=1}^{m} \mu_i \left[\sum_{\mathbf{s} \in \mathbf{S}(i)} x_{V(\mathbf{s})} - p \right] + \\ + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{i,j} \left[\sum_{\mathbf{s} \in \mathbf{S}(i,j)} x_{V(\mathbf{s})} - (p^2 + pqc_{i,j}) \right] \quad (35)$$

where the objective function is given by

$$\Phi(\mathbf{x}) = \frac{1}{2} \sum_{\mathbf{s} \in \mathbf{S}} \left[x_{V(\mathbf{s})} - p^{\sum_{i=1}^{m} v_i} q^{m - \sum_{i=1}^{m} v_i} \right]^2 .$$

Vector $\mathbf{x}$ is a probability vector of length 2^m. The subscript $V(\mathbf{s}) = \sum\limits_{i=1}^{m} 2^{m-i}(1 - v_i) + 1$ indicates the coordinate of $\mathbf{x}$ that corresponds to the probability of the voting outcome $\mathbf{s}$, so that the coordinates of $\mathbf{x}$ are indexed in the descending order of the decimals represented by the corresponding binary vectors of voting outcomes, starting from the vector of m ones. Indexing is necessary for taking a derivative of $\mathcal{L}(\mathbf{x})$.

The first-order condition $\frac{\partial \mathcal{L}(\mathbf{x})}{\partial \mathbf{x}} = 0$ implies

$$x_{V(\mathbf{s})} = p^{\sum_{i=1}^{m} v_i} q^{m - \sum_{i=1}^{m} v_i} - \lambda - \sum_{i=1}^{m} \mu_i v_i - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{i,j} v_i v_j. \quad (36)$$

Substitution into the first constraint yields

$$4\lambda + 2 \sum_{i=1}^{m} \mu_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{i,j} = 0. \quad (37)$$

When substituting (36) into the second set of constraints note that the sum is now taken over the set of all vectors having 1 as their i-th coordinate. We need to distinguish between coordinates to the left and the right of the i-th coordinate. Upon the substitution of (36) we have

$$4(\lambda+\mu_i)+2\left(\sum_{\substack{j=1\\ j\neq i}}^{m}\mu_j+\sum_{j=1}^{i-1}\kappa_{j,i}+\sum_{j=i+1}^{m}\kappa_{i,j}\right)+ \\ +\sum_{\substack{k=1\\ k\neq i}}^{m-1}\sum_{\substack{l=k+1\\ l\neq i}}^{m}\kappa_{k,l}=0, \quad (38)$$

which in view of (37) simplifies to

$$2\mu_i+\sum_{j=1}^{i-1}\kappa_{j,i}+\sum_{j=i+1}^{m}\kappa_{i,j}=0. \quad (39)$$

Similarly, the sum in the third set of constraints is taken over the set of all vectors having 1 as their i-th and j-th coordinates. Now we need to distinguish between coordinates to the left of the i-th coordinate, to the right of the j-th coordinate, and in between the two. Thus,

$$2^{4-m}pqc_{i,j}+4(\lambda+\mu_i+\mu_j+\kappa_{i,j})+2\left(\sum_{\substack{k=1\\ k\neq i,j}}^{m}\mu_k+\sum_{\substack{k=i+1\\ k\neq j}}^{m}\kappa_{i,k}+\right. \\ \left.+\sum_{k=1}^{i-1}\kappa_{k,i}+\sum_{l=j+1}^{m}\kappa_{j,l}+\sum_{\substack{l=1\\ l\neq i}}^{j-1}\kappa_{l,j}\right)+\sum_{\substack{k=1\\ k\neq i,j}}^{m-1}\sum_{\substack{l=k+1\\ l\neq i,j}}^{m}\kappa_{k,l}=0. \quad (40)$$

In view of (37) and (39) the above expression simplifies to

$$\kappa_{i,j}^* = -2^{4-m} pq c_{i,j}. \tag{41}$$

Plugging (41) into (37) and (39) yields all other Lagrangian multipliers and the solution $x_{V(\mathbf{s})}^*$

$$\mu_i^* = 2^{3-m} pq \left(\sum_{j=1}^{i-1} c_{j,i} + \sum_{j=i+1}^{m} c_{i,j} \right); \tag{42}$$

$$\lambda_i^* = -2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{i,j} \tag{43}$$

(after some algebraic manipulations);

$$x_{V(\mathbf{s})}^* = p^{\sum_{i=1}^m v_i} q^{m - \sum_{i=1}^m v_i} + 2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{i,j} -$$
$$- 2^{3-m} pq \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{i-1} c_{j,i} + \sum_{j=i+1}^{m} c_{i,j} \right) + 2^{4-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{i,j} v_i v_j$$
$$\text{for } \mathbf{s} \in \mathbf{S} \text{ and } 1 \leq i < j \leq m. \tag{44}$$

The $V(\mathbf{s})$-th coordinate of $x_{V(\mathbf{s})}^*$ represents the probability of occurrence of voting outcome $\mathbf{s}$.

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