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PARTISAN DRAWING OF ELECTORAL DISTRICTS: HOW BAD CAN IT BE?

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Gerrymandering – the artful and partisan manipulation of electoral districts – is a well known pathology of electoral systems, especially majoritarian ones. In this paper, we try to give theoretical and experimental answers to the following questions: 1) How much biased can the assignment of seats be under the effect of gerrymandering? 2) How effective is compactness as a remedy against gerrymandering? Accordingly, the paper is divided into two parts. In the first one, a highly stylized combinatorial model of gerrymandering is studied; in the second one, a more realistic multiobjective graph-partitioning model is adopted and local search techniques are exploited in order to find satisfactory district designs. In a nutshell, our results for the theoretical model mean that gerrymandering is as bad as one can think of and that compactness is as good as one can think of. These conclusions are confirmed to a large extent by the experimental results obtained with the latter model on some medium-large real-life test problems.

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1. Introduction

Gerrymandering — the partisan manipulation of electoral district boundaries — has plagued modern democracies since their early times. Far from being defeated, it keeps displaying its perverse effects even at present [1]. It was only with the rise of the electronic computer that researchers started thinking about neutral and rational procedures for political districting. Its nature of multicriteria decision problem was soon recognized. Suppose that the territory is subdivided into elementary administrative units (counties, townships, wards,..). The most commonly adopted districting criteria are the following: Integrity (no unit may be split between two or more districts); Contiguity (the units within the same district should be geographically contiguous); Population Equality (the district populations should be equal or nearly equal, especially in majoritarian systems); Compactness (each district should be compact, that is, "closely and neatly packed together" (Oxford Dictionary)); Conformity to Administrative Boundaries (the electoral district boundaries should not cross other administrative boundaries, such as those of regions, provinces, local or minority communities,..). Among these criteria, Compactness stands as a powerful weapon against gerrymandering, since it bans indented or elongated districts: a sunfish-shaped district is deemed to be compact, while an octopus-shaped or an eel-shaped one is not.

The present paper deals with the following two basic problems:

1) How bad can gerrymandering be?

2) How effective is compactness in preventing gerrymandering?

We shall give both theoretical and experimental answers to these two problems. Accordingly, our paper is divided into two parts. In the first one, an idealized combinatorial model is investigated; in the second part, a more realistic and flexible multicriteria graph-theoretic model is adopted, and computational results are presented for some medium-large real-life test problems.

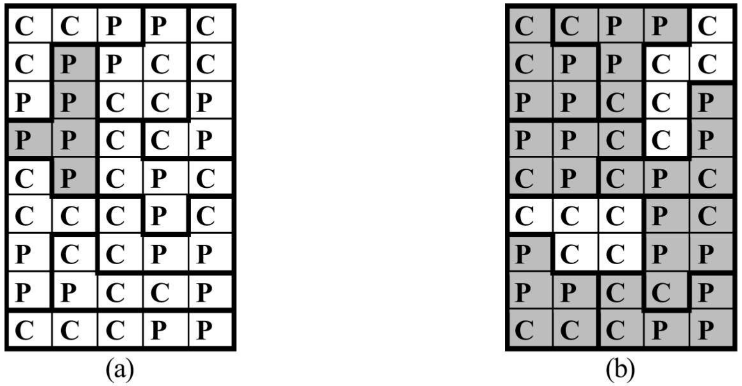


Figure 1. Example of Dixon and Plischke

2. A Combinatorial Gerrymandering Model

As a motivation for the present section we mention a striking artificial example of gerrymandering given by Dixon and Plischke [2]. Suppose that only two parties P and C compete under a first-past-the-post system and that, as in Figure 1, the territory is divided into elementary units having the same population and an homogeneous electoral behavior. If the district map of Figure 1 (a) is adopted, party C wins in 8 districts out of 9; however, if the alternative district map of Figure 1 (b) is adopted, party C wins only in 2 districts out of 9, so the outcome is drastically reversed.

A careful look at Figure 1 gives us a clue about an effective strategy for maximizing the number of districts won by either party: the districts should be designed so that every win should be close and every loss should be sweeping.

In this section we shall consider an idealized graph-theoretic formulation that captures the essence of the artificial example by Dixon and Plischke. Given a territory composed by territorial units, define the following integers:

- n is the number of territorial units;

- p is the number of districts;
- s is the common district size (number of territorial units in each district).

Clearly, the three parameters n, p, s must satisfy the relation $n = ps$.

We model the territory as an undirected graph $G = (V, E)$ with $|V| = n$, where the vertices represent territorial units and the edges represent adjacency between territorial units.

A *connected partition* of G is a partition of its set of vertices V such that each component induces a connected subgraph of G. We suppose that G is p-equipartitionable, that is, there exists a connected partition of G into p components of the same size s.

A *district design* is a connected partition of the graph into p components or *districts* of the same size. Notice that this definition takes into account the criteria of integrity, contiguity and population equality. A *vote outcome* is a bicoloring of the vertices that assigns to each vertex either the color blue or the color red: this means that all voters in the corresponding unit vote for the same party, *blue* or *red*, respectively. A vote outcome is *balanced* if the number of blue vertices is equal to the number of red ones.

A balanced vote outcome corresponds to a situation in which the electoral population is perfectly split among two parties. In our treatment, we shall consider only balanced vote outcomes.

From now on, except for the last section, we shall make the following assumptions on the integers n, s, and p:

- n is even: this is a necessary condition for the existence of balanced vote outcomes;
- s is odd and greater or equal to 3: this assumption forbids trivial cases and ties between the two parties;
- p is even: this follows from the relation $n = ps$.

If in a district D the number of blue vertices is greater than the number of red ones, we will say that D is a *blue district*. In a similar way we can define a *red district*. We will denote Π the set of all district designs and Ω the set of all possible balanced vote outcomes.

We define an *electoral competition* a pair (ω, π) such that $\omega \in \Omega$ and $\pi \in \Pi$. The functions $b(\omega, \pi)$ and $r(\omega, \pi)$, represent the number of blue and red districts, respectively, resulting from the electoral competition (ω, π). Let

$$B(G) = max_{\omega \in \Omega, \pi \in \Pi} b(\omega, \pi).$$

$B(G)$ is the maximum number of blue districts for all the electoral competitions $(\omega, \pi) \in \Omega \times \Pi$. In a similar way we can define $R(G)$.

Property 1. *Since, for any bicoloring, it is possible to switch the colors of the vertices so that the red vertices become the blue vertices and viceversa, any property of the blue party that does not explicitly depend on any given bicoloring must hold for the red party too. In particular we have that $B(G) = R(G)$.*

By this property we can define the function

$$W(G) = B(G) = R(G).$$

Moreover the results that we will provide for the blue party hold also for the red one.

Given an electoral competition $(\omega, \pi) \in \Omega \times \Pi$, for any district k, $k = 1, ..., p$, let

- b_k = number of blue vertices in district k,
- r_k = number of red vertices in district k.

Proposition 1. *Given a p-equipartitionable graph G, for any $(\omega, \pi) \in \Omega \times \Pi$ the following inequality holds:*

$$b(\omega, \pi) \leq \lfloor n/(s+1) \rfloor.$$

Proof. Given an electoral competition $(\omega, \pi) \in \Omega \times \Pi$, for each district k, let b_k and r_k be defined as above. Since ω is balanced , we may assume:

$$\sum_{k=1,...,p} (b_k - r_k) = 0.$$

Hence:

$$0 = \sum_{k=1,...,p} (b_k - r_k) = \sum_{k:b_k > r_k} (b_k - r_k) + \sum_{k:b_k < r_k} (b_k - r_k)$$

$$\geq b(\omega, \pi) - s(p - b(\omega, \pi)) = (s+1)b(\omega, \pi) - sp$$

Since $n = ps$ and $b(\omega, \pi)$ is a natural number we obtain:

$$b(\omega, \pi) \leq \lfloor n/(s+1) \rfloor.$$

□

Corollary 1. *If G is p-equipartitionable, then $W(G) = \lfloor n/(s+1) \rfloor$.*

Proof. Let $\pi \in \Pi$ be any district design. It is possible to color the vertices of the graph G in such a way that $\lfloor n/(s+1) \rfloor$ districts have at least $(s+1)/2$ blue vertices. In fact, in any balanced vote outcome, the number of blue vertices is $n/2$ and:

$$\frac{s+1}{2} \left\lfloor \frac{n}{s+1} \right\rfloor \leq \frac{n}{2}.$$

Since a district with $(s+1)/2$ blue vertices is blue, we obtain a vote outcome with at least $\lfloor n/(s+1) \rfloor$ blue districts. But, by Proposition 1., this is an upper bound for the number of blue districts, hence $W(G) = \lfloor n/(s+1) \rfloor$. □

Corollary 2. *If G is p-equipartitionable, and $p = q(s+1) + r$ with $1 \leq r \leq s+1$ then $W(G) = qs + r - 1$*[1].

[1]Notice that q and r might not coincide with the quotient and the remainder, respectively, of the division of p by $s+1$.

Proof. From Corollary 1. one has:

$$W(G) = \left\lfloor \frac{n}{s+1} \right\rfloor = qs + \left\lfloor \frac{rs}{s+1} \right\rfloor.$$

Since $r \leq s+1$,

$$\left\lfloor \frac{rs}{s+1} \right\rfloor = \left\lfloor r - \frac{r}{s+1} \right\rfloor = r - 1,$$

hence

$$W(G) = qs + r - 1.$$

□

Given a bicoloring $\omega \in \Omega$ and a partition $\pi \in \Pi$, we say that a district is (*blue*) *edgy* if it contains $(s+1)/2$ blue vertices and $(s-1)/2$ red vertices, while we will say that a district is (*blue*) *sweeping* if all its vertices are blue. Moreover we say that π is (*blue*) *extremal* if the number of blue districts $b(\omega,\pi)$ is equal to its upper bound $\lfloor n/(s+1) \rfloor$. Similar concepts can be introduced for the red party.

Remark 1. *If $p \leq s+1$, each blue extremal partition has $p-1$ blue districts and one red district.*

We are especially interested in the following optimization problem:

$$GAP(G) = max_{\omega\in\Omega}(max_{\pi\in\Pi} b(\omega,\pi) - min_{\pi\in\Pi} b(\omega,\pi)).$$

For a given graph G the function $GAP(G)$ is a measure of the maximum bias of an electoral outcome (in terms of number of seats in single member majority districts) due to gerrymandering.

Proposition 2. $GAP(G) \leq 2W(G) - p = 2\lfloor \frac{n}{s+1} \rfloor - p.$

Proof. Since $b(\omega,\pi) + r(\omega,\pi) = p$, then

$$\begin{aligned} GAP(G) &= max_{\omega\in\Omega}(max_{\pi\in\Pi} b(\omega,\pi) + max_{\pi\in\Pi} r(\omega,\pi)) - p \leq \\ &\leq max_{\omega\in\Omega} max_{\pi\in\Pi} b(\omega,\pi) + max_{\omega\in\Omega} max_{\pi\in\Pi} r(\omega,\pi) - p = \\ &= 2W(G) - p. \quad (1) \end{aligned}$$

□

For a given p-equipartitionable graph G we are interested in finding, if it exists, a bicoloring $\omega^* \in \Omega$ such that there are a blue extremal partition and a red extremal one, both w.r.t. ω^*. If such a bicoloring exists, we will say that G is *two-faced* and there exist two partitions $\pi_b, \pi_r \in \Pi$ such that:

$$b(\omega^*, \pi_b) = r(\omega^*, \pi_r) = W(G) = \lfloor n/(s+1) \rfloor.$$

Corollary 3. *We have*

$$GAP(G) = 2W(G) - p \quad (2)$$

if and only if G is two-faced.

Proof. Follows from (1). □

Two-faced graphs are those for which gerrymandering exhibits its worst case bias. There is an absolute threshold for the largest number of seats that a party can obtain when the vote outcome is balanced. In two-faced graphs, for a suitable balanced vote, both parties can achieve this threshold by artful gerrymandering.

3. Theoretical Results on Grid Graphs

The main result of this section is that, under the above assumptions on n, s, and p, any grid graph with an even number of vertices is two-faced.

Let G be a grid graph with M rows and N columns, with $n = MN$. Notice that a grid graph contains a hamiltonian path and so,

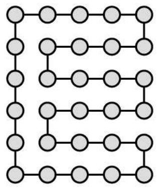


Figure 2. Hamiltonian cycle in a grid graph with an even number of rows

since $n = sp$, it is p-equipartitionable. Moreover, since we assume that n is even, at least one between M or N must be even. In the following we assume, without loss of generality, that M is even.

We start from the case $p = s+1$, where a blue extremal partition has exactly s edgy districts and one sweeping district. In fact, by Corollary 2. with $q = 0$ and $r = s+1$, the upper bound on the number of blue districts is s. These districts must be edgy since the number of blue vertices in G is $s(s+1)/2$. It follows that the remaining district is red sweeping. We will show how to construct such an extremal partition on a hamiltonian cycle H of G. In fact, since M is even, it is easy to show that G is hamiltonian (see Figure 2). We suppose that the vertices of H are consecutively numbered from 1 to n along the cycle (traversed clockwise).

A *boa* is a path with $(s+1)(s-1)/2$ vertices that can be partitioned into $(s+1)/2$ components having $(s-1)/2$ consecutive blue vertices and $(s-1)/2$ consecutive red vertices each. Boas have the following nice property: if one cuts the s-th, the $2s$-th, ... , the

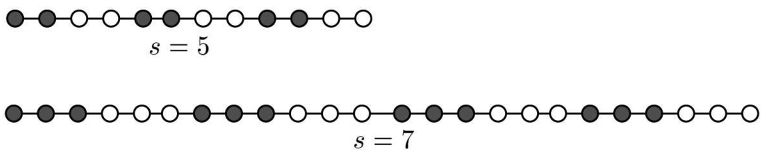


Figure 3. Examples of boas

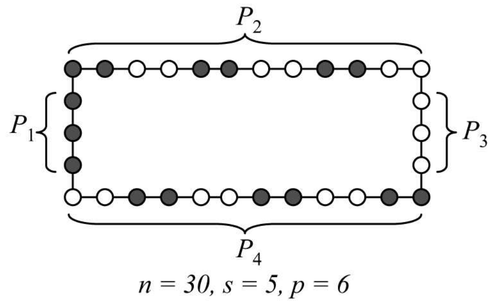


Figure 4. Bicoloring for the case $p = s+1$

$((s-1)s/2)$-th edge from left to right, one obtains $(s-1)/2$ red edgy districts and the remaining $(s-1)/2$ nodes are blue; a symmetrical property holds when one interchanges the two colors "red" and "blue", as well as "right" and "left".

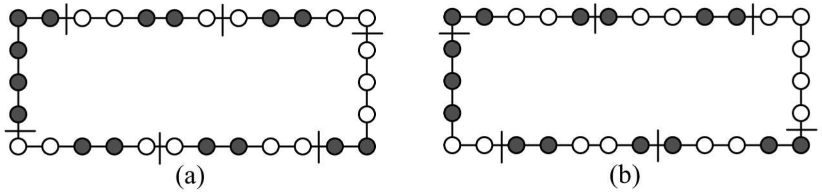
In Figure 3 the boas for $s = 5$ and $s = 7$ are shown. Here, as in all black and white figures in the sequel, blue vertices are displayed as white and red vertices as black.

In Figure 4 we consider the case $s = 5$ and we show how to use two boas in order to find a bicoloring of H for which there are both a blue extremal partition and a red extremal one. One obtains such bicoloring by splitting H into four consecutive subpaths that are colored in the following way:

- the first subpath P_1 extends from vertex 1 to vertex $(s+1)/2$ and all its vertices are red;
- the second subpath P_2 is a boa starting from vertex $(s+1)/2+1$, colored red, and ending at vertex $s(s+1)/2$;
- the third subpath P_3 extends from vertex $s(s+1)/2+1$ to vertex $(s+1)(s+1)/2$ and all its vertices are blue;
- the fourth subpath P_4 is a boa starting from vertex $(s+1)(s+1)/2+1$, colored red, and ending at vertex $s(s+1)$.

It is easy to verify that the number of blue vertices is equal to the number of red ones. Since H is a cycle, one can obtain an arbitrary partition into p connected components by cutting p edges. In Figure 5 the two extremal partitions are shown for the case $s = 5$. If the cut edges are $(s, s+1), (2s, 2s+1), ..., (s^2, s^2+1), ((s+1)s, 1)$ the district containing vertices from 1 to s is red sweeping and all the other ones are blue edgy (Figure 5 (a)). Thus the partition is blue extremal. By shifting each cut to its next edge (clockwise) $(s+1)/2$ times, we obtain a blue sweeping district from vertex $s(s+1)/2+1$ to vertex $s(s+1)/2+s$ and all the other districts are red edgy. So the partition is red extremal (Figure 5 (b)).

Let us consider now the case $p < s+1$. Since p is even and positive we can suppose $p = (s+1) - 2k$ for a given k such that $1 \leq k \leq (s-1)/2$. As shown in Figure 6 for the case $s = 5$ and $k = 1$, starting from the bicoloring of the case $p = s+1$ we delete from the subpath P_2 the last ks vertices and from the subpath P_4 the first ks vertices. We obtain a cycle with $s(s+1) - 2ks$ vertices where the number of blue vertices is equal to the number of red ones. If one cuts the edges as above, starting from $(s, s+1)$, the district containing vertices from 1 to s is red sweeping and all the other ones are blue edgy except the one containing the subpath P_3, which is not edgy because it contains $(s+1)/2+k$ blue vertices and $(s-1)/2-k$ red vertices. The obtained partition is blue extremal. By shifting the cuts as for the case $p = s+1$, the resulting partition is red extremal. In fact, in the district containing the subpath P_3,



$n = 30, s = 5, p = 6$

Figure 5. Partitions for the case $p = s+1$

$n = 20, s = 5, p = 4, k = 1$

Figure 6. Bicoloring and Partitions for the case $p < s+1$

the blue party wins since there are $s-k$ blue vertices and k red vertices, while all the other districts are red edgy.

Finally suppose that $p > s+1$.

Proposition 3. *Under the above assumptions on G, M, N, p and s, G can be decomposed into p grid subgraphs having s vertices each.*

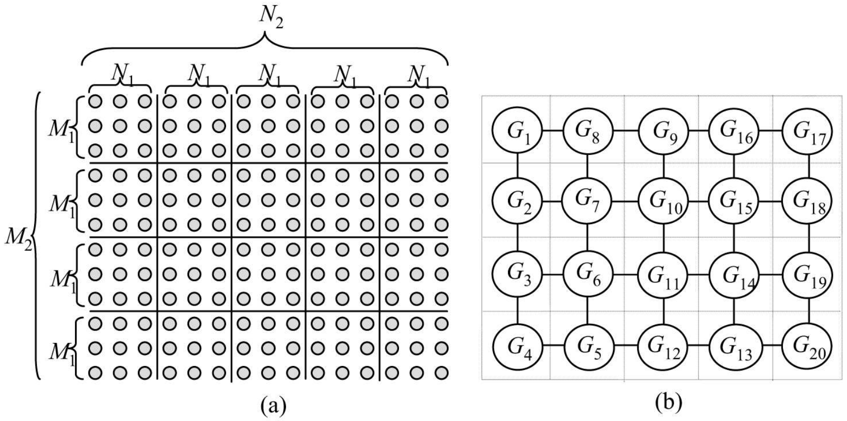
Proof. Since $MN = ps$ there exist four natural numbers M_1, M_2, N_1 and N_2 such that:

$$M = M_1 M_2,\ N = N_1 N_2,\ M_1 N_1 = s,\ M_2 N_2 = p.$$

As shown in Figure 7 (a), by partitioning the columns of G into N_2 components having N_1 columns each and the rows of G into M_2 components having M_1 columns each, one can decompose G into p grid subgraphs having M_1 rows and N_1 columns each. Notice that, since s is odd, also M_1 and N_1 are odd; hence, since M is even, also M_2 is even. □

As in Corollary 2., we suppose that $p = q(s+1) + r$, with $q \geq 1$ and $1 \leq r \leq s+1$. Notice that, since $s+1$ and p are even, also r must be even.

We represent the decomposition given in Proposition 3. by a grid graph $\overline{G}$, with M_2 rows and N_2 columns, whose vertices V_k, $k = 1, ..., p$, correspond to the grid subgraphs and there is an edge connecting the vertices V_k and V_j if some vertex of the grid corresponding to V_k is adjacent to some vertex of the grid corresponding to V_j (see Figure 7 (b)). Let us consider the hamiltonian path



$s = 9, p = 20, M = 12, N = 15, M_1 = 3, M_2 = 4, N_1 = 3, N_2 = 5$

Figure 7. Decomposition of G into p grid subgraphs

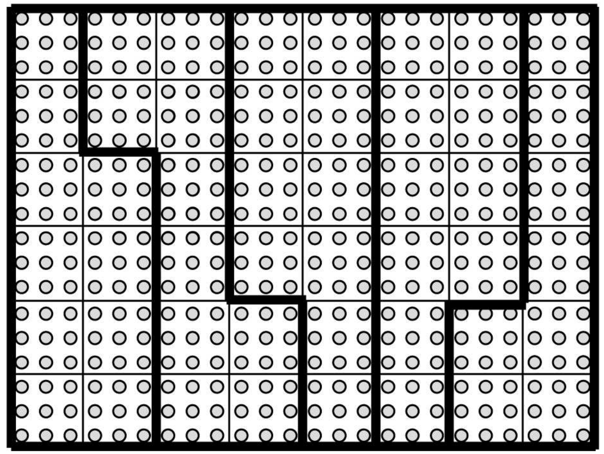
$\overline{P} = (V_1, V_2, ..., V_p)$ of $\overline{G}$ and partition it into q subpaths having $s+1$ vertices each and one subpath having r vertices. Let P_j be the j-th subpath of $\overline{P}$.

Lemma 1. *For each $j = 1, ..., q+1$, and for each column c of $\overline{G}$, the number of vertices of P_j in column c is even.*

Proof. The proof is based on the fact that the number of rows of $\overline{G}$, M_2, and the number of vertices in each subpath P_j, $s+1$ or r, are even. Let c_1 be the smallest numbered column whose intersection with some of the subpaths P_j is odd. Then c_1 must intersect in an odd number of nodes an even positive number of subpaths P_j. But then the smallest numbered such subpath, by the minimality assumption on c_1, would contain an odd number of nodes, a contradiction.

□

As shown in Figure 8, the subpaths P_j, $j = 1, ..., q+1$, define in G a decomposition into $q+1$ connected subgraphs $H_1, ..., H_{q+1}$.



$s = 9, p = 48, M = 18, N = 24, M_1 = 3, M_2 = 6\ N_1 = 3, N_2 = 8$

Figure 8. Decomposition of G into p grid subgraph

Proposition 4. *For each $j = 1, ..., q+1$, H_j is hamiltonian.*

Proof. As shown in Figure 8, each H_j can be decomposed into at most three grid subgraphs which, by Lemma 1., have an even number of rows. Hence it is possible to find a hamiltonian cycle of H_j as in the graph of Figure 9. □

Since H_j, $j = 1, ..., q+1$ is hamiltonian, then, as shown before, it is two-faced and so it is possible to find a bicoloring such that there exist a blue extremal partition and a red extremal one. By using the blue extremal partitions of the subgraphs H_j, one can obtain a partition of G having $qs + r - 1$ blue districts. In fact, by Corollary 2., in each of the q subgraphs having $s(s+1)$ vertices, there are s blue districts and in the subgraph having r vertices there are $r - 1$ blue districts. But, again by Corollary 2., $qs + r - 1$ is an upper bound on $W(G)$, hence the partition of G is blue extremal. The same arguments can be used for obtaining a red extremal partition. Then G is two-faced.

By the constructions shown for the cases $p = s+1$ and $p < s+1$ and the decomposition found for the case $p > s+1$, the following theorem holds.

Theorem 1. *Under the above assumptions on p and s, any grid graph with ps vertices is two-faced.*

Corollary 4. *If $G(s+1, s)$ is a grid graph with $s+1$ rows and s columns, then*

$$lim_{odd\ s\to\infty} \frac{GAP(G(s+1,s)}{s+1} = 1.$$

Proof. After Theorems 3. and 1., one has

$$\frac{GAP(G(s+1,s))}{s+1} = \frac{2W(G)-s-1}{s+1} = \frac{2s-s-1}{s+1} = \frac{s-1}{s+1}.$$

When s odd $\to \infty$, the thesis follows. □

Corollary 4. is stunning: it means that, for certain infinite families of grids, as the number and size of the districts grow, vicious gerrymandering can make the percentages of blue districts and red ones both arbitrarily close to 1 even under the assumptions that the vote outcome is the same and that the blue party and the red one get the same total number of votes.

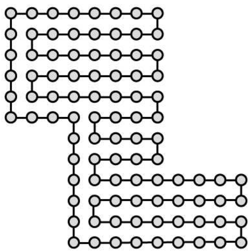


Figure 9. Hamiltonian cycle in a H_j subgraph of G

In conclusion, we have shown that for all even grids one can construct Dixon-Plischke-like examples where gerrymandering can heavily reverse the electoral result in terms of Parliament seats.

Our final result shows that for some highly symmetric colorings, on the one hand, there are blue and red extremal district designs; on the other hand, the most compact design, namely, the partition of the grid into square subgrids, yields the same number of blue and red districts.

To address the question we introduce the notion of *skew-symmetrical* coloring.

Let φ be the mapping of the grid onto itself that maps node (i, j) into $(M+1-i, N+1-j)$. Notice that φ is the product of two reflections, the first one around the y-axis, the second one around the x-axis. Since M is even, φ fixes no point of G. A coloring $\omega \in \Omega$ is *skew-symmetrical* if (i, j) and $\varphi(i, j)$ have opposite colors.

If a grid is skew-symmetrically colored, then $\varphi(G)$ is isomorphic to G the colors of its vertices being interchanged (in fact φ is an automorphism of the grid). In other words, up to the labels of the vertices, the effect of φ on G reduces to switching the colors of its vertices.

Theorem 2. *Let G be an $M \times N$ grid having ps vertices with $p \leq s+1$ and p even. One can always find a blue- and a red- extremal partition with respect to some skew-symmetric bicoloring of G.*

Proof. (Sketch). We can divide the grid into two equally sized parts, say L and R, of $\frac{ps}{2}$ vertices each, in such a way that: (i) $(i, j) \in L$ if and only if $\varphi(i, j) \in R$; (ii) both L and R induce subgraphs containing hamiltonian paths.

Let us consider the subgraph G_R induced by R. We can define a coloring of G_R and two connected partitions π'_R and π''_R into $p/2$ components such that: π'_R is a partition all whose districts are red edgy, π''_R is a partition all whose districts but one are blue edgy, the exceptional one being red (see Figure 10). Using φ we extend the coloring of G_R to the entire grid. By construction this coloring is skew-symmetrical. Moreover, if C is any component of either π'_R or π''_R, $\varphi(C)$ is a connected component of G_L (the graph induced by

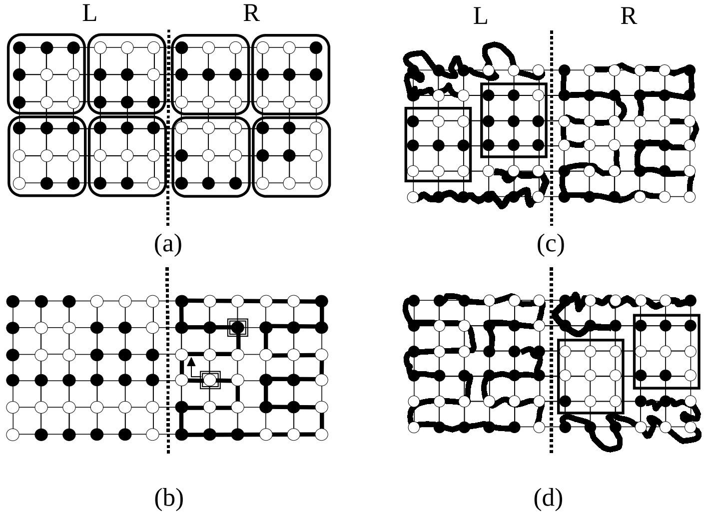


Figure 10. **(a)** The most compact and equitable partition of a 6×12 skew-symmetrically colored grid. **(b)** The hamiltonian cycle from which the two extremal partitions in (c) and (d) are generated. Starting from the framed blue (white) vertex, and cutting the 9th, 18th, 27th and 36th edges of the cycle (clockwise) the right hand side of the partition in (c) is generated (the left hand side of the partition in (d) can be obtained by symmetry). Similarly, the right hand side of the partition in (d) (and, by symmetry, the left hand side of the partition in (c)) is generated by starting from the framed red (black) vertex. **(c),(d)** Red and blue extremal partitions

L), isomorphic to C but with colors interchanged. It follows that if π'_L and π''_L are the partitions of G_L corresponding via φ to π'_R and π''_R, respectively, then $\pi'_R \cup \pi''_L$ and $\pi''_R \cup \pi'_L$ are extremal partitions of G. □

However, skew-symmetric colorings give rise not only to maximally biased designs, but also to minimally biased compact designs (see Figure 10).

Theorem 3. *Let G be a skew-symmetrically colored $M \times N$ grid. Suppose that G can be divided into squares of sides length $\sqrt{s}$ and let*

π be the s-partition formed by such squares. Then, in π, the number of red district equals the number of blue districts .

Theorem 2. shows that even highly symmetrical vote outcomes can be manipulated in a partisan way. Nevertheless, in view of Theorem 3., compactness can be considered (at least within the frame of our idealized model) as an effective remedy against gerrymandering.

4. Experimental Results on Real-life Test Problems

In this section we provide a graph partitioning model for political districting and we study combinatorial gerrymandering from an experimental point of view on real-life data. The graph-theoretic model of this section is different from the one introduced before: here, we adopt a more general formulation in order to adhere to reality as much as possible. Many restrictive assumptions introduced in the previous section are now dropped, such as, for example, the one imposing the same number of territorial units in each district.

As before, n denotes the total number of territorial units in the territory, $n = |V|$, and p, $1 \leq p \leq n$, is a positive integer denoting the number of districts. Let p_i, $\forall i \in V$, be positive integral node-weights, representing territorial unit populations and d_{ij}, $\forall i, j \in V$, be positive real distances defined for each unit pair (i, j). For each territorial unit, the list of all those administrative areas (regions, provinces,...) that contain the unit is known. Finally, with reference to political elections in Italy, for each territorial unit we introduce two positive integral node weights, vo_i and vp_i, $\forall i \in V$, representing the number of votes obtained in unit i by the Olive Tree and by the Pole of Liberties, respectively[2]. The general partitioning problem can be formulated as follows:

[2] In this application we consider the Italian (majoritarian) vote distribution of Political Elections of 1996. The Olive Tree and Pole of Liberties parties were the center-right and center-left coalitions, respectively, which were in competition at that time.

Given a graph G, partition its set of nodes into p subsets (districts) such that the subgraph induced by each subset is connected and a given function of the partition is minimized.

The objective function may measure different criteria. In the sequel, we use the term “district design” as a synonim of ”connected partition into p components” (we are no longer imposing the further restriction that the districts be equally sized).

Integrity and contiguity are automatically guaranteed by the graph-theoretic model. The remaining criteria of population equality, compactness and conformity to administrative boundaries are measured by proper indicators to be optimized. To this purpose, we have chosen the same indicators as in [3]. Actually, these indicators measure non-population equality, non-compactness and non-administrative conformity, therefore they must be minimized.

In addition, we consider a fourth objective function given by a convex combination of the other three. Moreover, in order to study how far gerrymandering can be pushed, we also consider a partisan criterion. The idea is that both Pole and Olive would like to УwinФ the election. To this purpose, if they each had the opportunity of designing their own political districts, they would try to find the district design that make them win as many seats as possible (gerrymandering). Given a political party, we compute a measure of the *utility* of a district design for that party and use it as the partisan objective function. This measure is obtained as the sum of district-utilities computed over all the districts and is computed for both the Pole and the Olive party. This provides our fifth and sixth objective functions.

A natural choice for the district utility would be the step function

$$h(\rho) = \begin{cases} 0, & \text{if} \quad \rho < 1 \\ 1, & \text{if} \quad \rho \geq 1, \end{cases} \tag{3}$$

where ρ is the ratio between the number of votes for the Pole and those for the Olive. However, when applying local search techniques,

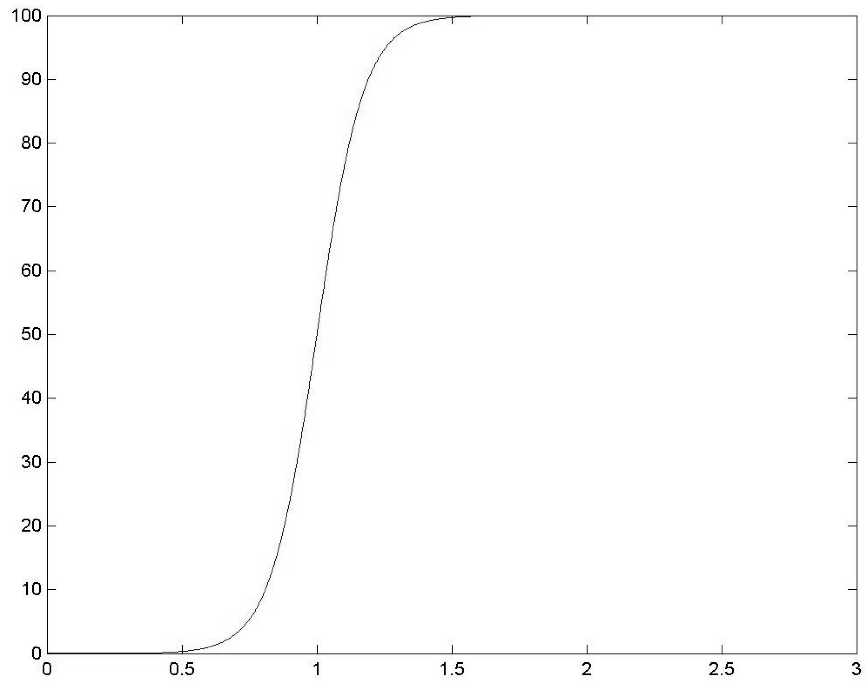


Figure 11. District-utility logistic function for $c = 100$ and $b = 11,51$

(3) is not sufficiently sensitive to the migration of a vertex from a district to another. This explains why we chose to replace the step function (3) by a smoother objective function. For a given party, say the Pole, in each district we compute the following district-utility *logistic* function for that party

$$g(\rho) = \frac{c}{1 + exp(b\ (1 - \rho))},$$

where c and b are suitably chosen in order to get the desired shape of the utility function. The idea is that the district-utility grows up rapidly when ρ is near 1 (see Figure 11).

Table 1. Graphs of the Italian Regions

Region	**N. of Nodes**	**N. of Edges**	**Density**	**N. of Districts**
Piedmont	1208	3527	2.92	28
Latium	374	1006	2.69	19
Abruzzi	305	847	2.78	11

Table 2. Piedmont

District Design	**PE**	**C**	**AC**	**MT**	**Pole seats**	**Olive seats**
Min PE	**0.075**	0.911	0.577	0.426	10	18
Min C	0.771	**0.531**	0.347	0.614	11	17
Min AC	0.940	0.643	**0.113**	0.686	12	16
Min MT	0.094	0.762	0.288	**0.334**	11	17
Max Pole	1.052	0.777	0.454	0.850	**21**	7
Max Olive	1.364	0.593	0.263	0.913	3	**25**
Institutional	0.105	0.859	0.143	0.339	11	17

In our experimental plan we used data of three Italian Regions, namely, Piedmont, Latium and Abruzzi, divided into census tracts. The weights p_i associated to territorial units correspond to the Italian population from 1991 Census, and we considered the real road distances between pairs of territorial units. In this application we considered the Italian (majoritarian) vote distribution of Political Elections of 1996. We used the Old Bachelor Acceptance metaheuristic [4] in order to find solutions that minimize the six different objectives. This metaheuristic has shown to perform well when applied to territorial political districting problems. For details, see [5].

Table 1 shows the main characteristics of the graphs representing the territories of three Italian regions.

Table 3. Latium

District Design	**PE**	**C**	**AC**	**MT**	**Pole seats**	**Olive seats**
Min PE	**0.046**	0.778	0.523	0.361	13	6
Min C	1.226	**0.166**	0.143	0.692	12	7
Min AC	1.072	0.620	**0.050**	0.732	13	6
Min MT	0.050	0.502	0.270	**0.230**	10	9
Max Pole	1.512	0.321	0.061	0.864	**19**	0
Max Olive	1.299	0.277	0.131	0.759	3	**16**
Institutional	0.060	0.683	0.202	0.275	10	9

Table 4. Abruzzi

District Design	**PE**	**C**	**AC**	**MT**	**Pole seats**	**Olive seats**
Min PE	**0.040**	0.744	0.508	0.345	4	7
Min C	0.668	**0.390**	0.288	0.508	4	7
Min AC	0.894	0.539	**0.056**	0.620	4	7
Min MT	0.113	0.442	0.263	**0.242**	4	7
Max Pole	1.217	0.425	0.320	0.800	**10**	1
Max Olive	1.129	0.473	0.328	0.772	1	**10**
Institutional	0.078	0.633	0.215	0.272	5	6

Tables 2—4 show our experimental results on the three different graphs: in the tables PE means “Population Equality”, C means “Compactness”, AC means “Administrative Conformity”, while MT refers to the “Mixed Target” which is defined as the following convex combination of PE, C and AC:

$$0.5PE + 0.3C + 0.2AC.$$

The last row of Tables 2—4 refers to the values of the six objectives computed for the Institutional district design adopted in Italy for the Political Elections of 1996.

On the basis of our experiments, we can state the following conclusions.

1. Given a vote distribution, gerrymandering is able to dramatically reverse the electoral outcome.

2. The districting bias produced by gerrymandering algorithms implies the deterioration of the values of all the traditional PD criteria.

3. It turns out that there is a substantial stability of the number of seats attributed to the Pole and to the Olive when the criteria of Population Equality, Compactness, Administrative Conformity and the Mixed one are optimized.

4. Compactness is a good shield against the practice of gerrymandering. On the other hand, in view of 3, and since gerrymandering deteriorates *all* the districting criteria, satisficing the other criteria helps in preventing gerrymandering. This is why the use of more than one traditional PD criteria is generally recommended.

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