Dynamic Moral Hazard with History-Dependent Participation Constraints[1]

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Abstract:

This paper considers a moral hazard problem in an infinite-horizon, principal-agent framework. In the model, both the principal and the agent can commit only to short-term (single-period) contracts and their reservation utilities are allowed to depend on some finite truncation of the history of observables. After existence is proved, the original problem of obtaining the optimal incentive-compatible self-enforcing contract is given an equivalent recursive representation on a properly defined state space. I construct an auxiliary version of the problem where the participation of the principal is not guaranteed. The endogenous state space of agent's expected discounted utilities which on a different dimension includes the set of truncated initial histories in order to account for their influence on the reservation utilities is proven to be the largest fixed point of a set operator. Then, the self-enforcing contract is shown to be recursively obtainable from the solution of the auxiliary problem by severely punishing any violation of the principal's participation constraint.

Keywords: principal-agent problem, moral hazard, dynamic contracts, limited commitment

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Динамическая модель морального риска с ограничениями участия, зависящими от предыстории

(на английском языке)

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Данное исследование рассматривает проблему морального риска в бесконечно повторяющейся модели принципала-агента. В этой модели и принципал, и агент не способны придерживаться заключенного контракта в долгосрочной перспективе, и их гарантированные полезности потенциально зависят от конечного отсечения предыстории наблюдаемых переменных. После того, как существование доказано, оригинальной проблеме получения оптимального самоподдерживающего контракта дано эквивалентное представление, которое является рекурсивным в подходящем пространстве состояний. Для этого была создана вспомогательная версия проблемы, в которой участие принципала не гарантировано. Доказано, что эндогенное пространство состояний, которое включает в себя как дисконтированные ожидаемые полезности агента, так и набор первоначальных историй, чтобы учесть их влияние на гарантированные полезности, является крупнейшей фиксированной точкой оператора над множеством. Продемонстрировано, можно вывести что самоподдерживающейся контракт рекурсивно из решения вспомогательной проблемы, строго наказывая любое нарушение ограничений участия принципала.

1 Introduction

During the last years, there has been a revived interest in the theory of dynamic contracting[3]. However, although most of the research incorporates some form of limited commitment/enforcement, little has been done in terms of extending the notion of commitment per se. In particular, there is no reason to believe that (the value of) the outside option is constant across the history of observables. For example, it is unrealistic to treat the reservation utility of a CEO as fixed regardless of the situation in his/her firm, industry, or the economy as a whole. The dependence could come through many channels- externalities, different types of agents, a certain structure of beliefs, but more importantly, it can significantly influence the nature of the relationship and the form of the optimal contract. It would be interesting to see how the agent is actually compensated for variability in the value of his/her outside options. When would his/her participation constraint bind? How is the agent's wealth affected in the short and the long run? In fact, would there be a limiting distribution and how would it depend on initial conditions? Such questions can only be analyzed in a generalized framework allowing for history-dependent reservation utilities. Moreover, extending the notion of commitment can bring some important insights into various contractual problems. For example, in order to address the wide use of broad-based stock option plans, Oyer (2004) builds a simple two-period model where adjusting compensation is costly and employee's outside opportunities are correlated with the firm's performance.

The current paper generalizes the notion of commitment by defining the outside options on the history observed in a dynamic contractual setting. I prove existence and obtain the first in the literature characterization of such an environment. The characterization is very general in terms of assumptions and, more importantly, is fully recursive. Its convergence properties make it perfect for computing the optimal contract for a general class of dynamic hidden action models.

I consider a moral hazard problem in an infinitely repeated principal-agent interaction while allowing the reservation utilities of both parties to vary across

[3] See, for example, Fernandes and Phelan (2000), Ligon, Thomas and Worrall (2000), Wang (2000), Phelan and Stacchetti (2001), Sleet and Yeltekin (2001), Ligon, Thomas and Worrall (2002), Ray (2002), Thomas and Worrall (2002), Doepke and Townsend (2004), Jarque (2005), Abraham and Pavoni (2008).

the history of observables. More precisely, to keep the model tractable, the reservation utilities are assumed to depend on some finite truncation of the publicly observed history. The rest of the model is standard in the sense that the principal wants to implement some sequence of actions which stochastically affect a variable of his/her interest, but suffers from the fact that the actions are unobservable. For this purpose, the optimal contract needs to provide the proper incentives for the agent to exercise the sequence of actions suggested by the principal. The incentives, however, are restricted by the inability of the parties to commit to a long-term relationship. It is here where the dynamics of the reservation utilities enters the relationship by reshaping the set of possible self-enforcing, incentive-compatible contracts.

In order to be able to characterize the optimal contract in such a setting, I construct a reduced stationary representation of the model in line with the dynamic insurance literature. The representation benefits from Green (1987)- the notion of temporary incentive compatibility, Spear and Srivastava (1987)- the recursive formulation of the problem with the agent's expected discounted utility taken as the state variable, and Phelan (1995)- the recursive structure with limited commitment, but is closest to Wang (1997) as far as the recursive form is concerned. Unlike Wang (1997), however, I formally introduce limited commitment on both sides and provide a rigorous treatment of its effect on the structure of the reduced computable version of the model. A parallel research by Aseff (2004) uses a similar general formulation[4], but via a transformation due to Grossman and Hart (1983) constructs a dual, cost-minimizing recursive form closer to Phelan (1995) in order to solve for the optimal contract. Such a procedure, however, exogenously imposes the optimality of a certain action on every possible contingency.

After existence is proved, the general form of the model is reduced to a more tractable, recursive form where the state is given by the agent's (promised) expected discounted utility. On a different dimension, the state space includes the set of possible truncated histories in order to account for their influence[5] on the reservation utilities. This recursive formulation does not rely on the first-order approach and is not based on Lagrange multipliers [cf. Marcet and Marimon, (1998)]. In fact, all I need is continuity of the momentary utilities. I

[4]His benchmark model is a full-commitment one, but he considers limited commitment on part of the agent as an extension.

[5]The relationship between the history of observables and the reservation utilities is predetermined since the reservation utilities are exogenous to the problem.

first consider an auxiliary version where the participation of the principal is not guaranteed. The solution of this problem can be computed through standard dynamic programming methods once the state space is determined. Following the approach of Abreu, Pearce and Stacchetti (1990), the state space is shown to be the fixed point of a set operator and can be obtained through successive iteration on this operator until convergence. Given the solution of the auxiliary problem, I resort to a procedure outlined by Rustichini (1998) in order to solve for the optimal incentive compatible, two-side participation guaranteed supercontract. This is achieved by severely punishing the principal for any violation of his/her participation constraint. The procedure allows of recovering the subspace of agent's expected discounted utilities supportable by a self-enforcing incentive-compatible contract.

The rest of the paper is structured as follows. Section 2 presents the dynamic model. Section 3 derives the reduced recursive formulation. Section 4 concludes. Appendix 1 contains all the proofs.

2 Dynamic model

The model considers a moral hazard problem in an infinite horizon principal-agent framework with limited commitment on both sides. Each period, the principal needs the agent to implement some action that stochastically affects a variable of principal's interest, but suffers from the fact that the action is observable only by the agent. Given that the variable of interest to the principal is publicly observable, the principal may want to condition the wage of the agent on the realization of this variable instead. However, the issue of inducing the proper incentives is further complicated by the lack of commitment to a long-term relationship. The commitment problem is structured very generally in the sense that the reservation utilities are allowed to depend on some truncation of the publicly observed history.

Consider, for example, the interaction between the firm's shareholders (the principal) and its CEO (the agent). The CEO may exert a different amount of effort which on its turn randomly affects the success of the corporation illustrated by its observed gross profit. Both the principal and the agent have some outside options: the firm may close, while the agent may quit and start working for another employer. These options are represented by reservation utilities which may vary on the history of observables (in this case, the history of firm's realized gross profits).

First, I will introduce some notation. Let $\mathbb{Z}$ be the set of integers with $\mathbb{Z}_{++}$ and $\mathbb{Z}_{+}$ denoting the sets of positive and respectively nonnegative integers. Time is discrete and indexed by $t \in \mathbb{Z}$. Let y_t denote a particular realization of the variable of interest to the principal in period t. This outcome is realized and observed by both the principal and the agent at the end of the period. As a matter of fact, at the beginning of period t there is a stream of previously realized outcomes which we denote by y^{t-1}. Given that the end-of-period-t realization is y_t, the history of outcomes at the beginning of period $t+1$ is simply $y^t = \left(y^{t-1}, y_t\right)$. The set of possible outcomes is assumed a time- and history-invariant, finite set of real numbers which is denoted by Y. For concreteness, we assume that it consists of $n > 1$ distinct elements.

There is an initial period of contracting which we normalize to 0. At the beginning of this initial period, an outcome history of length $\theta \in \mathbb{Z}_{+}$ is observed. Therefore, a period-0 contracting problem should be defined on n^{θ} initial history nodes.[6] Both the principal and the agent can only commit to short term contracts, therefore it is natural to start with a series of single-period contracts defined on all possible contingencies stemming from some initial node. Each such contract is history dependent and specifies an action and a monetary transfer from the principal to the agent contingent on the particular outcome observed at the end of the period. The timing is as follows. A short-term contract is signed at the beginning of the period. Then, the agent implements some action which is unobserved by the principal and may not be the one specified in the contract. Nature observes the action and draws a particular element of the set of possible outcomes according to some probability distribution. The outcome is observed by both parties and the agent receives the transfer corresponding to this particular outcome.[7] Then, a new period starts, a new short-term contract

[6] As it will become clear afterwards, history will not matter at the initial period of contracting unless the reservation utility of either the principal or the agent is history dependent. Since in order to keep the problem tractable, I allow the reservation utilities to vary across a finite truncation of the observed history with length θ (Assumption 2), it would be natural to consider the contracting problem as defined on n^{θ} initial nodes. As for the existence of an initial period of contracting, note that we can modify the period-0 contracting problem [PP] so that the principal should provide the agent with a given initial (expected discounted) utility level resulting from a previous round of long-term contracting.

[7] Given the above setup, the principal's ability to commit to a short-term contract should be understood as an ability to commit to providing the agent with the promised monetary

is signed, and so on.

Formally, at the beginning of each period $t \in \mathbb{Z}_+$, after a particular history y^{t-1} has been publicly observed,[8] a single-period contract $c_t\left(y^{t-1}\right) := \left\{a_t\left(y^{t-1}\right), w_t\left(y^{t-1}, y\right) : y \in Y\right\}$ is signed between the principal and the agent. Hereafter, for the sake of simplicity, I will often denote such a contract by c_t with the clear understanding that it is defined on a particular history y^{t-1}. The contract specifies an action a_t to be implemented by the agent. To make the analysis tractable, the action is assumed one-dimensional and the action space is taken compact, time- and history-invariant. Formally, $a_t \in A$, where $A \subset \mathbb{R}$ compact. The contract also specifies a compensation scheme $\{w_t(., y) : y \in Y\}$ under which the agent will receive a monetary payoff $w_t(., y)$ in the end of the period if the (end-of-period) outcome is y for any $y \in Y$. The space of possible wages, W, is assumed a compact, time- and history-invariant subset of $\mathbb{R}$.[9] After the contract is signed, the agent exercises action $a_t' \in A$ which is not necessarily the one prescribed by the contract. Then, outcome y_t is realized and the agent receives $w_t(., y_t)$. At the beginning of period $t+1$, contract $c_{t+1}(y^t)$ is signed and so on.

Hereafter, I will refer to any sequence of outcomes, actions, or wages as admissible if all their elements belong to Y, A, or W, respectively.

In order to simplify the analysis, I assume that the probability distribution of

transfer. Indeed, the transfer specified in the short-term contract signed at the beginning of the period occurs at the end of the same period.

[8] You may note that an outcome history y^{t-1} consists of θ elements corresponding to the initial history observed at the beginning of period 0 and t elements from period 0 to period $t-1$.

[9] The compactness assumption can easily be defended by economic considerations. Consider $W = [\underline{w}, \overline{w}] \subset \mathbb{R}$, where $\underline{w}$ may either be zero or a higher number that corresponds to the legally established minimum wage, while $\overline{w}$ is some finite number reflecting the boundedness of the principal's total wealth (the discounted sum of maximum possible income flows). For example, if we treat y as profits, then $\overline{w}$ may be taken equal to $\frac{\max Y}{1-\beta_P}$, where β_P is the relevant discount factor, or to a lower number reflecting restrictions on the principal's ability to borrow against future profits. In Morfov (2010a), a minimum wage level is assumed and from there a theoretical upper bound on the wage is derived in Proposition 1. In the same paper, two other possibilities are considered. The first deals with the case where the principal can borrow up to $\max Y - y$ units of consumption, where y is current gross profit. Then, we can take $\overline{w} = \max Y$. The second case assumes that the principal is prohibited from borrowing, so the wage cannot exceed the current gross profit realization. Note that we can easily extend this case to the environment described here, by taking $\overline{w} = \max Y$ and additionally requiring $w_t(., y) \leq y$, $\forall y \in Y$.

the variable of interest to the principal depends only on the action taken (earlier) in the same period[10] and that each value in the admissible set Y is reached with a strictly positive probability.

Assumption 1 *For any period $t \in \mathbb{Z}$, any admissible outcome history $y^t = (y^{t-1}, y_t)$, and any admissible action sequence $a^t = (a^{t-1}, a_t)$, the probability that y_t is realized given y^{t-1} has been observed and a^t has been implemented equals $\pi(y_t, a_t)$ where $\pi : Y \times A \rightarrow (0,1)$ such that $\forall a \in A$, $\sum_{y \in Y} \pi(y, a) = 1$ and $\forall y \in Y$, $\pi(y, .)$ continuous on A.*

The continuity of π in its second argument is a regularity condition which is trivially satisfied if A is finite.

The principal's (end-of-)period-t utility is denoted by $u(w_t, y_t)$, where $u : W \times Y \longrightarrow \mathbb{R}$ is assumed continuous, decreasing in the agent's wage, and increasing in the outcome. The principal discounts the future by a factor $\beta_P \in (0,1)$. The agent's (end-of-)period-t utility is given by $\nu(w_t, a_t)$ with $\nu : W \times A \longrightarrow \mathbb{R}$ continuous, increasing in wage, and decreasing in the implemented action.[11] The agent discounts the future by a factor $\beta_A \in (0,1)$. Note that given our assumptions, the expected discounted utilities of both the agent and the principal are bounded at any node.

As already mentioned, the agent need not necessarily implement the action specified in the contract. Indeed, if another action brings the agent strictly higher utility, he/she will find it profitable to deviate. Therefore, the contract

[10] While the framework can be modified to include some form of "action" persistence [see, for example, Fernandes and Phelan (2000) and Jarque (2005)], such an extension will be of little value here since the current paper aims to characterize the effect of a generalized form of limited commitment on the optimal dynamic contract. Given that the reservation utilities are allowed to vary across the history of observables, we have another form of persistence which should be analyzed in isolation from potential long-term effects coming from agent's action choice.

[11] Note that we effectively prohibit the agent from borrowing or saving. While extending the model in that direction is possible, introducing such a behavior would shift the focus to incentive-compatibility, while in the current research I seek to analyze the role of the participation constraints in the optimal contract. Moreover, without a set of strong assumptions justifying the first-order approach, such an extension would be very hard to deal with on a practical level given the increase in the dimensionality of the state space of the recursive form.

should provide the proper incentives to the agent in order for him/her to exercise exactly the action recommended by the principal.

Limited commitment is assumed on both parts in the sense that both the principal and the agent can commit only to short-term (single-period) contracts. This assumption is intended to reflect legal issues on the enforcement of long-term contracts. However, at the initial period the principal can offer a long term contract (a supercontract) that neither he/she, nor the agent would like to renege on,[12] and that would provide the necessary incentives for the agent to exercise the sequence of actions proposed by the principal. We will refer to this supercontract as a self-enforcing, incentive-compatible contract and would concentrate on the one maximizing the utility of the principal.

Regarding the issue of commitment, the reservation utilities take values in $\mathbb{R}$ and are allowed to vary across the history of observables. Since it is not practical to define reservation utilities on infinite histories, I make the following assumption.

Assumption 2 *The reservation utilities of both the principal and the agent exogenously depend on the previous θ outcomes.*

The assumption says that the reservation utilities are finite-history dependent, but time independent. Note that the history dependence is truncated to the realizations in the previous θ periods. This is no coincidence. Analogously, we could have started with potentially infinite histories in period 0, introduced finite-history dependence of different length: θ_P for the principal and θ_A for the agent, and then considered finite truncations with length $\theta := \max\{\theta_P, \theta_A\}$ of the infinite histories observed in period 0.

Let Y^θ denote the set of possible outcome streams of length θ periods, or, alternatively, the set of possible initial histories observed at the beginning of period 0. For concreteness, let us enumerate this set using some bijective function $l : Y^\theta \to L$, where $L := \left\{1, ..., n^\theta\right\}$. Hereafter, all functions and correspondences with domain Y^θ will be considered as vectors or Cartesian products of sets indexed by L. Moreover, we will often abuse the notation and use l as its inverse, namely, as the initial history to which the particular index corresponds.

[12] That is, a self-enforcing contract extending the definition of Phelan (1995) to my generalized notion of limited commitment.

Given this indexing, we will denote the reservation utilities of the principal and the agent at node $y^{t-1} \in Y^t \times l$ as $\underline{U}_l$ and $\underline{V}_l$ respectively, $\forall t \in \mathbb{Z}_+$, $\forall l \in L$. For example, if the history observed in the previous θ periods has been $(y_{t-\theta}, ..., y_{t-1})$, the principal's reservation utility in the current period will be $\underline{U}_l$, where $l = l\left(y_{t-\theta}, ..., y_{t-1}\right)$ is the index of the particular outcome stream.

For any $l \in L$, we will define a long-term contract (a supercontract), $c := (a, w)$, where $a := \left\{a_t\left(y^{t-1}\right) : y^{t-1} \in l \times Y^t\right\}_{t=0}^{\infty}$ and $w := \{w_t\left(y^{t-1}, y_t\right) : \left(y^{t-1}, y_t\right) \in l \times Y^t \times Y\}_{t=0}^{\infty}$ are the plan of actions and respectively the sequence of wages defined the whole tree of contingencies stemming from an initial history l.[13] The supercontract prescribes a single action at every node, but specifies the agent's compensation as further dependent on the end-of-period outcome, i.e., as a function with domain Y, or alternatively, as a vector of n elements.[14] Let $V_\tau\left(c, y^{\tau-1}\right)$ and $U_\tau\left(c, y^{\tau-1}\right)$ be the expected discounted utilities of the agent and respectively the principal at node $y^{\tau-1}$ given a supercontract c, i.e.:

$$V_\tau\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_A^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} \nu\left(w_t, a_t\right) \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right),$$
$$U_\tau\left(c, y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} u\left(w_t, y_t\right) \prod_{i=\tau}^{t} \pi\left(y_i, a_i\left(y^{i-1}\right)\right).$$

At time 0, after a truncated history l has been observed, the principal is solving the following problem:

[PP]

$\sup_c U_0\left(c, l\right)$ s.t.:

$$a_\tau \in A,\ \forall nai(l) \tag{1}$$

$$w_\tau\left(., y\right) \in W,\ \forall y \in Y,\ \forall nai(l) \tag{2}$$

$$V_\tau\left(a, w, y^{\tau-1}\right) \geq V_\tau\left(a', w, y^{\tau-1}\right),\ \forall (a' : \forall nai(y^{\tau-1}),\ a'_t \in A),\ \forall nai(l) \tag{3}$$

[13] Note that the supercontract depends on the initial history, but to ease up the exposition, I suppress this dependence notationally.

[14] Remember that Y is finite with cardinality n.

$$V_\tau\left(c, y^{\tau-\theta-1}, \widetilde{l}\right) \geq \underline{V}_{\widetilde{l}}, \forall nai(l) \tag{4}$$

$$U_\tau\left(c, y^{\tau-\theta-1}, \widetilde{l}\right) \geq \underline{U}_{\widetilde{l}}, \forall nai(l) \tag{5}$$

where "$\forall nai(l)$" should be understood as "for any *node after* and *including* l", that is $\forall y^{\tau-1} \in l \times Y^\tau$, $\forall \tau \in \mathbb{Z}_+$.

This is a time-0, history-l contracting problem that mimics dynamic contracting from this node on. That is, at l, the principal solves for a sequence of future strategies on all possible contingencies, so at each node the continuation strategy needs to be self-enforcing and incentive compatible. As in the standard model of dynamic contracting, these strategies are history-dependent. Here, we additionally have that each decision node is characterized by a specific pair of reservation utilities which depend on the history of observables. Nevertheless, as the next section shows, the problem does possess a recursive representation in the spirit of Spear and Srivastava (1987).

Constraints (1) and (2) guarantee that the action plan and respectively the wage scheme are admissible. That is, at any node of the tree stemming from l, the supercontract prescribes an action from A and specifies a compensation scheme that maps Y to W. (3) guarantees that the contract is incentive compatible on any node. For example, at the initial node l, it requires that the action plan of the principal should make the agent weakly better off in terms of period-0 expected discounted utility than any other sequence of admissible actions.[15] (4) and (5) are the participation constraints of the agent and respectively the principal which due to limited commitment should hold at any node. These constraints guarantee the participation of both parties at each contingency. For example, at node $y^{\tau-1} = \left(y^{\tau-\theta-1}, \widetilde{l}\right)$, the expected discounted utility of the agent should be no less than his/her respective reservation utility at this node, $\underline{V}_{\widetilde{l}}$, and the expected discounted utility of the principal should be greater or equal to $\underline{U}_{\widetilde{l}}$.

For future reference, we denote the problem above as [PP] and its supremum as U_l^{**}.

[15] In our framework, we actually have that incentive compatibility on any node is equivalent to initial (time-0) incentive compatibility (see Lemma 1 in the Appendix).

The solution of [PP], if such a solution exists, would be the self-enforcing, incentive-compatible contract that maximizes the utility of the principal at the initial period of contracting.

Let $\Gamma_{y^{\tau-1}} := \left\{c : (1) - (5) \text{ hold after } y^{\tau-1}\right\}$. This is the set of admissible, incentive-compatible, self enforcing contracts that can be signed at node $y^{\tau-1}$. In particular, consider Γ_l, the set of such contracts available at an initial history l. We shall assume that this set is non-empty for any $l \in L$.[16]

Assumption 3 *$\forall l \in L$, $\Gamma_l \neq \emptyset$.*

3 Recursive Form

In this section, we will prove existence and construct an equivalent recursive representation of [PP]. We start by establishing the equivalence of incentive compatibility at all contingencies to Green (1987)'s temporary incentive compatibility at all contingencies.

Proposition 1 *Let (1) and (2) hold after $l \in L$. Then, (3) $\Leftrightarrow$*

$$\forall nai\,(l)\,,\; V_\tau\left(a, w, y^{\tau-1}\right) \geq V_\tau\left(a', w, y^{\tau-1}\right),$$
$$\forall a' : a'_\tau\left(y^{\tau-1}\right) \in A, \text{ and } \forall y \in Y,\; \forall nai\left(y^{\tau-1}, y\right),\; a'_t\,(.) = a_t\,(.) \qquad (6)$$

[16] If the set is empty for some initial history, then there does not exists an incentive-compatible, self-enforcing supercontract at this node. As our numerical estimates in Morfov (2010a) demonstrate, this is hardly the case: in fact there is a wide interval of possible utility promises to the agent that can be supported by a contract of such type for any initial history node. Also note that for suitably chosen reservation utility values, the incentive compatible contract will behave as a full-commitment one, so any violation of Assumption 3 will directly imply the non-existence of the latter. Therefore, it is more a problem of choosing the "proper" (not too high) reservation utilities than anything else. Nevertheless, Morfov (2010b) considers an extension that allows for permanent separations and does not require an assumption of this sort.

The proposition says that constraint (3) is equivalent to requiring that at any date τ, after any history $y^{\tau-1}$, there is no profitable deviation in the current period which will make the agent strictly better off (in expected utility terms) given that he/she fully complies to the plan in the future. The proposition allows us to focus on single-period deviations, which is the first step towards a recursive structure.

Consider two types of supercontracts. The first, hereafter referred to as a 2P contract, is an incentive-compatible supercontract which is self-enforcing, i.e., guarantees the participation of both the agent and the principal. The second, hereafter referred to as an AP contract, is an incentive-compatible supercontract which guarantees the participation of the agent, but not necessarily the one of the principal.[17] Note that the set of possible 2P contracts that can be signed at node $y^{\tau-1}$ was already denoted by $\Gamma_{y^{\tau-1}}$. Let $\Gamma^{AP}_{y^{\tau-1}}$ be the set of possible AP contracts that can be signed at node $y^{\tau-1}$. Formally, $\Gamma^{AP}_{y^{\tau-1}} := \left\{c : (1) - (4) \text{ hold after } y^{\tau-1}\right\}$. Now, we are going to consider the sets of agent's initial utilities that can be guaranteed/supported by a 2P and respectively an AP contract.

Let l be some initial history node. Take an arbitrary period τ and a history $y^{\tau-1}$ stemming from l, i.e., $y^{\tau-1} \in l \times Y^{\tau}$. Let $V^{2P}_{\tau}\left(y^{\tau-1}\right)$ be the set of admissible values for the expected discounted utility of the agent signing at date τ after a history $y^{\tau-1}$ a 2P contract with the principal. Formally, $V^{2P}_{\tau}\left(y^{\tau-1}\right) := \{V \in \mathbb{R} : \exists c \in \Gamma_{y^{\tau-1}} \text{ such that } V_{\tau}\left(c, y^{\tau-1}\right) = V\}$. Let us also introduce another set, $V^{AP}_{\tau}\left(y^{\tau-1}\right)$, which gives us the possible discounted utilities of the agent signing at date τ after a history $y^{\tau-1}$ an AP contract with the principal. Formally, $V^{AP}_{\tau}\left(y^{\tau-1}\right) := \{V \in \mathbb{R} : \exists c \in \Gamma^{AP}_{y^{\tau-1}} \text{ such that } V_{\tau}\left(c, y^{\tau-1}\right) = V\}$. Since every 2P contract is an AP contract, the agent's utilities supportable by a 2P contract will be a subset of the agent's utilities supportable by an AP contract. Formally, $V^{2P}_{\tau}\left(y^{\tau-1}\right) \subset V^{AP}_{\tau}\left(y^{\tau-1}\right)$ for any $l \in L$, $\tau \in \mathbb{Z}_{+}$, and $y^{\tau-1} \in l \times Y^{\tau}$. Now, we are ready to introduce the sets of principal's initial utilities that can be supported by a 2P and respectively an AP contract promising a certain initial utility to the agent.

For any $V \in V^{2P}_{\tau}\left(y^{\tau-1}\right)$, let $U^{2P}_{\tau}\left(V, y^{\tau-1}\right)$ be the set of possible values for the expected discounted utility of the principal signing at node $y^{\tau-1}$ at time τ a 2P contract that would give the agent an initial expected discounted utility

[17] It may be easier to remember the abbreviations in the following way: AP="agent participates"; 2P= "two [...] participate", i.e., both the agent and the principal participate.

of V, i.e., $U_\tau^{2P}\left(V, y^{\tau-1}\right) := \{U \in \mathbb{R} : \exists c \in \Gamma_{y^{\tau-1}}$ such that $V_\tau\left(c, y^{\tau-1}\right) = V$ and $U_\tau\left(c, y^{\tau-1}\right) = U\}$. For any $V \in V_\tau^{AP}\left(y^{\tau-1}\right)$, let $U_\tau^{AP}\left(V, y^{\tau-1}\right)$ be the corresponding set (defined accordingly) in case the principal is signing an AP contract instead. Then, for any $V \in V_\tau^{2P}\left(y^{\tau-1}\right)$, we have $U_\tau^{2P}\left(V, y^{\tau-1}\right) \subset U_\tau^{AP}\left(V, y^{\tau-1}\right)$, while for $V \in V_\tau^{AP}\left(y^{\tau-1}\right) \setminus V_\tau^{2P}\left(y^{\tau-1}\right)$, $U_\tau^{2P}\left(V, y^{\tau-1}\right)$ is not defined.

Proposition 2 *Let $l \in L$ and $i \in \{2P, AP\}$. Then, for any $\tau \in \mathbb{Z}_+$ and $y^{\tau-1} \in Y^\tau \times l$: (a) $V_\tau^i\left(y^{\tau-1}\right) = V_0^i(l)$ compact; (b) $\forall V \in V_\tau^i\left(y^{\tau-1}\right)$, $U_\tau^i\left(V, y^{\tau-1}\right) = U_0^i(V, l)$ compact.*

Part (a) of the proposition says that the sets of possible expected discounted utility values for the agent signing a 2P or AP contract are time invariant and compact. Furthermore, the history dependence of these sets is restricted only to the previous (as of signing) θ realizations. As part (b) indicates, the result also applies to the set of possible expected discounted utilities of the principal signing a 2P or AP contract guaranteeing a particular initial utility to the agent.

To ease up the notation, we will hereafter refer to these sets as $V^{2P}(l)$, $V^{AP}(l)$, $U^{2P}(V, l)$, and $U^{AP}(V, l)$.

Remember that U_l^{**} is the supremum of the principal's problem [PP]. We state the following result.

Proposition 3 *(Existence of an optimal contract): For any $l \in L$, $\exists c_l \in \Gamma_l$ s.t. $U_l^{**} = U_0(c_l, l)$.*

The proposition establishes the existence of an optimal 2P contract. However, due to the complexity of the problem, the optimal contract cannot be derived analytically. Nevertheless, I show that it can be characterized and given a computable representation. In the spirit of Spear and Srivastava (1987), this is done by constructing a recursive version of [PP] taking the agent's expected discounted utility as a state variable. Up to certain qualifications, this new formulation of the problem can be addressed by dynamic programming routines.

I will first establish a useful result which is related to the transformation of the dynamic principal's problem to a series of static problems defined on an endogenously obtained state space.

Fix $l \in L$. By Proposition 2 (b) for any $V \in V^{2P}(l)$, $U^{2P}(V,l)$ is compact and therefore, we can define $U^{*}(V,l) := \max U^{2P}(V,l)$ as the maximum utility the principal can get by signing a 2P supercontract offering V to the agent. Furthermore, let $U_l^* := \sup_{V \in V^{2P}(l)} U^*(V,l)$.

Proposition 4 $\forall l \in L$, $U_l^{**} = U_l^* = \max_{V \in V^{2P}(l)} U^*(V,l)$.

This proposition shows that the principal is indifferent between directly maximizing his/her utility given l, or first finding the maximum utility he/she can obtain by guaranteeing the agent a certain initial level of utility and then maximizing over the resulting set.[18]

Let $l_+ : L \times Y \to L$ give the index of the initial history tomorrow given the index of the initial history today and the new realization of the variable. Consider, for example, that we are in period t. At the beginning of t, an initial history $(y_{t-\theta}, ..., y_{t-1})$ with index l has been observed. At the end of the period, an outcome y_{t-1} is realized. Then, at the beginning of period $t+1$, the observed initial history will be $(y_{t-\theta+1}, ..., y_t)$ and will have an index $l_+(l, y_t)$. We will often abuse the notation and use l_+ instead of $l^{-1}(l_+)$, i.e., replace the initial history tomorrow by its index.

Consider the state space $\left\{(V,l) : V \in V^{AP}(l)\right\}_{l \in L}$ that matches initial histories of outcomes with initial utility promises supportable by an AP contract.[19] Let $c_R(V,l) = \{(a_-(V,l), w_+(V,l,y), V_+(V,l,y)) : y \in Y\}$ be a stationary contract defined on a point (V,l) of the state space, where $a_-(.)$ is the agent's

[18] Note that the original problem can be set as the principal maximizing expected discounted utility given an initial truncated history l at period 0, where the maximum is taken over a set of 2P supercontracts promising the agent an initial expected discounted utility of V_l for any $l \in L$ and $V_l \in V^{2P}(l)$. The promise should be consistent (in a sense that will soon become clear; see (9)) and can be considered a leftover from a (remote) previous round of contracting. Then, the original problem is defined on $\{V^{2P}(l) : l \in L\}$ and the recursive representation will be equivalent to the one obtained here without the need to maximize $U^*(.,l)$ over $V^{2P}(l)$. Namely, we would have $U^{**}(.,l) = U^*(.,l)$ over $V^{2P}(l)$. Since the static form characterizing both [PP] and the problem described here is the same, I choose to present the former because of the more involved description and notation of the latter.

[19] The possible initial histories (of length θ) enter the picture because they could potentially affect the reservation utility values.

action in the beginning of the period, $w_+(.,y)$ is the wage the agent will receive in the end of the period if the realization of the variable of interest to the principal is y, $\forall y \in Y$, and $V_+(.,y)$ is the end-of-period expected discounted utility of the agent in case of realization y, $\forall y \in Y$. Since the realization of the variable in question is not known when this contract is signed, the wage and the end-of-period utility of the agent are specified for all possible outcomes, Y. Although the stationary contract depends on the initial history and the particular expected discounted utility of the agent in the beginning of the period, I will often suppress this dependence notationally and refer to the contract simply as $c_R = \{(a_-, w_+(y), V_+(y)) : y \in Y\}$. Let $USCB_l$ denote the space of bounded upper semicontinuous (usc) functions from $V^{AP}(l)$ to $\mathbb{R}$ endowed with the sup metric. Define $V^{AP} := \left\{V^{AP}(l)\right\}$ as the set of possible initial discounted utilities of the agent signing an AP contract ordered by initial history. Since L is finite, this set inherits the properties of $V^{AP}(l)$ established in Proposition 2 (a). Then, for any $U = \{U_l\}$ with $U_l \in USCB_l$, $\forall l \in L$, define the operator T as follows. For any $V = \{V_l\} \in \left\{V^{AP}\right\}$, $T(U)_{(V)} := \left\{T_l(U)_{(V_l)}\right\}$, where:

$$T_l(U)_{(V_l)} := \max_{c_R}\{\sum_{y\in Y}[u(w_+(y), y) + \beta_P U_{l_+(l,y)}(V_+(y))]\pi(y, a_-)\} \text{ s.t.:}$$

$$a_- \in A \tag{7}$$

$$w_+(y) \in W, \forall y \in Y \tag{8}$$

$$\sum_{y\in Y}[\nu(w_+(y), a_-) + \beta_A V_+(y)]\pi(y, a_-) = V_l \tag{9}$$

$$\sum_{y\in Y}[\nu(w_+(y), a'_-) + \beta_A V_+(y)]\pi(y, a'_-) \le V_l, \forall a'_- \in A \tag{10}$$

$$V_+(y) \in V^{AP}(l_+(l,y)), \forall y \in Y \tag{11}$$

Notice that the maximization above is over a set of static contracts at a particular point (V_l, l) of the state space.[20] Also note that if the initial history today

[20]This may not show up directly since I have simplified the notation by suppressing the dependence of c_R on the initial history l and the particular initial utility V_l promised to the

is l and the end-of-the-period realization is y, then the initial history tomorrow will be l_+ which in general will be different from l. Therefore, it is important that we keep track of the initial history update and so each T_l is applied to U, not just to U_l.[21] The use of max instead of sup in the definition of T is justified by the fact that we are maximizing an usc function over a compact set. Constraints (7), (8), and (10) are the stationary versions of (1), (2), and (6) respectively. In particular, (7) guarantees that the action is admissible (i.e., an element of A), (8) guarantees that the compensation scheme is admissible (i.e., mapping Y to W), and (10) is temporary incentive compatibility.[22] (9) is a promise keeping constraint[23] which guarantees the agent an expected discounted utility of V_l today. It is a requirement on the static contract that makes the principal's initial utility promise to the agent at node l, V_l, consistent with the future promise given the proposed action and compensation scheme. (11) is another consistency constraint requiring that the discounted expected utility that the agent will get next period can be supported by an AP supercontract. Note that (11) implies that the agent's continuation utility should not fall below the reservation level on any respectively updated initial history, i.e., $V_+(.,y) \geq \underline{V}_{l_+(l,y)}$, $\forall y \in Y$. In fact, constraints (9) and (11) guarantee the dynamic consistency of the series of static contracts generated by iterating on the operator T.

For any $l \in L$ and $V \in V^{AP}(l)$, we have that $U^{AP}(V,l)$ is compact by Proposition 3 (b). Then, we can define $U^{AP^*}(V,l) := \max U^{AP}(V,l)$ as the maximum utility the principal can get by signing an AP supercontract offering V to the agent. For any $V \in V^{AP}$, let $U^{AP^*}(V) = \left\{U^{AP^*}(V_l,l)\right\}$ be the vector of these maximum utilities indexed by initial history. Next, I will show that

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agent.

Notice also that $\{(V_l,l) : V_l \in V^{AP}(l)\}$ is endogenous to the model, so one may doubt the usefulness of defining the operator T on an unknown state space as well as the practical benefit of constraint (11). Further in this section, however, we will demonstrate that V^{AP} can be recovered from the primitives by a recursive procedure in the spirit of Abreu, Pearce and Stacchetti (1990).

[21] Of course, if $\theta = 0$, the reservation utilities will be constant at all nodes, so the initial history will be immaterial for the static contract. The state space will shrink to a single dimension; namely, the set of expected discounted utilities that can be promised to the agent will be the same at every node. Then, the history update will prove irrelevant since U will be defined on the one-dimensional V^{AP}.

[22] Note that we have made use of (9) when stating temporary incentive compatibility as (10). Namely, given (9) holds, temporary incentive compatibility is equivalent to (10).

[23] It is referred to as a re-generation constraint in Spear and Srivastava (1987).

$U^{AP^*}: V^{AP} \to \mathbb{R}^{n^\theta}$ is the unique fixed point of the operator T and can be obtained as the limit of successively iterating on T. I start with a proposition that establishes some useful properties of U^{AP^*}.

Proposition 5 *For any $l \in L$, $U^{AP^*}(.,l)$ is usc and bounded on $V^{AP}(l)$.*

Note that these properties can directly be translated to U^{AP^*}, say with the sup metric over Y^θ.

Proposition 6 $T\left(U^{AP^*}\right) = U^{AP^*}$.

The proposition says that U^{AP^*} is a fixed point of the operator T.

For the purposes of the next proposition, I introduce some additional notation. Let $ß_l$ denote the space of bounded functions from $V^{AP}(l)$ to $\mathbb{R}$ endowed with the sup metric. For any $U', U'' \in \{USCB_l\}$, define the metric $\mu(U', U'') := \sup_{l \in L} \mu_l(U', U'')$, where $\mu_l(U', U'') := \sup_{V_l \in V^{AP}(l)} |U'_l(V_l) - U''_l(V_l)|$, $\forall l \in L$. Note that both suprema in the above definition are achieved.

Proposition 7 *(a) T maps $(\{USCB_l\}, \mu)$ into itself; (b) T is a contraction mapping with modulus β_P in terms of the metric μ; (c) Let $\widetilde{U} \in (\{ß_l\}, \mu)$: $T\left(\widetilde{U}\right) = \widetilde{U}$. Then, $\widetilde{U} = U^{AP^*}$; (d) $\forall U \in (\{USCB_l\}, \mu)$, $\mu\left(T^n(U), U^{AP^*}\right) \underset{n\to\infty}{\to} 0$, where $T^n(U) := T(T^{n-1}(U))$ for any $n \in \mathbb{Z}_{++}$ with $T^0(U) := U$.*

This proposition shows that the fixed point of T is unique and can be obtained as a limit of successive iterations on T. Consequently, we can use standard dynamic programming techniques in order to solve for the optimal AP contract.

However, what we are ultimately interested in is solving for the optimal 2P contract. For this purpose, I resort again to dynamic programming using a method outlined by Rustichini (1998).

First, I will introduce some notation. For any $l \in L$ and $V_l \in V^{AP}(l)$, let $\Gamma_R(V_l, U, l) := \{c_R : (7) - (11) \text{ hold at } (V_l, l) \text{ and } U_{l_+(l,y)}(V_+(y)) \geq \underline{U}_{l_+(l,y)}, \forall y \in Y\}$ for some function $U : V^{AP} \to (\mathbb{R} \cup \{-\infty\})^{n^\theta}$. Additionally, let

$$\Lambda_R(V_l, U, l) := \begin{cases} \Gamma_R(V_l, U, l) & \text{if } U_l(V_l) \geq \underline{U}_l \\ \Lambda_R(V_l, U, l) := \emptyset & \text{otherwise.} \end{cases}$$

Denote by $USCBA_l$ the space of usc, bounded from above functions from $V^{AP}(l)$ to $\mathbb{R} \cup \{-\infty\}$. Then, for any $U = \{U_l\}$ with $U_l \in USCBA_l$, $\forall l \in L$, define the operator $\underline{T}$ as follows. For any $V \in V^{AP}$, $\underline{T}(U)_{(V)} := \left\{\underline{T}_l(U)_{(V_l)}\right\}$, where

$$\underline{T}_l(U)_{(V_l)} := \max_{\substack{c_R \in \\ \Lambda_R(V_l, U, l)}} \{\sum_{y \in Y} [u(w_+(y), y) + \beta_P U_{l_+(l,y)}(V_+(y))] \pi(y, a_-)\}$$

following the convention that $\underline{T}_l(U)_{(V_l)} = -\infty$ if $\Lambda_R(V_l, U, l) = \emptyset$.

This operator encompasses the lower bounds on the utility of the principal in the form of additional constraints. The only difference with T is that in case U is lower than the reservation utility of the principal today or at any possible contingency tomorrow, $\underline{T}$ becomes $-\infty$. The idea is that any violation of the constraints in this stationary framework is punished severely making the contract in question non-optimal. What remains to be shown is that iterating on this operator will indeed lead us to the optimal dynamic contract.

Proposition 8 $\underline{T}$ *maps* $\{USCBA_l\}$ *into itself.*

For any $V \in V^{AP}$, let $D_0(V) := U^{AP^*}(V)$ and $D_{i+1}(V) := \underline{T}(D_i)$, $\forall i \in \mathbb{Z}_+$. Note that by Proposition 8 and the fact that $\Gamma_R(V_l, U, l)$ is compact if non-empty for any $V_l \in V^{AP}(l)$, $U \in \{USCBA_l\}$, and $l \in L$ (trivial), D_i is well defined on V^{AP} for any $i \in \mathbb{Z}_+$.

Proposition 9 *(a)* $\{D_i\}_{i=1}^{\infty}$ *is a weakly decreasing sequence and* $\exists D_\infty \in \{USCBA_l\} : D_i(V_l, l) \underset{i\to\infty}{\to} D_\infty(V_l, l)$*,* $\forall V_l \in V^{AP}(l)$*,* $\forall l \in L$*; (b)* $\underline{T}(D_\infty) = D_\infty$*; and (c) if* $\exists D' \in \{USCBA_l\} : \underline{T}(D') = D'$*, then* $D' \leq D_\infty$*.*

This proposition says that if we start iterating on the operator $\underline{T}$ taking U^{AP^*} as an initial guess, we will ultimately converge (pointwise) to D_∞, the largest fixed point of $\underline{T}$. Next, I establish the relationship between U^* and D_∞.

In the subsequent analysis it will be useful to extend U^* on V^{AP}. For any $V \in V^{AP}$, let $\widehat{U}^*(V) := \left\{\widehat{U}^*(V_l, l)\right\}$ with $\widehat{U}^*(V_l, l) := U^*(V_l, l)$ if $V_l \in V^{2P}(l)$ and $\widehat{U}^*(V_l, l) := -\infty$ otherwise.

Proposition 10 $\underline{T}\left(\widehat{U}^*\right) = \widehat{U}^*$*.*

This proposition establishes that the extension of U^* on V^{AP} is a fixed point of $\underline{T}$. What remains to be shown is how to recover U^* from D_∞. The next proposition gives the answer.

Proposition 11 *For any* $V \in V^{AP}$*,* $\widehat{U}^*(V) = D_\infty(V)$*.*

The proposition provides a straight-forward method of solving for the optimal 2P supercontract. After we have found the optimal AP contract we take it as an initial guess and start iterating on the operator $\underline{T}$ until convergence is reached. Note that convergence here is pointwise and is meant to be on $\mathbb{R} \cup \{-\infty\}$. After we have obtained the limit function D_∞, we can recover the set of possible values for the expected discounted utility of the agent signing a 2P contract by taking the subset of the domain of D_∞ on which the limit function takes finite values. More precisely, for any $l \in L$ we can restrict ourselves only to values of $D_\infty(., l)$ above $\underline{U}_l$. Formally, $V^{2P}(l) := \left\{V \in V^{AP}(l) : D_\infty(V, l) \geq \underline{U}_{-\theta_y}\right\}$. Then, for any$V \in V^{2P}(l)$, we have $U^*(V; l) = D_\infty(V, l)$.

However, note that the state space of the recursive problem constructed for computing the optimal AP contract, V^{AP}, is endogenous. Nevertheless, it is the largest fixed point of a set operator and can be obtained through successive iterations in a procedure introduced by Abreu, Pearce and Stacchetti (1990).

Choose some $\widehat{V} \in \mathbb{R} : \widehat{V} \geq \max_{l \in L} \{\max V^{AP}(l)\}$, where the right-hand side of the inequality is well defined given $V^{AP}(l)$ compact, $\forall l \in L$ and L finite. Note that given Assumption *3*, $\left[\underline{V}_l, \widehat{V}\right] \neq \emptyset, \forall l \in L$. Then, for any $X = \{X_l\} : X_l \in \mathbb{R}$, $\forall l \in L$ let $B(X) := \{B_l(X)\}$ with

$$B_l(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R : (7) - (10) \text{ and } (12) \text{ hold at } (V_l, l)\},$$

where (12) is defined as:

$$V_+(V_l, y) \in X_{l_+(l,y)} \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right), \forall y \in Y \tag{12}$$

Note that $B_l(X)$ gives the set of agent's initial utilities that are not below the reservation level $\left(\text{which follows from } V_l \in \left[\underline{V}_l, \widehat{V}\right]\right)$ and that can be supported by a single-round (stationary) contract at l that is admissible [i.e., (7) and (8) hold], consistent [i.e., satisfies (9)], temporary incentive-compatible [i.e., satisfies (10)] and has continuation utilities which are taken from X and are not below the relevant reservation level [i.e., (12) holds]. In short, B maps continuation utilities to relevant initial utilities. It is this operator that will help us recover the endogenous state space of T, V^{AP}.

Proposition 12 *(a)* $B\left(V^{AP}\right) = V^{AP}$*; and (b) if* $\exists X \subset \mathbb{R}^{n^\theta} : B(X) = X$*, then* $X \subset V^{AP}$*.*

This proposition establishes that the set of agent's expected discounted utilities supportable by an AP supercontract, V^{AP}, is the largest fixed point of B.

Proposition 13 *Let X_0 compact : $V^{AP} \subset X_0 \subset \mathbb{R}^{n^\theta}$ and $B(X_0) \subset X_0$. Define $X_{i+1} := B(X_i)$, $\forall i \in \mathbb{Z}_+$. Then, $X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ and $X_\infty := \lim_{i\to\infty} X_i = V^{AP}$.*

The proposition says that if we start iterating on B taking as an initial guess some compact set X_0 that contains both $B(X_0)$ and V^{AP}, we will ultimately converge to the largest fixed point of the operator, V^{AP}. This is sufficient for obtaining V^{AP} since we can always take $X_0 = \{X_{0,l}\} : \left[\underline{V}_l, \widehat{V}\right] \subset X_{0,l} \subset \mathbb{R}$ with $X_{0,l}$ compact, $\forall l \in L$. However, an even more computationally efficient result exists.

Let us modify the operator B as follows. For any $X = \{X_l\} : X_l \in \mathbb{R}, \forall l \in L$ let $\widetilde{B}(X) := \left\{\widetilde{B}_l(X)\right\}$ with

$$\widetilde{B}_l(X) := \{V_l \in X_l : \exists c_R : \text{(7)-(10) and (13) hold at } (V_l, l)\},$$

where (13) is defined as:

$$V_+(y) \in X_{l_+(l,y)}, \forall y \in Y \tag{13}$$

Note that the operator $\widetilde{B}$ does not require that the agent should commit to the contract. Namely, we do not impose a constraint keeping the continuation values for the utility of the agent above the lower bound given by the reservation utility. From a computational point of view, we are increasing the efficiency since we are relaxing the set of constraints.

Proposition 14 *(a) Take $\widetilde{X}_0 := \left\{\widetilde{X}_{0,l}\right\}$ with $\widetilde{X}_{0,l} = \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ and let $\widetilde{X}_{i+1} := \widetilde{B}\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. Then, $\widetilde{X}_{i+1} \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$ and $\widetilde{X}_\infty := \lim_{i\to\infty} \widetilde{X}_i = V^{AP}$. (b) $\widetilde{B}\left(V^{AP}\right) = V^{AP}$; and (c) if $\exists X : \emptyset \neq X \subset \widetilde{X}_0$ and $\widetilde{B}(X) = X$, then $X \subset V^{AP}$.*

This proposition outlines a practical way of obtaining V^{AP}. Namely, we start with the set $\left\{\left[\underline{V}_l, \widehat{V}\right]\right\}$ and iterate on the set operator $\widetilde{B}$ until convergence in a properly defined sense is attained. Note that we can always take $\widehat{V} = \frac{\nu(\max\{W\},\min\{A\})}{1-\beta_A}$.

4 Conclusion

This paper builds a framework for analyzing dynamic moral hazard problems characterized by limited commitment and history-dependent reservation utilities. This is achieved by constructing an equivalent recursive representation that is stationary on a properly defined state space. The state space which contains the expected discounted utilities of the agent on one dimension and the initial histories on the other is characterized by a generalized Bellman equation. Given the state space, the optimal AP contract is recursively characterized by standard dynamic programming routines on bounded usc functions and in the same time is used as an initial guess for the optimal 2P in a procedure severely punishing any violation of the principal's participation constraint.

This general setting can be used to address multiple dynamic problems including but not limited to executive compensation, stock option packages, tenure decisions, optimal insurance, and investment. It would also be interesting to try to endogenize the external options in a model directly providing the link between fundamentals/beliefs and reservation utilities.

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APPENDIX

Lemma 1 *Let (1) and (2) hold after $l \in Y^{\theta}$. Then, (3) $\Leftrightarrow$*

$$V_0(a,w,l) \geq V_0(a',w,l), \forall (a' : \forall nai(l), \; a'_t \in A) \tag{14}$$

Proof. It is trivial to show (3)$\Rightarrow$(14). Just take $\tau = 0$. In the other direction, let (14) hold, but assume that (3) is not satisfied, i.e., there is a node $y^{\tau-1}$ s.t. $\exists a'$ admissible $\forall \text{nai}(y^{\tau-1})$ and $V_\tau(a',w,y^{\tau-1}) > V_\tau(a,w,y^{\tau-1})$. Let a'' : $\forall \text{nai}(y^{\tau-1})$, $a''_t = a'_t$, with $a''_t = a_t$ elsewhere. Given (1) and (2), ν is continuous on a compact set and, therefore bounded. Then, we obtain:

$$\begin{aligned} &V_0(a'',w,l) = \\ &\sum_{t=0}^{\tau-1} \beta_A^t \sum_{y_t \in Y} ... \sum_{y_0 \in Y} \nu\left(w_t\left(y^t\right), a''_t\left(y^{t-1}\right)\right) \prod_{i=0}^{t} \pi\left(y_i, a''_i\left(y^{i-1}\right)\right) + \\ &\beta_A^\tau \sum_{y_{t-1} \in Y} ... \sum_{y_0 \in Y} V_\tau\left(a'',w,y^{\tau-1}\right) \prod_{i=0}^{t-1} \pi\left(y_i, a''_i\left(y^{i-1}\right)\right) \\ &> V_0(a,w,l), \end{aligned} \tag{A1}$$

where the inequality follows from the construction of a'' since $V_\tau(a',w,y^{\tau-1}) > V_\tau(a,w,y^{\tau-1})$ and $\pi > 0$ by Assumption *1*. Given that a'' is admissible after l by construction, (A1) contradicts (14). ■

This proposition shows that incentive compatibility at an initial node ${}^{-\theta}y$ is equivalent to incentive compatibility at all the nodes following l.

Proof of Proposition 1. It is trivial that (3) implies (6). In the other direction, assume (6) holds at every node, but $\exists$ an admissible plan a' : $V_0(a',w,l) > V_0(a',w,l)$. We have:

$$V_0\left(a',w,l\right) =$$

$$\sum_{t=0}^{T}\beta_A^t\sum_{y_t\in Y}...\sum_{y_0\in Y}\nu\left(w_t\left(y^t\right),a'_t\left(y^{t-1}\right)\right)\prod_{i=0}^{t}\pi\left(y_i,a'_i\left(y^{i-1}\right)\right)+$$

$$\beta_A^{T+1}\sum_{y_T\in Y}...\sum_{y_0\in Y}V_{T+1}\left(a',w,y^T\right)\prod_{i=0}^{T}\pi\left(y_i,a'_i\left(y^{i-1}\right)\right),$$

where the second term on the right-hand side can be made arbitrarily small by choosing T big enough given (1), (2) and the assumptions on β_A, ν, A, W. Therefore, $\exists T\in\mathbb{Z}_+$ and an admissible plan $a'' : a''_t\left(y^{t-1}\right)=a'_t\left(y^{t-1}\right)$, $\forall y^{t-1}\in l\times Y^t$, $\forall t\leq T$, and $a''_t=a_t$ elsewhere, s.t. $V_0\left(a'',w,l\right)>V_0\left(a,w,l\right)$ Then, take $\tau\in\mathbb{Z}_+ : \tau\leq T$ s.t. $\exists y^{\tau-1} : a''_\tau\left(y^{\tau-1}\right)\neq a_\tau\left(y^{\tau-1}\right)$ and $\nexists\tau'\in\mathbb{Z}_{++} : \tau<\tau'\leq T : a''_{\tau'}\left(y^{\tau'-1}\right)\neq a_{\tau'}\left(y^{\tau'-1}\right)$ for some $y^{\tau'-1}\in l\times Y^{\tau'}$. If we define an admissible plan $a''' : a'''_\tau\left(y^{\tau-1}\right)=a_\tau\left(y^{\tau-1}\right),\forall y^{\tau-1}\in l\times Y^\tau$ and $a'''_t=a''_t$ elsewhere, by (6) at $\forall y^{\tau-1}\in l\times Y^\tau$, we have that $V_\tau\left(a''',w,y^{\tau-1}\right)\geq V_\tau\left(a'',w,y^{\tau-1}\right)$, from where $V_0\left(a''',w,l\right)\geq V_0\left(a'',w,l\right)$. Proceeding in this way we can eliminate all the deviations (note that $\tau\in\mathbb{Z}_+ : \tau\leq T$) to obtain $V_0\left(a,w,l\right)\geq V_0\left(a'',w,l\right)$, i.e., a contradiction. Therefore, we obtain (6)⇒(14), which by Lemma 1 results in (6)⇒(3). ■

For any $l\in L$, let $C_l := \{c : \text{(1) and (2) hold after } l\}$.

Proof of Proposition 2. (a) Fix $l\in L$. Take $\tau',\tau''\in\mathbb{Z}_+ : \tau'\leq\tau''$ and arbitrary $y^{\tau'-1}\in Y^{\tau'}\times l$ and $y^{\tau''-1}\in Y^{\tau''}\times l$. Take an arbitrary $V'\in V^{2P}_{\tau'}\left(y^{\tau'-1}\right)$. Then, there exists a contract $c'=(a',w')\in\Gamma_{y^{\tau'-1}} : V_{\tau'}\left(c',y^{\tau'-1}\right)=V'$. Define $c''=(a'',w'')$ such that for any $\widetilde{y}^t\in y^{\tau''-1}\times Y^{t-\tau''+1}$ with $t\geq\tau''$, $a''_t\left(\widetilde{y}^{t-1}\right):=a'_{\tau'+t-\tau''}\left(y^{\tau'-1},\widetilde{y}_{\tau''},...,\widetilde{y}_{t-1}\right)$, $w''_t\left(\widetilde{y}^t\right):=w'_{\tau'+t-\tau''}\left({}^{-\theta}y^{\tau'-1},\widetilde{y}_{\tau''},...,\widetilde{y}_t\right)$. It is straight-forward that $V_{\tau''}\left(c'',y^{\tau''-1}\right)=V'$ and $c''\in\Gamma_{y^{\tau''-1}}$. Therefore, we have that $V'\in V^{2P}_{\tau''}\left(y^{\tau''-1}\right)$. The same argument holds in the other direction, so we have proven that $V^{2P}_{\tau''}\left(y^{\tau'-1}\right)=V^{2P}_{\tau''}\left(y^{\tau''-1}\right)$.

Fix $l \in L$. $V^{2P}(l)$ is bounded given (1) and (2). Regarding the compactness of $V^{2P}(l)$, we should also prove that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i=1}^{\infty} : V_i \in V^{2P}(l)$, $\forall i \in \mathbb{Z}_{++}$ with limit V_∞. We need to show that $V_\infty \in V^{2P}(l)$. By the construction of the sequence, for any $i \in \mathbb{Z}_{++}$, $\exists c_i \in \Gamma_l$ such that $V_0(c_i, l) = V_i$. Then, for any $i \in \mathbb{Z}_{++}$, $c_i \in C_l$. Let us endow C_l with the product topology. C_l is compact as a product of compact spaces. Consequently, there exists a convergent subsequence $\{c_{i_k}\}_{k=1}^{\infty}$ of $\{c_i\}_{i=1}^{\infty}$ such that $c_\infty := \lim_{k \to \infty}(c_{i_k}) \in C_l$, from where c_∞ satisfies (1) and (2) after l. For any $T \in \mathbb{Z}_+ : T \geq \tau$, let $V_\tau^T(c, y^{\tau-1}) := \sum_{t=\tau}^{T} \beta_A^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} [\nu(w_t(y^t), a_t(y^{t-1}))] \prod_{i=\tau}^{t} \pi(y_i, a_i(y^{i-1}))$. Notice that $V_\tau(c, y^{\tau-1}) - V_\tau^T(c, y^{\tau-1}) = \beta_A^{T+1} \sum_{y_T \in Y} ... \sum_{y_\tau \in Y} V_{T+1}(c, y^T) \prod_{i=\tau}^{T} \pi(y_i, a_i(y^{i-1})) \in [\beta_A^{T+1} \frac{\upsilon(\min W, \max A)}{1-\beta_P}, \beta_A^{T+1} \frac{\upsilon(\max W, \min A)}{1-\beta_P}]$, $\forall T \in \mathbb{Z}_+ : T \geq \tau$, $\forall c \in C_l$, $\forall y^{\tau-1} \in l \times Y^\tau$, $\forall \tau \in \mathbb{Z}_+$. Moreover, $V_\tau^T(., y^{\tau-1})$ is continuous on C_l. Then, $V_\tau(., y^{\tau-1})$ is continuous on C_l. Analogously, we can show that $U_\tau(., y^{\tau-1})$ is continuous on C_l. As a result, we have that c_∞ satisfies (4), (5), (6) after l and $V_0(c_\infty, l) = V_\infty$.

Following the same logic, we can show that $V_\tau^{AP}(y^{\tau-1})$ is time invariant and compact and depends only on the last θ observations prior to signing.

(b) Analogous to the proof of (a). ■

Proof of Proposition 3. Fix $l \in L$. We have $\Gamma_l \subset C_l$. Let's endow C_l with a metric inducing the product topology. Then, following the argument of the proof of Proposition 2, we obtain that Γ_l is compact and $U_0(., l)$ is continuous on C_l. ■

Proof of Proposition 4. Fix $l \in L$. By Proposition 3, we have that $\exists c \in \Gamma_l$ and $U_0(c, l) = U_l^{**}$. Let $V^{**} := V_0(c, l)$. By Proposition 1, $V^{**} \in V^{2P}(l)$ and $U_l^{**} \in U(V^{**}, l)$. Therefore, $U_l^* \geq U_l^{**}$. Suppose $U_l^* > U_l^{**}$. Then, $\exists V^* \in V^{2P}(l) : U_l^{**} < U^*(V^*, l) \leq U_l^*$. Since $U^*(V^*, l) \in U(V^*, l)$, $\exists c^* \in \Gamma_l$, $V_0(c^*, l) = V^*$ and $U_0(c^*, l) = U^*(V^*, l)$. Then, by the definition of U_l^{**} and Proposition 1 we have that $U_l^{**} \geq U^*(V^*, l)$, i.e., a contradiction is reached. Consequently, $U_l^* = U_l^{**}$ and the supremum in the definition of U_l^* is achieved. ■

For any $l \in L$ and $\forall V \in V^{AP}(l)$, define $\Gamma_l^{AP}(V) := \{c$:(1), (2), (4), (6) hold after l and $V_0(c,l) = V\}$ and $G_l^{AP}(V) := \{c \in \Gamma_l^{AP}(V) : U_0(c,l) = U^{AP^*}(V,l)\}$.

Lemma 2 *For any $l \in L$, $\Gamma_l^{AP}(.)$ is upper hemi-continuous (uhc) on $V^{AP}(l)$.*

Proof. Fix $l \in L$ and $V \in V^{AP}(l)$ and note that $\Gamma_l^{AP}(V)$ is non-empty and compact. Take a sequence $\{V_i\}_{i=1}^{\infty}$ s.t. $V_i \in V^{AP}(l)$, $\forall i \in \mathbb{Z}_{++}$ and $V_i \underset{i\to\infty}{\to} V$. Let $c_i \in \Gamma_l^{AP}(V_i)$, $\forall i \in \mathbb{Z}_{++}$. Note that $\Gamma_l^{AP}(V_i) \subset C_l$, $\forall i \in \mathbb{Z}_{++}$ with C_l compact . Then, $\exists$ a subsequence $\{c_{i_j}\}_{j=1}^{\infty}$ of $\{c_i\}_{i=1}^{\infty} : c_{i_j} \underset{j\to\infty}{\to} c \in C_l$. Since $V_\tau\left(.,^{-\theta} y^{\tau-1}\right)$ is continuous on C_l, c satisfies (4) and (6) after l and $V_0(c,l) = V$. Therefore, $c \in \Gamma_l^{AP}(V)$. ■

Proof of Proposition 5. Fix $l \in L$ and $V \in V^{AP}(l)$. Take a sequence $\{V_i\}_{i=1}^{\infty}$ s.t. $V_i \in V^{AP}(l)$, $\forall i \in \mathbb{Z}_{++}$ and $V_i \underset{i\to\infty}{\to} V$. Let $c_i \in G_l^{AP}(V_i)$, $\forall i \in \mathbb{Z}_{++}$. Define $\overline{U_l^{AP^*}} := \overline{\lim_{i\to\infty}} U^{AP^*}(V_i,l)$. $\exists$ a subsequence $\{c_{i_j}\}_{j=1}^{\infty}$ of $\{c_i\}_{i=1}^{\infty} : \lim_{j\to\infty} U_0\left(c_{i_j},l\right) = \overline{U_l^{AP^*}}$. Since $G_l^{AP}(.) \subset \Gamma_l^{AP}(.)$ and $\Gamma_l^{AP}(.)$ is uhc from Lemma 2, $\exists$ a subsequence $\{c_{i_{j_n}}\}_{n=1}^{\infty}$ of $\{c_{i_j}\}_{j=1}^{\infty} : c_{i_{j_n}} \underset{n\to\infty}{\to} c$ with $c \in \Gamma_l^{AP}(V)$. Then, $\overline{U_l^{AP^*}} = \lim_{n\to\infty} U_0\left(c_{i_{j_n}},l\right) = U_0(c,l) \leq U^{AP^*}(V,l)$ where the first equality comes from the fact that $\{c_{i_{j_n}}\}_{n=1}^{\infty}$ is a subsequence of $\{c_{i_j}\}_{j=1}^{\infty}$ and $\lim_{j\to\infty} U_0\left(c_{i_j},l\right) = \overline{U_l^{AP^*}}$, the second follows from the continuity of $U_0(.,l)$ and the third obtains directly from $c \in \Gamma_l^{AP}(V)$ and the definition of $U^{AP^*}(V,l)$. Therefore, $U^{AP^*}(.,l)$ is usc on $V^{AP}(l)$.

Regarding the boundedness of $U^{AP^*}(.,l)$, note that for any $V \in V^{AP}(l)$, $U^{AP^*}(V,l) = U_0(c_V,l)$ for some $c_V \in \Gamma_l^{AP}(V) \subset C_l$ with C_l non-empty and compact. Since $U_0(.,l) : C_l \to \mathbb{R}$ is continuous on a compact set, it is also bounded. Consequently, $U^{AP^*}(.,l)$ is bounded on $V^{AP}(l)$. ■

Lemma 3 *Fix arbitrary $l \in L$ and $V \in V^{AP}(l)$, and let $c \in G_l^{AP}(V)$. Then, $U_\tau\left(c,.,\widetilde{l}^{-1}\right) = U^{AP^*}\left(V_\tau\left(c^*,.,\widetilde{l}^{-1}\right),\widetilde{l}^{-1}\right)$, $\forall nai(l)$.*

Proof. Note that $\forall nai(l)$, $V_\tau\left(c,.,\widetilde{l}^{-1}\right) \in V^{AP}\left(\widetilde{l}^{-1}\right)$ and, therefore, $U^{AP^*}\left(V_\tau\left(c,.,\widetilde{l}^{-1}\right),\widetilde{l}^{-1}\right)$ is well defined. Since for $\tau = 0$, the result is trivial, take arbitrary $\tau \in \mathbb{Z}_{++}$ and $y^{\tau-1} = \left(y^{\tau-\theta-1},\widetilde{l}^{-1}\right) \in l \times Y^\tau$, and assume that the lemma does not hold. Then, there exists a supercontract $c' \in \Gamma^{AP}_{\widetilde{l}^{-1}}\left(V_\tau\left(c,y^{\tau-1}\right)\right) : U_0\left(c',\widetilde{l}^{-1}\right) > U_\tau\left(c,y^{\tau-1}\right)$. Let us construct a supercontract c'' after l s.t. $\left(a''_t\left(y^{t-1}\right),w''_t\left(y^{t-1},y_t\right)\right) = (a'_{t-\tau}(\widetilde{l}^{-1},y_\tau,...,y_{t-1}), w'_{t-\tau}(\widetilde{l}^{-1},y_\tau,...,y_{t-1},y_t))$, $\forall \text{nai}(y^{\tau-1})$, with $\left(a''_t\left(y^{t-1}\right),w''_t\left(y^{t-1},y_t\right)\right) = \left(a_t\left(y^{t-1}\right),w_t\left(y^{t-1},y_t\right)\right)$ elsewhere. By the definition of c and the construction of c'' we have that c'' satisfies (1), (2), (4), (6) after l and $V_0\left(c'',l\right) = V_0\left(c,l\right) = V$. Then, $U_0\left(c'',l\right) \in U^{AP}\left(V,l\right)$. However, since $U_\tau\left(c'',y^{\tau-1}\right) > U_\tau\left(c,y^{\tau-1}\right)$, we have that $U_0\left(c'',l\right) > U_0\left(c,l\right)$, which contradicts the fact that $U_0\left(c,l\right) = U^{AP^*}\left(V,l\right)$. ■

The lemma says that at any contingency, the expected discounted utility of the principal who has signed the AP supercontract maximizing his/her utility at period 0 while guaranteeing the agent particular initial expected discounted utility also gives the maximum initial utility the principal can get by signing a new AP supercontract guaranteeing the agent an initial utility equal to the utility the agent would receive in that contingency under the previous contract. In other words, at the optimum the principal can neither lose nor gain by breaching the original contract and signing a new one guaranteeing the same utility stream to the agent.

For any $l \in L$ and $V_l \in V^{AP}(l)$, define $\Gamma^{AP}_R\left(V_l,l\right) := \{c_R : (7)-(11) \text{ hold at } (V_l,l)\}$.

Proof of Proposition 6. Take an arbitrary $V = \{V_l\} \in V^{AP}$. Fix $l \in L$. Given the existence of $U^{AP^*}\left(V_l,l\right)$, $\exists\ c \in \Gamma_l : V_0\left(c,l\right) = V_l$ and $U_0\left(c,l\right) = U^{AP^*}\left(V_l,l\right)$. For any $y \in Y$, let $a_- := a_0\left(l\right)$, $w_+\left(y\right) := w_0\left(l,y\right)$, and $V_+\left(y\right) := V_1\left(c,l,y\right)$. Then, we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). As in the proof of Proposition 2 (a), for any $y \in Y$, we can construct $c'_y \in \Gamma_{l_+(l,y)} : V_0\left(c'_y,l_+\left(l,y\right)\right) = V_1\left(c,l,y\right)$, from where (11) also holds. By Lemma 3, for any $y \in Y$, we have $U_1\left(c,l,y\right) = U^{AP^*}\left(V_1\left(c,l,y\right),l_+\left(l,y\right)\right) = U^{AP^*}\left(V_+\left(y\right),l_+\left(l,y\right)\right)$. Consequently, $U^{AP^*}\left(V_l,l\right) = U_0\left(c,l\right) =$

$\sum_{y \in Y} [u(w_0(l,y), y) + \beta_P U_1(c,l,y)]\pi(y, a_0(l)) = \sum_{y \in Y} [u(w_+(y), y) + \beta_P U^{AP^*}(V_+(y), l_+(l,y))]\pi(y, a_-)$, where U^{AP^*} is usc and bounded from Proposition 5. Then, by the definition of $T(.)$, we have that $T_l\left(U^{AP^*}\right)_{(V_l)} \geq U^{AP^*}(V_l, l)$. Since V and l were chosen randomly, the result generalizes to $T\left(U^{AP^*}\right) \geq U^{AP^*}$.

Fix arbitrary $V = \{V_l\} \in V^{AP}$ and $l \in L$. We have demonstrated above that $\Gamma_R^{AP}(V_l, l) \neq \emptyset$. Then, since $\Gamma_R^{AP}(V_l, l)$ can be shown to be compact and U^{AP^*} is usc and bounded, there exists $c_R^* \in \Gamma_R^{AP}(V_l, l) : T_l\left(U^{AP^*}\right)_{(V_l)} = \sum_{y \in Y} [u\left(w_+^*(y), y\right) + \beta_P U^{AP^*}\left(V_+^*(y), l_+(l,y)\right)]\pi\left(y, a_-^*\right)\}$. By (11), for any $y \in Y$, $V_+^*(y) \in V^{AP}(l_+(l,y))$, from where there exists $c_y^* \in \Gamma_{l_+(l,y)}^{AP}\left(V_+^*(y)\right)$: $U_0\left(c_y^*, l_+(l,y)\right) = U^{AP^*}\left(V_+^*(y), l_+(l,y)\right)$. Then, let c^{**} be a supercontract s.t.: $(a_0^{**}(l), w_0^{**}(l,y)) = \left(a_-^*, w_+^*(y)\right)$ and $\forall \text{nai}(l,y)$, $(a_t^{**}(l,y,.), w_t^{**}(l,y,.)) = (a_{y,t-1}^*(l_+(l,y),.), w_{y,t-1}^*(l_+(l,y),.))$, $\forall y \in Y$. It is immediate that c^{**} satisfies (1), (2), (4), (6) after (l,y), $\forall y \in Y$. Moreover, (7)$\Rightarrow$ $a_0^{**}(l) \in A$, (8)$\Rightarrow$ $w_0^{**}(l,y) \in W$, $\forall y \in Y$. By construction and (10), we have that (6) holds at l. By (9), we obtain that $V_0(c^{**}, l) = V_l \in V^{AP}(l)$, from where (4) is satisfied at l. Finally, we have that $T_l\left(U^{AP^*}\right)_{(V_l)} = U_0(c^{**}, l) \in U^{AP}(V_l, l)$, from where $U^{AP^*}(V_l, l) \geq T_l\left(U^{AP^*}\right)_{(V_l)}$. As before, this immediately generalizes to $T\left(U^{AP^*}\right) \geq U^{AP^*}$. ■

Proof of Proposition 7. (a) Analogously to the proof of Lemma 2, we can show that for any $l \in L$, $\Gamma_R^{AP}(.,l)$ is uhc on $V^{AP}(l)$. Then, following an argument similar to the proof of Proposition 5, we conclude that $T(U)_{(.)}$ is usc on V^{AP}. It is trivial to show that $T(U)_{(.)}$ is also bounded.

(b) The result follows by the argument of Theorem 3.3 in Stokey and Lucas (1989) since it is trivial that T satisfies the Blackwell's sufficient conditions.

(c) Assume on the contrary that $\mu\left(\widetilde{U}, U^{AP^*}\right) > 0$. We have that $\mu\left(\widetilde{U}, U^{AP^*}\right) = \mu\left(T\left(\widetilde{U}\right), T\left(U^{AP^*}\right)\right) \leq \beta_P \mu\left(\widetilde{U}, U^{AP^*}\right)$, where the equality follows from the fact that both $\widetilde{U}$ and U^{AP^*} are fixed points of T (the first - by assumption, the second - by Proposition 6) and the inequality obtains by (b). However, this contradicts $\beta_P \in (0,1)$. Consequently, $\mu\left(\widetilde{U}, U^{AP^*}\right) = 0$.

(d) Since by (a) T maps $(\{USCB_l\}, \mu)$ into itself, the existence of $T^n(U)$ is guaranteed for any $n \in \mathbb{Z}_+$. Using Proposition 6 and successively applying (b), we obtain $\mu\left(T^n(U), U^{AP^*}\right) \leq \beta_P^n \mu\left(U, U^{AP^*}\right)$. Note that $\mu\left(U, U^{AP^*}\right) < \infty$ since U is bounded by assumption and U^{AP^*} is bounded by Proposition 5. Therefore, given $\beta_P \in (0,1)$, the result follows. ■

Proof of Proposition 8. Take arbitrary $U \in \{USCBA_l\}$, $l \in L$, $V_\infty \in V^{AP}(l)$ and $\{V_i\}_{i=1}^{\infty}$ s.t. $V_i \in V^{AP}(l)$, for any $i \in \mathbb{Z}_{++}$ and $V_i \underset{i\to\infty}{\to} V_\infty$. If $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} = -\infty$, the result is trivial. If $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} > -\infty$, we can always extract a subsequence $\{V_{i_k}\}_{k=1}^{\infty}$ of $\{V_i\}_{i=1}^{\infty}$ s.t. $\underline{T}_l(U)_{(V_{i_k})} > -\infty$, $\forall k \in \mathbb{Z}_{++}$ and $\lim_{k\to\infty} \underline{T}_l(U)_{(V_{i_k})} = \overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)}$. Since $\Lambda_R(V_{i_k}, U, l) \neq \emptyset$, $\forall k \in \mathbb{Z}_{++}$, we can apply the argument used in the proof of Proposition 5 to obtain $\overline{\lim_{i\to\infty}} \underline{T}_l(U)_{(V_i)} \leq \underline{T}_l(U)_{(V_\infty)}$. ■

Proof of Proposition 9. (a) Notice that $U^{AP^*} \in \{USCB_l\} \subset \{USCBA_l\}$. Then, directly from the definition of T and $\underline{T}$, we have $\underline{T}\left(U^{AP^*}\right) \leq T\left(U^{AP^*}\right) = U^{AP^*}$, where the equality follows from Proposition 6. Since $\underline{T}_l$ is monotonic for any $l \in L$, $\{D_i\}_{i\in Z_+}$ is a weakly decreasing sequence of bounded from above usc functions, therefore $\exists D_\infty \in \{USCBA_l\} : D_i(V_l, l) \underset{i\to\infty}{\to} D_\infty(V_l, l)$, $\forall V_l \in V^{AP}(l)$, $\forall l \in L$.

(b) First we are going to prove $\underline{T}(D_\infty) \geq D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)$. Let us assume that $D_\infty(V_l, l) > -\infty$ because otherwise the result is trivial. Since $D_\infty(V_l, l)$ is a limit of a weakly decreasing sequence, we have that $D_i(V_l, l) > -\infty$, $\forall i \in \mathbb{Z}_+$. Consequently, $D_i(V_l, l) \geq \underline{U}_l$, $\forall i \in \mathbb{Z}_+$ since $D_i(V_l, l) < \underline{U}_l \Rightarrow \Lambda_R(V_l, D_i, l) = \emptyset \Rightarrow D_{i+1}(V_l, l) = -\infty$. This immediately implies that $D_\infty(V_l, l) \geq \underline{U}_l$. Moreover, $\Gamma_R(V_l, D_{i-1}, l) \neq \emptyset$, $\forall i \in \mathbb{Z}_{++}$ since if $\Gamma_R(V_l, D_{i-1}, l) = \emptyset$, we would have $D_i(V_l, l) = -\infty$. Then, for any $i \in \mathbb{Z}_{++}$, since D_{i-1} is usc and $\Gamma_R(V_l, D_{i-1}, l)$ is compact (trivial given D_{i-1} is usc), $\exists c_{R,i} \in \Gamma_R(V_l, D_{i-1}, l)$ such that $D_i(V_l, l) = \sum_{y\in Y} [u(w_{+,i}(y), y) + \beta_P D_{i-1}(V_{+,i}(y), l_+(l, y))]\pi(y, a_{-,i}) \geq \underline{U}_l$. Since $\forall i \in \mathbb{Z}_{++}$, $\Gamma_R(V_l, D_{i-1}, l) \subset$

$\Gamma_R^{AP}(V_l,l)$ and $\Gamma_R^{AP}(V_l,l)$ is compact, $\exists$ a convergent subsequence of $\{c_{R,i}\}_{i=1}^{\infty}$, $\{c_{R,i_k}\}_{k=1}^{\infty}$, s.t. $c_{R,\infty} := \lim_{k\to\infty} c_{R,i_k} \in \Gamma_R^{AP}(V_l,l)$. Fix an arbitrary $y \in Y$. Then, we have:

$$\begin{gathered}D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right) = \\ \lim_{j\to\infty} D_{i_j-1}\left(V_{+,\infty}(y), l_+(l,y)\right) \geq \\ \lim_{j\to\infty}\overline{\lim_{k\to\infty}} D_{i_j-1}\left(V_{+,i_k}(y), l_+(l,y)\right) \geq \\ \lim_{j\to\infty}\overline{\lim_{k\to\infty}} D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) = \\ \overline{\lim_{k\to\infty}} D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right),\end{gathered}$$

where the first equality follows from $\left\{D_{i_j-1}\right\}_{j=1}^{\infty}$ being a subsequence of a sequence converging to D_∞ by (a), the first inequality results from the upper semicontinuity of D_{i_j-1}, the second inequality derives from the fact that $\{D_i\}_{i=0}^{\infty}$ is weakly decreasing, hence $D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) \leq D_{i_j-1}\left(V_{+,i_k}(y), l_+(l,y)\right)$, $\forall k \geq j$, and the last equality is trivial. Notice that $D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, $\forall k \in \mathbb{Z}_{++}$ since by construction $c_{R,i_k} \in \Gamma_R(V_l, D_{i_k-1}, l) \neq \emptyset$. Then, $D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, from where $c_{R,\infty}(V_l) \in \Gamma_R(V_l, D_\infty, l)$. Finally,

$$\begin{gathered}\underline{T}_l\left(D_\infty\right)_{(V_l)} = \\ \max_{\substack{c_R\in \\ \Gamma_R(V_l,D_\infty,l)}} \sum_{y\in Y}[u\left(w_+(y), y\right) + \beta_P D_\infty\left(V_+(y), l_+(l,y)\right)]\pi\left(y, a_-\right) \geq \\ \sum_{y\in Y}[u\left(w_{+,\infty}(y), y\right) + \beta_P D_\infty\left(V_{+,\infty}(y), l_+(l,y)\right)]\pi\left(y, a_{-,\infty}\right) \geq \\ \overline{\lim_{k\to\infty}}\sum_{y\in Y}[u\left(w_{+,i_k}(y), y\right) + \beta_P D_{i_k-1}\left(V_{+,i_k}(y), l_+(l,y)\right)]\pi\left(y, a_{-,i_k}\right) = \\ \overline{\lim_{k\to\infty}} D_{i_k}\left(V_l, l\right) =\end{gathered}$$

$$D_\infty(V_l, l),$$

where the first equality follows from the fact that $D_\infty(V_l, l) \geq \underline{U}_l$, D_∞ is usc, $\Gamma_R(V_l, D_\infty, l)$ is non-empty and compact, the first inequality - from $c_{R,\infty}(V_l) \in \Gamma_R(V_l, D_\infty, l)$, the second inequality - by using the result obtained earlier by developing for $D_\infty(V_{+,\infty}(y), l_+(l, y))$, the following equality - by construction, and the last equality - by construction and (a).

To conclude the proof, we need to show that $\underline{T}(D_\infty) \leq D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)$. If $\underline{T}_l(D_\infty)_{(V_l)} = -\infty$, the result is trivial, so assume $\underline{T}_l(D_\infty)_{(V_l)} > -\infty \Rightarrow \Lambda_R(V_l, D_\infty, l) \neq \emptyset$. From (a), we have that for any $i \in \mathbb{Z}_+$, $D_\infty \leq D_i$, from where $\Lambda_R(V_l, D_\infty, l) \subset \Lambda_R(V_l, D_i, l)$, $\forall i \in \mathbb{Z}_+$. Then, for any $i \in \mathbb{Z}_+$:

$$\underline{T}_l(D_\infty)_{(V_l)} =$$

$$\max_{\substack{c_R \in \\ \Lambda_R(V_l, D_\infty, l)}} \sum_{y \in Y} [u(w_+(y), y) + \beta_P D_\infty(V_+(y), l_+(l, y))] \pi(y, a_-) \leq$$

$$\max_{\substack{c_R \in \\ \Lambda_R(V_l, D_i, l)}} \sum_{y \in Y} [u(w_+(y), y) + \beta_P D_i(V_+(y), l_+(l, y))] \pi(y, a_-) =$$

$$D_{i+1}(V_l, l)$$

Consequently, $\underline{T}_l(D_\infty)_{(V_l)} \leq \lim_{i \to \infty} D_{i+1}(V_l, l) = D_\infty(V_l, l)$.

(c) Let $D' \in \{USCBA_l\} : \underline{T}(D') = D'$. Note that $\exists \overline{D} \in \{USCB_l\} : \overline{D} \geq D'$. Consequently, $T(\overline{D}) \geq T(D') \geq D'$, where the first inequality follows from the monotonicity of T, while the second comes from $T \geq \underline{T}$ and the fact that $\underline{T}(D') = D'$. Repeating the argument, we obtain $T^n(\overline{D}) \geq D'$ for any $n \in \mathbb{Z}_+$. Then, by Proposition 7 (d) we have that $U^{AP^*} = \lim_{n \to \infty} T_U^n(\overline{D}) \geq D'$, where the convergence is in terms of μ. Fix $l \in L$ and $V_l \in V^{AP}(l)$. By the monotonicity of $\underline{T}$, we have $D_i = \underline{T}^i(U^{AP^*}) \geq \underline{T}^i(D') = D'$, $\forall i \in \mathbb{Z}_+$. Therefore, $D_\infty \geq D'$. ■

Lemma 4 $\underline{T}\left(\widehat{U}^*\right) \geq \widehat{U}^*$.

Proof. [Adapted from the first part of the proof of Proposition 6] Fix an arbitrary $l \in L$. If $V_l \in V^{AP}(l) \backslash V^{2P}(l)$, $\widehat{U}^*(V_l, l) = -\infty$ and the result is trivial. Therefore, take $V_l \in V^{2P}(l)$. Then, $\widehat{U}^*(V_l, l) = U^*(V_l, l)$. Given the existence of $U^*(V_l, l)$, we have that $\exists c \in \Gamma_l : V_0(c, l) = V_l$ and $U_0(c, l) = U^*(V_l, l)$. For any $y \in Y$, let $a_- := a_0(l)$, $w_+(y) := w_0(l, y)$, $V_+(y) := V_1(c, (l, y))$. Then we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). For any $y \in Y$, we can construct $c' \in \Gamma_{l_+(l,y)}$: $V_0(c', l_+(l, y)) = V_1(c, l, y)$, from where we have that $V_+(y) \in V^{2P}(l_+(l, y)) \subset V^{AP}(l_+(l, y))$, i.e., (11) holds. From (5), we have $U^*(V_l, l) \geq \underline{U}_l$. Furthermore, by slightly modifying the argument of Lemma 3, we have that for any $y \in Y$, $U^*(V_+(y), l_+(l, y)) = U^*(V_1(c, l, y), l_+(l, y)) = U_1(c, l, y)$. Then, from (5) we obtain that $U^*(V_+(y), l_+(l, y)) \geq \underline{U}_{l+(l,y)}$, $\forall y \in Y$. Finally, $\widehat{U}^*$ is usc (by the argument used in the proof of Proposition 5 given the qualifications stated in the proof of Proposition 8) and bounded from above. Then, by the definition of $\underline{T}$, we have that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \geq \widehat{U}^*(V_l, l)$. ■

Lemma 5 $\underline{T}\left(\widehat{U}^*\right) \leq \widehat{U}^*$.

Proof. [Adapted from the second part of the proof of Proposition 6] Take $l \in L$ and $V_l \in V^{AP}(l)$. If $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = -\infty$, the result is trivial; therefore, assume that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} > -\infty$. This implies that $\widehat{U}^*(V_l, l) \geq \underline{U}_l$, from where we immediately have that $V_l \in V^{2P}(l)$ and $\widehat{U}^*(V_l, l) = U^*(V_l, l)$. Since we would trivially obtain $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \leq \widehat{U}^*(V_l, l)$ if $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} \leq \underline{U}_l$, assume $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} > \underline{U}_l$. Also note that $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = \max_{\substack{c_R \in \\ \Gamma_R(V_l, \widehat{U}^*, l)}} \sum_{y \in Y} [u(w_+(y), y) + \beta_P \widehat{U}^*(V_+(y), l_+(l, y))]\pi(y, a_-)$ with $\Gamma_R\left(V_l, \widehat{U}^*, l\right) \neq \emptyset$. Given that $\widehat{U}^*$ is usc and $\Gamma_R\left(V_l, \widehat{U}^*, l\right)$ is compact, there exists a contract c_R^* such that (7) – (11)

hold at (V_l, l), $\widehat{U}^*\left(V_+^*(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, $\forall y \in Y$ and $\underline{T}_l\left(\widehat{U}^*\right)_{(V_l)} = \sum_{y\in Y}[u\left(w_+^*(y), y\right) + \beta_P \widehat{U}^*\left(V_+^*(y), l_+(l,y)\right)]\pi\left(y, a_-^*\right)$. By (11), $V_+^*(y) \in V^{AP}(l_+(l,y))$, which together with $\widehat{U}^*\left(V_+^*(y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$ implies $V_+^*(y) \in V^{2P}(l_+(l,y))$. Since $\widehat{U}^*\left(V_+^*(y), l_+(l,y)\right) = U^*\left(V_+^*(y), l_+(l,y)\right)$, $\exists c_y^* \in \Gamma_{l_+(l,y)}$: $V_0\left(c_y^*, l_+(l,y)\right) = V_+^*(y)$ and $U_0\left(c_y^*, l_+(l,y)\right) = \widehat{U}^*\left(V_+^*(y), l_+(l,y)\right)$. Note that this is true for any $y \in Y$. Then, let c^{**} be defined as follows: $(a_0^{**}(l), w_0^{**}(l,y)) = \left(a_-^*, w_+^*(y)\right)$ and $(a_t^{**}(l,y,.), w_t^{**}(l,y,.)) = (a_{y,t-1}^*(l_+(l,y),.)), w_{y,t-1}^*(l_+(l,y),.))$, $\forall$nai(l,y), $\forall y \in Y$. It is immediate that $c^{**} \in \Gamma_{l,y}$, $\forall y \in Y$. Moreover, (7) $\Rightarrow a_0^{**}(l) \in A$, (8) $\Rightarrow w_0^{**}(l,y) \in W$, $\forall y \in Y$. By construction and (10), we have that (6) holds at l. By (9), we obtain that $V_0(c^{**}, l) = V_l \in V^{2P}(l)$, from where (4) is satisfied at l. Furthermore, we have that $U_0(c^{**}, l) = T_l\left(\widehat{U}^*\right)_{(V_l)} > \underline{U}_l$. Therefore, $T_l\left(\widehat{U}^*\right)_{(V_l)} \in U^{2P}(V_l, l)$, from where $\widehat{U}^*(V_l, l) = U^*(V_l, l) \geq T_l\left(\widehat{U}^*\right)_{(V_l)}$. ■

Proof of Proposition 10. From Lemmas 4 and 5. ■

Proof of Proposition 11. Since $\widehat{U}^* \in \{USCBA_l\}$, by Propositions 9 (c) and 10 we obtain $\widehat{U}^* \leq D_\infty$. What remains to be shown is that $\widehat{U}^* \geq D_\infty$. Fix $l \in L$ and $V_l \in V^{AP}(l)\}$. If $D_\infty(V_l, l) = -\infty$, the result is trivial; therefore, assume $D_\infty(V_l, l) > -\infty$. Then, $D_\infty(V_l, l) = \underline{T}_l(D_\infty)_{(V_l)} = \max_{\substack{c_{R,l}(V_l)\in \\ \Gamma_R(V_l, D_\infty, l)}} \sum_{y\in Y}[u(w_+(V_l,y), y) + \beta_P D_\infty(V_+(V_l,y), l_+(l,y))]\pi(y, a_-(V_l))$ with $\Gamma_R(V_l, D_\infty, l)$ nonempty and $D_\infty(V_l, l) \geq \underline{U}_l$ since otherwise we would have $D_\infty(V_l, l) = -\infty$. Since D_∞ is usc and $\Gamma_R(V_l, D_\infty, l)$ is compact, we have that $\exists c_R^*(V_l, l) \in \Gamma_R(V_l, D_\infty, l)$ such.that. $\underline{T}_l(D_\infty)_{(V_l)} = \sum_{y\in Y}[u\left(w_+^*(V_l,y), y\right) + \beta_P D_\infty\left(V_+^*(V_l,y), l_+(l,y)\right)]\pi\left(y, a_-^*(V_l)\right)$. Since $D_\infty\left(V_+^*(V_l,y), l_+(l,y)\right) \geq \underline{U}_{l_+(l,y)}$, $\forall y \in Y$, we have:

$$D_\infty\left(V_+^*(V_l,y), l_+(l,y)\right) = \underline{T}_{l_+(l,y)}(D_\infty)_{V_+^*(V_l,y)} =$$

$$\max_{\substack{c_R\left(V_+^*(V_l,y), l_+(l,y)\right)\in \\ \Gamma_R\left(V_+^*(V_l,y), D_\infty, l_+(l,y)\right)}} \sum_{y'\in Y}[u\left(w_+\left(V_+^*(V_l,y), y'\right), y'\right) +$$

$$\beta_P D_\infty \left(V_+ \left(V_+^* \left(V_l, y\right), y'\right), l_+ \left(l_+ \left(l, y\right), y'\right)\right)]\pi \left(y', a_- \left(V_+^* \left(V_l, y\right)\right)\right)$$

with $\Gamma_R \left(V_+^* \left(V_l, y\right), D_\infty, l_+ \left(l, y\right)\right)$ nonempty, so the previous analysis applies. Proceeding in this way, we can construct a supercontract c such that $\forall$nai(l), $a_t \left(y^{t-1}\right) := a_-^* \left(V_+^{*\,t} \left(V_l, y^{t-1}\right), y^{t-1}\right)$, $w_t \left(y^{t-1}, y_t\right) := w_+^* \left(V_+^{*\,t} \left(V_l, y^{t-1}\right), y^{t-1}, y_t\right)$, where $V_+^{*\,t} \left(V_l, l, y_0, ..., y_{t-1}\right) := V_{+\,\langle y_{t-1}\rangle}^* \circ ... \circ V_{+\,\langle y_0\rangle}^* \left(V_l, l\right)$ for any $t \in \mathbb{Z}_{++}$ and $V_+^{*\,0} \left(V_l, l\right) := V_l$ with $V_{+\,\langle y_\tau\rangle}^* \left(V, y^{\tau-1}\right) := V_+^* \left(V, y^{\tau-1}, y_\tau\right)$, $\forall y_\tau \in Y$, $V \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, $y^{\tau-1} \in l \times Y^\tau$, $\tau \in \mathbb{Z}_+$. We immediately have (7) $\Rightarrow$ (1) and (8) $\Rightarrow$ (2). Moreover, by construction and successively applying (9), we obtain that $\forall$nai(l):

$$V_t \left(c, y^{t-1}\right) - V_+^{*\,t} \left(V_l, y^{t-1}\right) =$$

$$\lim_{T\to\infty} \beta_A{}^T \sum_{y_{t+T-1}\in Y} ... \sum_{y_t \in Y} [V_{t+T} \left(c, y^{t+T-1}\right) -$$

$$V_+^{*\,t+T} \left(V_l, y^{t+T-1}\right)] \prod_{i=t}^{t+T-1} \pi \left(y_i, a_i \left(y^{i-1}\right)\right)$$

Consequently, $\forall$nai(l), $V_t \left(c, y^{t-1}\right) = V_+^{*\,t} \left(V_l, y^{t-1}\right)$ since by (11) $V_+^{*\,t} \left(V_l, y^{t-1}\right) \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$ and is therefore bounded, while $V_t \left(c, y^{t-1}\right)$ is bounded given (7) and (8). In particular, $V_0 \left(c, y^{\tau-1}\right) = V_l$. Then (10) $\Rightarrow$ (6). Furthermore, $V_t \left(c, y^{t-1}\right) \in V^{AP} \left(l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, implies that (4) holds after l. Since $U_\tau \left(c, y^{t-1}\right)$ is bounded given (7) and (8) and $D_\infty \left(V_+^{*\,t} \left(V_l, y^{t-1}\right), l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$ is bounded from above by Proposition 9 (a).and from below by $\min_{l\in L} \underline{U}_l$ (well defined given L finite), we also have that $U_t \left(c, y^{t-1}\right) = D_\infty \left(V_+^{*\,t} \left(V_l, y^{t-1}\right), l \left(y_{\tau-\theta}, ..., y_{\tau-1}\right)\right)$, $\forall$nai(l). In particular, $U_0 \left(c, l\right) = D_\infty \left(V_l, l\right)$. Then, (5) is satisfied at any node. Therefore, $V_l \in V^{2P} \left(l\right)$ and $D_\infty \left(V_l, l\right) \in U^{2P} \left(V_l, l\right)$. Then, $\widehat{U}^* \left(V_l, l\right) = U^* \left(V_l, l\right) \geq D_\infty \left(V_l, l\right)$. ■

Lemma 6 $V^{AP} \subset B\left(V^{AP}\right)$.

Proof. Let $V \in V^{AP}$ and fix an arbitrary $l \in L$. From $V_l \in V^{AP}(l)$, $\exists c \in \Gamma_l^{AP}(V_l)$. By construction, $V_l \in \left[\underline{V}_l, \widehat{V}\right]$. For any $y \in Y$, let $a_- := a_0(l)$, $w_+(y) := w_0(l,y)$, and $V_+(y) := V_1(c,l,y)$. Given these choices, we immediately have that (9) holds. Moreover, (1) $\Rightarrow$ (7), (2) $\Rightarrow$ (8), (6) $\Rightarrow$ (10). Note that for any $y \in Y$, $V^{AP}(l_+(l,y)) \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right) = V^{AP}(l_+(l,y))$. Since for any $y \in Y$ we can construct a supercontract $c'_y \in \Gamma^{AP}_{l_+(l,y)}(V_1(c,(l,y))$, we have that (12) is satisfied. Therefore, $V_l \in B_l\left(V^{AP}\right)$. Since $l \in L$ was chosen randomly, this generalizes to $V \in B\left(V^{AP}\right)$. ■

The lemma establishes that V^{AP} is self-generating in the terminology of Abreu, Pearce and Stacchetti (1990).

Lemma 7 *Assume* $X = \{X_l\} : \emptyset \neq X_l \subset B_l(X)$, $\forall l \in L$. *Then,* $B(X) \subset V^{AP}$.

Proof. Let the condition of the lemma hold and take $V \in B(X)$. Fix an arbitrary $l \in L$. Since $V_l \in B_l(X)$, $\exists c_{R,l}(V_l)$: (7)-(10) and (12) hold at l. By (12) and $X_{l_+(l,y)} \subset B_{l_+(l,y)}(X)$, we obtain that $V_{+,l}(V_l, y) \in B_{l_+(l,y)}(X)$. Then, $\forall y \in Y$, $\exists c_R$: (7)-(10) and (12) hold at $(V_{+,l}(V_l,y), l_+(l,y))$. Proceeding this way, as in the proof of Proposition 11, we can consecutively construct a supercontract c after l s.t. $c \in \Gamma_l^{AP}(V_l)$. Here, it deserves noting that while (12) implies (4) on every node but the first, $V_l \in B_l(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, from where (4) is also satisfied at l. Therefore, $V_l \in V^{AP}(l)$, which generalizes to $V \in V^{AP}$. ■

The lemma says that the image of every nonempty, self-generating set is a subset of V^{AP}.

Proof of Proposition 12. (a) By Assumption *3* and Lemma 6, V^{AP} satisfies the condition of Lemma 7. Therefore, $B\left(V^{AP}\right) \subset V^{AP}$, which together with Lemma 6 implies the result.

(b) It follows by Lemma 7. ■

Lemma 8 *Assume* $X' = \{X'_l\}$ *and* $X'' = \{X''_l\} : X'_l \subset X''_l \subset \mathbb{R}$, $\forall l \in L$. *Then,* $B_l(X') \neq \emptyset \Rightarrow B_l(X') \subset B_l(X'')$, $\forall l \in L$.

Proof. Trivial. ■

Lemma 9 *Assume $X = \{X_l\} : X_l \subset \mathbb{R}$ compact, $\forall l \in L$. Then, $B_l(X) \neq \emptyset \Rightarrow B_l(X)$ compact, $\forall l \in L$.*

Proof. Let the condition of the lemma hold and assume $B_l(X) \neq \emptyset$ for some $l \in L$. Note that $B_l(X) \subset \left[\underline{V}_l, \widehat{V}\right] \subset \mathbb{R}$ is bounded by definition. We should also show that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i=1}^{\infty} : V_i \in B_l(X)$, $\forall i \in \mathbb{Z}_{++}$ with $V_i \underset{i\to\infty}{\rightarrow} V_\infty$. We need to prove that $V_\infty \in B_l(X)$. By construction, we have that for any $i \in \mathbb{Z}_{++}$, $V_i \in \left[\underline{V}_l, \widehat{V}\right]$ and $\exists c_{R,i}$:(7)-(10), (12) hold at (V_i, l). By $V_i \in \left[\underline{V}_l, \widehat{V}\right]$, $\forall i \in \mathbb{Z}_{++}$, we obtain $V_\infty \in \left[\underline{V}_l, \widehat{V}\right]$. By (7), (8), (12), Assumption 2, L finite, and $X_l \subset \mathbb{R}$ compact for any $l \in L$, we have that $\{c_{R,i}\}_{i=1}^{\infty}$ is uniformly bounded, therefore $\exists$ a subsequence $\{c_{R,i_k}\}_{k=1}^{\infty}$ of $\{c_{R,i}\}_{i=1}^{\infty} : c_{R,i_k} \underset{k\to\infty}{\rightarrow} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies (7)-(10), (12) at (V_∞, l). ■

Proof of Proposition 13. For any $l \in L$ and $i \in \mathbb{Z}_+$, denote by $X_{i,l}$ the element of X_i corresponding to initial history l. By the condition of the Proposition and Assumption *3*, we have that $\emptyset \neq V^{AP}(l) \subset X_{0,l} \subset \mathbb{R}$, $\forall l \in L$. Since by Proposition 12 (a) $B_l\left(V^{AP}\right) = V^{AP}(l)$, we can apply Lemma 8 to obtain $\emptyset \neq V^{AP}(l) \subset X_{1,l} \subset \mathbb{R}$, $\forall l \in L$. Using $X_1 \subset X_0$ and repeating the argument, we reach $V^{AP} \subset X_{i+1} \subset X_i$, $\forall i \in \mathbb{Z}_+$ Then, $\{X_i\}_{i=0}^{\infty}$ is a sequence of non-empty, compact (by Lemma 9 since X_0 compact), monotonically decreasing (nested) sets; therefore it converges to $X_\infty = \bigcap_{i=0}^{\infty} X_i \supset V^{AP}$ with X_∞ compact.

What remains to be shown is that $X_\infty \subset V^{AP}$. By Lemma 7, it is enough to show that $X_\infty \subset B(X_\infty)$. Let $V \in X_\infty$. This implies that $V \in X_i$, $\forall i \in \mathbb{Z}_+$. Fix an arbitrary $l \in L$. We have that $\exists c_{R,i}$: (7)-(10), (12) hold at (V_l, l). By (7), (8), (12), Assumption 2, L finite, and $X_i \subset X_0 \subset \mathbb{R}^{n^\theta}$ compact, $\forall i \in \mathbb{Z}_+$, we have that $\{c_{R,i}\}_{i\in\mathbb{Z}_+}$ is uniformly bounded; therefore, $\exists$ a subsequence $\{c_{R,i_k}\}_{k=1}^{\infty}$ of $\{c_{R,i}\}_{i=1}^{\infty} : c_{R,i_k} \underset{k\to\infty}{\rightarrow} c_{R,\infty}$. It is immediate that $c_{R,\infty}$ satisfies (7)-(10) at (V_l, l). Moreover, $V_{+,\infty}(y) \geq \underline{V}_{l_+(l,y)}$, $\forall y \in Y$. We also need to show that for any $y \in Y$, $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$. Fix an arbitrary $y \in Y$

and assume, on the contrary, that $V_{+,\infty}(y) \notin X_{\infty,l_+(l,y)}$. Since $X_{\infty,l_+(l,y)} = \bigcap_{i=0}^{\infty} X_{i,l_+(l,y)} = \bigcap_{k=0}^{\infty} X_{i_k,l_+(l,y)}$, we have that $\exists k' \in \mathbb{Z}_+ : V_{+,\infty}(y) \notin X_{i_{k'},l_+(l,y)}$. Furthermore, $\left\{X_{i_{k'}}\right\}_{k=0}^{\infty}$ was shown to be a monotonically decreasing (nested) sequence, from where $V_{+,i_k}(y) \in X_{i_k,l_+(l,y)} \subset X_{i_{k'},l_+(l,y)}$, $\forall k \in \mathbb{Z}_+ : k \geq k'$. Since $X_{i_{k'},l_+(l,y)}$ is closed and $V_{+,i_k}(y) \underset{k\to\infty}{\to} V_{+,\infty}(y)$, we obtain that $V_{+,\infty}(y) \in X_{i_{k'},l_+(l,y)}$, i.e., a contradiction is reached. This proves $V_{+,\infty}(y) \in X_{\infty,l_+(l,y)}$, $\forall y \in Y$. Consequently, (12) holds for $c_{R,\infty}$. Finally, note that $V_l \in \left[\underline{V}_l, \widehat{V}\right]$ follows immediately from $V_l \in X_{1,l}$. Therefore, $V_l \in B_l(X_\infty)$, which generalizes to $V \in B(X_\infty)$. ■

For any $X = \{X_l\} : X_l \in \mathbb{R}$, $\forall l \in L$ let $B'(X) := \{B'_l(X)\}$ with

$$B'_l(X) := \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R : (7) - (10) \text{ and } (13) \text{ hold at } (V_l, l)\}.$$

Note that the only difference between this operator and operator $\widetilde{B}$ defined in Section 3 is that $B'_l(X) \subset \left[\underline{V}_l, \widehat{V}\right]$, while $\widetilde{B}_l(X) \subset X_l$.

Lemma 10 *Take $X'_0 := \left\{X'_{0,l}\right\}$ with $X'_{0,l} := \left[\underline{V}_l, \widehat{V}\right]$, $\forall l \in L$ and let $X'_{i+1} := B'(X'_i)$, $\forall i \in \mathbb{Z}_+$. Then, $X'_{i+1} \subset X'_i$, $\forall i \in \mathbb{Z}_+$ and $X'_\infty := \lim_{i\to\infty} X'_i = V^{AP}$.*

Proof. We have that X'_0 is compact and $V^{AP} \subset X'_0 \subset \mathbb{R}^{n^\theta}$. Note that for any $X \subset \mathbb{R}^{n^\theta} : B_l(X) \neq \emptyset$, we have $B_l(X) \subset B'_l(X)$. Then, by Lemma 8 and Proposition 12 (a), we obtain $V^{AP} \subset B(X'_0) \subset B'(X'_0)$. Using the same arguments plus the monotonicity of B' (trivial), we have $V^{AP} \subset X'_i$, $\forall i \in \mathbb{Z}_+$. Moreover, by construction $B'(X'_0) \subset X'_0$. Then, the condition $B(X'_0) \subset X'_0$ is satisfied. Observe that for any $l \in L$, $X'_{1,l} = \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R \text{ s.t. } (7) - (10), (13) \text{ hold at } (V_l, l)\} = \{V_l \in \left[\underline{V}_l, \widehat{V}\right] : \exists c_R \text{ s.t. } (7) - (10), (12) \text{ hold at } (V_l, l)\} = B_l(X'_0)$ since, by construction, we have that $X'_{0,l_+(l,y)} \cap \left[\underline{V}_{l_+(l,y)}, +\infty\right) = X'_{0,l_+(l,y)}$,

$\forall y \in Y$. Furthermore, by $X_1' \subset X_0'$ and the monotonicity of B', we obtain $X_{i+1}' \subset X_i'$, $\forall i \in \mathbb{Z}_+$. Then, it is trivial that $X_{i+1}' = B(X_i')$, $\forall i \in \mathbb{Z}_+$ Therefore, Proposition 13 applies to $\{X_i'\}_{i=1}^{\infty}$. ■

Lemma 11 *Let* $\{X_i'\}_{i=1}^{\infty}$ *be defined as in Lemma 10. Take* $\widetilde{X}_0 := X_0'$ *and let* $\widetilde{X}_{i+1} := \widetilde{B}\left(\widetilde{X}_i\right)$, $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{X}_i = X_i'$, $\forall i \in \mathbb{Z}_+$.

Proof. Assume $\widetilde{X}_{i-1} = X_{i-1}'$ for some $i \in \mathbb{Z}_{++}$. By Lemma 10, $\emptyset \neq X_i' \subset X_{i-1}'$. Fix $l \in L$ and let $V \in X_{i,l}'$ Then, we have $V \in \widetilde{X}_{i-1,l}$, which together with $V \in B_l'\left(\widetilde{X}_{i-1}\right)$ implies that $V \in \widetilde{B}_l\left(\widetilde{X}_{i-1}\right)$. Since l and V were chosen randomly, this generalizes to $X_i' \subset \widetilde{X}_i$. Then, $\widetilde{X}_i$ is non-empty. Note that $\widetilde{X}_{i-1} = X_{i-1}' \subset X_0'$ by Lemma 10. Consequently, $\emptyset \neq \widetilde{B}_l\left(\widetilde{X}_{i-1}\right) \subset B_l'\left(\widetilde{X}_{i-1}\right)$, i.e., $\widetilde{X}_i \subset X_i'$.

We have that $\widetilde{X}_0 = X_0'$ by definition and have just shown that $\widetilde{X}_{i-1} = X_{i-1}'$ would imply $\widetilde{X}_i = X_i'$; therefore, by induction we obtain that $\widetilde{X}_i = X_i'$ for any $i \in \mathbb{Z}_+$. ■

Proof of Proposition 14. (a) From Lemmas 10 and 11.

(b) Similarly to the proof of Lemma 6, we can show that $V^{AP} \subset \widetilde{B}\left(V^{AP}\right)$. Since $\widetilde{B}\left(V^{AP}\right)$ is nonempty, it can easily be obtained that $\widetilde{B}\left(V^{AP}\right) \subset V^{AP}$.

(c) Since $\emptyset \neq X \subset \widetilde{X}_0$, we can use the monotonicity of $\widetilde{B}$ and $\widetilde{B}(X) = X$ to obtain $X \subset \widetilde{X}_i$, $\forall i \in \mathbb{Z}_+$. Then, by (a), we have $X \subset \widetilde{X}_\infty = V^{AP}$. ■