On the Consistent Measurement of Attainment and Shortfall Inequality

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Inequality of Attainment and Shortfall

- Is the bottle half full also half empty?
- What if there are many bottles, some full, some empty, some half empty, and some one-third full...
- In measuring income inequality, income is assumed to be unbounded, at least no upper bound. In measuring inequality of variables such as health level, however, variables are often bounded.
- For a bounded variable, a distinction can be made between measuring the inequality of attainments and the inequality of shortfalls.

Attainment-Shortfall Consistency

- Should there be some connection between the inequality of attainments and the inequality of shortfalls?
- Clarke et al. (2002) argue that attainments and shortfalls are "different sides of the same coin" and thus should mirror each other. Using data from Australia and Sweden, however, they find that the inequality of attainments and the inequality of shortfalls may not "mirror" each other.
- Erreygers (2009) and Erreygers et al. (2010) reinforce Clarke et al.'s view. They propose a "strongest form" of the mirror property (complementarity): inequality of attainments = inequality of shortfalls. By confining to two linear and quadratic-linear inequality frameworks, Erreygers (2009) characterize the Gini index and the variance as two satisfactory inequality indices.

Two Issues of Measurement

Conceptually,

- The complementarity condition may be too strong, an attainment-shortfall consistency condition is more appropriate: Country A has more attainment inequality than country B if and only if country A has more shortfall inequality than country B.
- The linear and quadratic-linear frameworks may be too narrow: are there new attainment-shortfall consistent inequality indices outside the frameworks?

Attainment and Shortfall

Denote x as attainment and y as shortfall. Denote α the maximum possible level of attainment. Then $y = \alpha - x$. For two countries, A and B, the distributions of attainments are $\mathbf{x}_A = (x_1^A, x_2^A, ..., x_n^A)$ and $\mathbf{x}_B = (x_1^B, x_2^B, ..., x_n^B)$, and the distributions of shortfalls are $\mathbf{y}_A = (y_1^A, y_2^A, ..., y_n^A)$ and $\mathbf{y}_B = (y_1^B, y_2^B, ..., y_n^B)$. $n \geq 2$.

Consistency: Partial Inequality Orderings

Unambiguous comparisons of inequality in attainments, and in shortfalls, between two populations can be made using a partial inequality ordering. The most general form of such an ordering is perhaps the one proposed by Zoli (1999). For Zoli, a vector **z** is unambiguously less unequal than another such vector **w** if and only if

$$\frac{\mathbf{z} - m(\mathbf{z})\mathbf{1}}{[\mu m(\mathbf{z}) + (1-\mu)]^{\lambda}} \text{ GL dominates } \frac{\mathbf{w} - m(\mathbf{w})\mathbf{1}}{[\mu m(\mathbf{w}) + (1-\mu)]^{\lambda}}$$

where $m(\cdot)$ is the mean income, $\lambda \in [0,1]$ and $\mu \in [0,1]$.

Consistency: Partial Inequality Orderings

Theorem 1. If $\mathbf{x}_A$ and $\mathbf{x}_B$ are inequality equivalent for the Zoli partial inequality ordering, then
(a) if $\lambda\mu = 0$, $\mathbf{y}_A$ and $\mathbf{y}_B$ are inequality equivalent;
(b) if $\lambda\mu \neq 0$, $\mathbf{y}_A$ has unambiguously less inequality than $\mathbf{y}_B$ if $m(\mathbf{x}_A) < m(\mathbf{x}_B)$ and unambiguously more inequality than $\mathbf{y}_B$ if $m(\mathbf{x}_A) > m(\mathbf{x}_B)$.

- Inequality equivalence of attainments guarantees inequality equivalence of shortfalls (and vice versa) *only for the absolute inequality concept* within the Zoli framework.
- Neither for the relative inequality concept nor for any intermediate concept does an unambiguous comparison of inequality in attainments guarantee an unambiguous comparison of inequality in shortfalls (or vice versa).

Consistency: Partial Inequality Orderings

In respect of the absolute inequality concept, an unambiguous comparison of attainment inequality guarantees a similarly unambiguous comparison of shortfall inequality (and vice versa):

Theorem 2. For the absolute inequality concept, $\mathbf{x}_A$ is unambiguously less unequal than $\mathbf{x}_B$ if and only if $\mathbf{y}_A$ is unambiguously less unequal than $\mathbf{y}_B$.

Consistency: Summary Inequality Indices

- It follows that no relative inequality index, nor any inequality index which respects one of the intermediate inequality partial orderings, measures attainment inequality and shortfall inequality consistently.
- The situation for indices of absolute inequality is very different. If $\mathbf{x}_A$ (or $\mathbf{y}_A$) is unambiguously less unequal than $\mathbf{x}_B$ (or $\mathbf{y}_B$), then the consistency condition is automatically satisfied for any absolute inequality index $I(\cdot)$, i.e., $I(\mathbf{x}_A) < I(\mathbf{x}_B)$ and $I(\mathbf{y}_A) < I(\mathbf{y}_B)$. But in the absence of dominance, it is possible for an absolute inequality index to indicate $I(\mathbf{x}_A) \leq I(\mathbf{x}_B)$ while $I(\mathbf{y}_A) > I(\mathbf{y}_B)$.

Two Classes of Absolute Inequality Indices

- A rank-independent index of absolute inequality I_{RI} takes the form

$$I_{RI}(\mathbf{x}) = \frac{1}{n}\sum_i u[x_i - m(\mathbf{x})]$$

where u is strictly convex and twice differentiable and $u(0) = 0$;

- A rank-dependent index of absolute inequality I_{RD} takes the form

$$I_{RD}(\mathbf{x}) = \frac{1}{n}\sum_i \omega(p_i)[x_i - m(\mathbf{x})]$$

where $p_i = \frac{2i-1}{2n}$ and $\omega(\cdot)$ is strictly increasing.

Consistency: Summary Inequality Indices

Theorem 4.

(a) $I_{RI}(\cdot)$ is attainment-shortfall consistent if and only if $u(z) = u(-z)$ for all $z \neq 0$, and moreover, $I_{RI}(\mathbf{x}) = I_{RI}(\mathbf{y})$.

(b) $I_{RD}(\cdot)$ is attainment-shortfall consistent if and only if $\omega(1-p) = -\omega(p)$ for all $p \in [0,1]$, and moreover, $I_{RD}(\mathbf{x}) = I_{RD}(\mathbf{y})$.

Consistency: Decomposable Inequality Indices

A generic subgroup decomposable inequality index takes the form

$$I_{GSD}(\mathbf{x}) = \frac{1}{n\lambda[m(\mathbf{x})]} \sum_i \{\phi(x_i) - \phi[m(\mathbf{x})]\}$$

where ϕ and λ are differentiable and ϕ is strictly convex.

Theorem 5. For $n \geq 3$, $I_{GSD}(\cdot)$ is attainment-shortfall consistent for a variable α if and only if $\phi(z) = az^2 + bz + c$ for some constants a, b and c, and $\lambda(m)$ is constant for all m. Hence the variance is the only subgroup decomposable index that can be attainment-shortfall consistent.

Attainment-Shortfall Consistency and Transfer-Sensitivity

Why only the variance? It is known that the variance satisfies only the transfer axiom not transfer sensitivity axiom. Can there be an inequality index outside the decomposable framework that satisfies the transfer sensitivity axiom?

Theorem 6. No inequality index $I_{RI}(\cdot)$ can satisfy both the transfer sensitivity axiom and be attainment-shortfall consistent.

The reason: attainment is a good thing while shortfall is a bad thing. The transfer axiom makes sense for the distributions of both "goods" and "bads" while the transfer sensitivity axiom does not.

Conclusion

- Only the absolute notion of inequality measurement can respect the consistency condition. We have identified, in two general classes of absolute inequality indices, the necessary and sufficient conditions of consistency.
- Demonstrated the cutting power of the consistency condition on a decomposable inequality index: only the variance can be consistent, among decomposable inequality indices of all types, in ranking attainment and shortfall inequality.
- Shown that, if decomposability is not a desideratum, then the inequality index chosen by the analyst should have distribution sensitivity limited to the level of transfer axiom.