Satisfaction Approval Voting

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Overview

Satisfaction approval voting (SAV) is a voting system applicable to *multiwinner* elections (e.g., to a council or legislature). It uses an *approval ballot*, whereby voters can approve of as many candidates as they like (no rankings).

A voter's *satisfaction score* is the fraction of his or her approved candidates who are elected. If *k* candidates are to be elected, SAV elects the set of *k* candidates that maximizes the *sum* of all voters' satisfaction scores. SAV has several desirable features:

• It is independent of the number of candidates a voter approves of—it works equally well for voters who are discriminating and not-so-discriminating in their choices.

• It tends to elect a more "representative" set of candidates than approval voting (AV)—in fact, SAV and AV may elect disjoint subsets—and discourages clones.

• In the 2003 election of the Game Theory Society, SAV would have elected a more representative council.

• It can be applied to party-list systems, wherein it gives parties approximate proportional representation (PR).

• In party-list systems, SAV favors larger parties, giving parties incentives to share support, form alliances, or even merge— perhaps into as few as two broad coalitions—and renders them responsive to voter preferences.

SAV vs. AV

Proposition 1. *SAV and AV can elect disjoint sets of candidates.*

Example: 10 voters, 4 candidates, 2 winners

4 voters: *ab*
3 voters: *c*
3 voters: *d.*

AV outcome: {*a*, *b*} (4 votes each)
SAV outcome: {*c*, *d*}: The satisfaction (*s*) of each pair is

$s(a, b) = s(b) = 4(½) + 4(½) = 4$
$s(a, c) = s(a, d) = s(b, c) = s(b, d) = 4(½) + 3(1) = 5$
$s(c, d) = 3(1) + 3(1) = 6.$

Whereas *c* and *d* give total satisfaction 6, every other pair—because at least one voter receives satisfaction ½—yields less total satisfaction.

Additivity

A *candidate's* satisfaction score—as opposed to a voter's—is the *sum* of the satisfaction scores of voters who approve of him or her. For example, if candidate x receives 3 voters from bullet voters, 2 from voters who approve of two candidates, and 5 from voters who approve of three candidates,

$$s(x) = 3(1) + 2(½) + 5(1/3) = 5\ 2/3.$$

Satisfaction is *additive*, which means that the satisfaction of a subset of candidates is the sum of the satisfaction scores of the candidates. This property makes the satisfaction of any subset easy to compute.

In particular, gaining the support of an additional voter always increases a candidate's score by $1/n$, where n is the number of candidates approved of by that voter. This is a *consequence* of the goal of maximizing total voter satisfaction, not an assumption about how approval votes are to be divided.

Representativeness

For any subset *S* of the candidates, we say that *S* *represents* voter *i* if and only if voter *i* approves of at least one candidate in *S*. How representative is the set of candidates who win under SAV or AV—that is, how many voters approve of at least one elected candidate?

SAV winners *usually* represent at least as many, and often more, voters than the set of AV winners, as illustrated by the previous example, in which SAV represents 6 of the 10 voters and AV only 4 voters.

SAV winners *c* and *d* appeal to distinctive voters, who are more numerous and so win under SAV, whereas AV winners *a* and *b* appeal to the same voters, but each receives more approval and so wins under AV.

Proposition 2. *There can be subsets that represent more voters than either the SAV or the AV outcome.*

4 voters: *ab*
4 voters: *acd*
3 voters: *ade*
1 voter: *e*.

If 2 candidates are to be elected, the AV outcome is {*a*, *d*} (11 votes for *a* and 7 votes for d). Coincidentally, the SAV outcome is also {*a*, *d*}. But while {*a*, *d*} represents 11 of the 12 voters, {*a*, *e*} represents all 12 voters.

Minimal Representative Sets

A *minimal representative set* is a subset such that: (i) every voter approves at least one candidate in the subset, and (ii) there are no smaller subsets with property (i). The so-called *greedy algorithm* (for representativeness) would selects $\{a, e\}$, but it is no panacea.

Proposition 3. *SAV, AV, and the greedy algorithm can* <u>*all*</u> *fail to find a unique minimal representative set.*

3 voters: *ab*
3 voters: *ac*
2 voters: *b*
1 voters: *c*

AV and the greedy algorithm give $\{a, b\}$ (6 votes to *a*, 5 to *b*), but so does SAV because

$s(a) = 3(½) + 3(½) = 3$
$s(b) = 3(½) + 2(1) = 3½$
$s(c) = 3(½) + 1(1) = 2½$.

However, $\{b, c\}$ is the unique minimal representative set, giving representation to all 9 voters.

Clones

AV may create incentives for *clones* to form, because it gives a full weight of 1 to every candidate a voter approves. To illustrate, assume that 2 candidates are to be elected in the following 12-voter, 3-candidate example:

5 voters: *a*
4 voters: *b*
3 voters: *c*

Under both AV and SAV, {*a*, *b*} is elected, representing 9 of the 12 voters.

But if candidate *a* splits into two clones, a_1 and a_2, and the 5 supporters of *a* approve of both clones, they would win under AV, representing only 5 of the 12 voters.

Under SAV, however,they would lose, because

$s(a_1) = s(a_2) = 5(½) = 2½$
$s(b) = 4(1) = 4$
$s(c) = 3(1) = 3.$

Instead, the SAV outcome would be {*b*, *c*}, which does represents a majority of 7 of the 12 voters.

Because SAV divides 1 vote equally among all candidates of whom a voter approves, it discourages the formation of clones.

The Game Theory Society Election

In 2003, the Game Theory Society used AV for the first time to elect 12 new council members from a list of 24 candidates. (The council comprises 36 members, with 12 elected each year to serve 3-year terms.) The 161 voters approved an average 9.8 candidates and a median 10.

Two of the 12 AV winners would not have been elected under SAV. Each set of winners is given below—ordered from most approved on the left to the least approved on the right—with differences between those who were elected under AV and those who would have been elected under SAV underscored:

AV: 111111111<u>1</u><u>100</u>000000000000
SAV: 111111111<u>0</u>1<u>011</u>0000000000

The AV winners who came in 10th (70 votes) and 12th (69 votes) would have been displaced under SAV by the candidates who came in 13th (66 votes) and 14th (62 votes).

The SAV outcome is more representative, because the elected subset under SAV represents all but 2 of the 161 voters, whereas the elected subset under AV failed to give representation to 5 of the 161 voters.

Although the SAV outcome is more representative, neither is the best possible: There are several subsets of only 8 candidates who would represent all 161 voters—but they do not maximize total voter satisfaction (the most satisfying of these includes the 24th-ranked candidate!)

Strategy under SAV

In a 2-winner election with 3 candidates, it is easy to see that if only two candidates are competitive, a player's strategy of voting for his or her preferred candidate is dominant. The situation is not so clear if all three candidates are competitive and a voter can be decisive.

There are 19 contingencies. In the following tables, the candidates are *a*, *b*, and *c*. The focal voter's preference is fixed at $a > b > c$.

Contingencies, such as (1, 1, 0) indicate the number of votes by all voters other than the focal voter for candidates *a*, *b*, and *c*, respectively. They are normalized according to the smallest number of votes for any of the three candidates. For example, in (1, 1, 0) candidates *a* and *b* are tied, one vote ahead of candidate *c*, and the focal voter is about to vote.

Notation: for each contingency
*** = best outcome**
Underscored = uniquely best outcome

Strategy *a* produces the best outcome 9 times
Strategy *ab* produces the best outcome 8 times
Strategy *b* produces the best outcome 6 times

Undominated strategies: *a, ab, b*

Table 1. Strategies and Outcomes for19 Contingencies in 3-Candidate, 2-Winner Elections

Contingency

Voter	1	2	3	4	5	6	7	8	9	10
Strategy	1,1,0	1,0,1	0,1,1	1,½,0	1,0,½	½,1,0	0,1,½	½,0,1	0, ½,1	1,0,0
A	*ab**	*ac**	*a-b/c**	*ab**	*ac*	*ab**	*ab**	*ac**	*ac**	*a-b/c*
ab	*ab**	*ac**	*bc*	*ab**	*a-b/c*	*ab**	*b- a/c*	*ac**	*bc*	*ab**
b	*ab**	*a/b/c**	*bc*	*ab**	*ab**	*ab**	*bc*	*bc*	*bc*	*ab**
bc	*ab**	*ac**	*bc*	*ab**	*ac*	*b-a/c*	*bc*	*c-a/b*	*bc*	*a-b/c*
c	*a/b/c*	*ac**	*bc*	*ac*	*ac*	*bc*	*bc*	*ac*	*bc*	*ac*
ac	*ab**	*ac**	*bc*	*a-b/c*	*ac*	*ab**	*bc*	*ac*	*c-a/b*	*ac*

Voter	11	12	13	14	15	16	17	18	19
Strategy	0,1,0	0,0,1	½,½,0	½,0,½	0,½,½	½,0,0	0,½,0	0,0,½	0,0,0
a	*ab**	*ac**	*ab**	*ac*	*a-b/c**	*a-b/c*	*ab**	*ac**	*a-b/c*
ab	*ab**	*c-a/b*	*ab**	*a-b/c**	*b-a/c*	*ab**	*ab**	*a/b/c**	*ab**
b	*b-a/c*	*bc*	*ab**	*b-a/c*	*bc*	*ab**	*b-a/c*	*bc*	*b-a/c*
bc	*bc*	*bc*	*b-a/c*	*c-a/b*	*bc*	*a/b/c*	*bc*	*bc*	*bc*
c	*bc*	*c-a/b*	*c-a/b*	*ac*	*bc*	*ac*	*bc*	*c-a/b*	*c-a/b*
ac	*b-a/c*	*ac**	*a-b/c*	*ac*	*c-a/b*	*ac*	*a/b/c*	*ac**	*ac*

SAV: Voting for Political Parties

In most parliamentary democracies, voters vote for political parties—not candidates—which win seats in a parliament in proportion to the number of votes they receive.

Under SAV, voters would *not* be restricted to voting for one party but could vote for as many parties as they like.

Because no party typically wins a majority of seats, it would seem that voters would have an incentive to vote for multiple parties to try to ensure that a favorite *coalition* of parties wins a majority of seats in order help it to become the governing coalition.

Example: 3 parties, 11 voters, 3 seats are to be filled

Bullet Voting

5 voters: party *A*
4 voters: party *B*
2 voters: party *C*.

Party *i*'s *quota*, q_i, is its proportion of votes times the number of seats to be apportioned:

$q_A = (5/11)(3) \approx 1.364$
$q_B = (4/11)(3) \approx 1.091$
$q_C = (2/11)(3) \approx 0.545.$

Under SAV, we assume that each party proposes a number of candidates equal to its *upper quota* (i.e., its quota rounded up), so *A*, *B*, and *C* nominate 2, 2, and 1 candidates, respectively, or 5 in all—2 more than the number of candidates to be elected.

SAV finds apportionments of seats to parties that (i) maximize total voter satisfaction and (ii) are *monotonic*: A party that receives more votes than another cannot receive fewer seats. In the example, there are two monotonic apportionments —(2, 1, 0) and (1, 1, 1) to parties (*A*, *B*, *C*)—giving *s* values of

$$s(2, 1, 0) = 5(1) + 4(½) + 2(0) = 7$$
$$s(1, 1, 1) = 5(½) + 4(½) + 2(1) = 6½.$$

Notice that apportionment (2, 1, 0) maximizes *s* by giving

- 5 *A* voters satisfaction of 1
- 4 *B* voters satisfaction of ½
- 2 *C* voters satisfaction of 0.

Also note that a voter's satisfaction is the fraction of his or her party's *proposed candidates* who are elected. Each party will get either its upper quota or *lower quota* (i.e., its quota rounded down) of nominees elected.

Multiple-Party Voting

If a voter votes for multiple parties, his or her approval is equally divided among all his or her approved parties.

In the example, suppose parties *B* and *C* reach an agreement on policy issues, and their 4 and 2 supporters,

respectively, approve of both parties. In contrast, the 5 party *A* supporters continue to vote for just *A*.

Now *B* and *C* receive a total of 6(½) = 3 votes, which are equally divided between them, making the quotas of the three parties the following:

$q_A = (5/11)(3) \approx 1.364$
$q_B = (3/11)(3) \approx 0.818$
$q_C = (3/11)(3) \approx 0.818.$

These quotas allow for the three monotonic apportionments —shown on the left sides of the equations below—which yield the following satisfaction scores for each apportionment:

$s(2, 1, 0) = 5(1) + 4(½) + 2(½) = 5(1) + 6(½) = 8$
$s(2, 0, 1) = 5(1) + 4(½) + 2(½) = 5(1) + 6(½) = 8$
$s(1, 1, 1) = 5(½) + 4(1) + 2(1) = 8½$

The SAV apportionment is now (1, 1, 1). Compared with apportionment (2, 1, 0) earlier with bullet voting, *A* loses a seat, *B* stays the same, and *C* gains a seat. Thereby *B* and *C* ensure themselves of a majority of seats that only *A* previously obtained.

Proposition 4. *SAV gives the same apportionment as the Jefferson/d'Hondt apportionment method with a quota restriction.*

Of the five so-called divisor methods of apportionment (Balinski and Young, 1982/2001), Jefferson/d'Hondt most favors large parties. Unlike (unrestricted) Jefferson/d'Hondt, SAV apportionments satisfy upper

quota, because parties cannot propose, and therefore cannot receive, more seats than their quotas rounded up.

Because Jefferson/d'Hondt apportionments always satisfy lower quota (Balinski and Young, 1982/2001), SAV apportionments satisfy quota (i.e., both lower and upper). However, they are not Hamilton apportionments.

A Paradox

Proposition 4 notwithstanding, the supporters of *B* and *C* may *not* approve of each other's party, because B does not *individually* benefit from doing so.

Therefore, despite the fact that *B* and *C* supporters can together ensure themselves of a majority of seats through mutual approval, they may still go their separate ways.

A possible way around this paradox is for *B* and *C* to become one party, reducing the party system to just two parties. Because the combination of *B* and *C* has more supporters than *A* does, this combined party would win a majority of seats.

Insofar as SAV encourages compromises that reflect voter preferences, PR systems are likely to become less fractious and more responsive, enhancing their stability.

Conclusions

1. SAV is applicable to multiwinner elections. It uses an approval ballot—whereby voters can approve of as many candidates or parties as they like—but they are not elected based on the number of approval votes they receive.

2. SAV measures the satisfaction of a *voter* by the fraction of his or her approved candidates who are elected. The set of candidates that maximizes the sum of voter satisfaction scores is selected.

3. This measure is independent of the number of candidates a voter approves of—it works equally well for voters who approve of few or many candidates—and so, in a sense, mirrors a voter's *personal* tastes. SAV may elect a completely different set of candidates from AV.

4. The satisfaction score of a *candidate* is the sum of the satisfaction contributions he or she receives from all voters. This is $1/n$ from each voter who approves of him or her, where n is the number of candidates approved of by the voter.

5. These equal contributions of voters to candidates make the winning candidates those with the highest individual satisfaction scores, rendering SAV outcomes easy to compute—unlike an algorithm that finds minimal representative sets.

6. SAV tends to elect candidates that give more voters either partial or complete satisfaction—and thus representation—than does AV. It also discourages clones.

7. Because bullet voting is risky when voting for individual candidates (a voter's satisfaction score will be either 0 or 1), a risk-averse voter may be inclined to approve of multiple candidates.

8. A decision-theoretic analysis in which there are 3 competitive candidates, and 2 are to be elected, shows that voting for one's top candidate, and voting for one's top two candidates, are best choices in approximately the same number of contingencies.

9. When SAV is applied to party-list systems in parliamentary democracies, the satisfaction score of a voter is the fraction of each party's *proposed* candidates—its quota rounded up—the voter approves of who are elected.

10. The number of seats apportioned to a party is never greater than a party's quota rounded up. SAV mimics the Jefferson/d'Hondt divisor method with a quota requirement, which favors larger parties.

11. Individually, parties may be hurt when their supporters approve of other parties. Collectively, however, they may be able to increase their combined seat share by forming coalitions—whose supporters approve of all parties in it—or even merging.

12. The coordination of policies and the formation of coalitions may reduce the party system to two broad left-of-center and right-of-center parties, or coalitions of parties.

13. Alternatively, a third moderate party might emerge (e.g., Kadima in Israel) that peels away supporters from the left and the right. This seems all very democratic, making coalitions fluid and responsive to voter sentiment.

14. More coordination by the parties would give voters a better idea of what to expect when they decide which parties to support, compared with the situation today when voters can never be sure about what parties will join in a governing coalition and what its policies will be.

15. Because SAV makes it easier for voters to know what parties to approve of, and for party coalitions that reflect voter interests to form, SAV should lead to more informed voting and more responsive government.