Generalized obligation rules for minimum cost spanning tree problems

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SCW 2010
Moscow, July 2010
Minimum cost spanning tree problems
1. Minimum cost spanning tree problems

2. Generalized obligation rules
   - Obligation rules
   - Generalized obligation rules
Outline

1. Minimum cost spanning tree problems

2. Generalized obligation rules
   - Obligation rules
   - Generalized obligation rules

3. Results
   - Kruskal’s sharing rules
   - Characterization of obligation rules
   - Characterizations of the folk rule
A minimum cost spanning tree problem (mcstp) is a pair \((N_0, C)\) where:

\[
N_0 = N \cup \{0\}
\]

- \(N\) is the set of agents.
- \(0\) is the source.

\[
C = (c_{ij})_{i,j \in N_0}
\]

- \(c_{ij}\) is the cost of direct link between agents \(i\) and \(j\).
- \(c_{ij} = c_{ji}\) for all \(i, j \in N_0\).
- \(c_{ii} = 0\) for all \(i \in N_0\).
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An **obligation function** for \( N \ (N \subset \mathcal{N}) \), is a map \( o \) assigning to each \( S \in 2^N \setminus \{\emptyset\} \) a vector \( o(S) \) satisfying the following properties:

\( (o\text{-}i) \quad o(S) \in \Delta (S) \)

The aggregate responsibility is normalized to 1

\( (o\text{-}ii) \) For each \( S, T \in 2^N \setminus \{\emptyset\}, S \subset T \) and \( i \in S \),

\[ o_i (S) \geq o_i (T) \]

If a group expands then any of its existing members should not suffer due to the expansion

**Example of obligation function**

For each \( S \subset N \), \( o_i(S) = \frac{1}{|S|} \).
Using an obligation function, we can obtain a cost allocation as follows:

1. At each stage of Kruskal algorithm an arc is added to the network.
2. The cost of this arc will be paid by the agents who benefit from adding this arc.
3. Each of these agents pays the difference between the obligation before the arc is added to the network and after it is added.
Formally, given an obligation function $o$ we define the obligation rule $f^o$.

For each $i \in N$,

$$f^o_i (N_0, C) = \sum_{p=1}^{\lvert N \rvert} c_{i^p j^p} (o_i (S (P (g^{p-1}) , i)) - o_i (S (P (g^p) , i)))$$

where,

- $(i^p, j^p)$ is the arc selected at Stage $p$ by Kruskal’s algorithm.
- $g^p = \{(i^1, j^1), ..., (i^p, j^p)\}$ is the graph constructed at stage $p$.
- $P (g^p)$ is the partition induced by $g^p$ in $N_0$.
- $(P (g^p) , i)$ is the element of $P (g^p)$ to which $i$ belongs to.
- By convention, $o_i (T) = 0$ if $0 \in T$. 
Example 1

**AGENT 1:**

<table>
<thead>
<tr>
<th>((i^p, j^p))</th>
<th>(c_{i^p j^p})</th>
<th>(o_i(g^{p-1}, i) - o_i(g^p, i))</th>
</tr>
</thead>
<tbody>
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Obligation rules: Example 1

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TOTAL PAYOFF: \( 4 \frac{1}{2} + 6 \frac{1}{6} + 12 \frac{1}{3} = 7 \)
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3. Results
PARTITIONS:

- $P(N_0)$ is the set of all partitions over $N_0$. $P = \{S_0, S_1, \ldots, S_m\}$ is an element of $P(N_0)$ such that $0 \in S_0$.
- Given $P, P' \in P(N_0)$, we say that $P$ is finer than $P'$ if $P'$ is obtained from $P$ joining several elements of $P$.
- We say that $P$ is 1-finer than $P'$ if $P'$ is obtained from $P$ joining two elements of $P$.

$P$ is 1-finer than $P'$
Generalized obligation rules

Generalized obligation functions

A generalized obligation function $\theta$ is a map that assigns to each partition $P \in P(N_0)$ a vector $\theta(P) \in \mathbb{R}^N$ satisfying that:

- $\theta_i(P) \geq 0$ for all $i \in N$.
- $\sum_{i \in N} \theta_i(P) = m$.
- If $P$ is finer than $P'$ then, $\theta_i(P) \geq \theta_i(P')$ for all $i \in N$.

We can associate a generalized obligation rule $f^\theta$ with each generalized obligation function $\theta$:

$$f_i^\theta(N_0, C) = \sum_{p=1}^{\lfloor N \rfloor} c_{ipj} \left( \theta_i(P(g^p-1)) - \theta_i(P(g^p)) \right)$$
Generalized obligation rules

Given an obligation function $o$, $P \in P(N)$, and $i \in S \in P$, we define $\theta^o_i(P) = o_i(S)$.

**Proposition.** $\theta^o$ is a generalized obligation function.

- Given $i \in S \in P$, if $o$ is an obligation function $o_i$ only depends on $S$.
- If $\theta$ is a generalized obligation function, $\theta_i$ depends on $S$ but also on the rest of the agents $(N \setminus S)$.
- We can think in obligation functions as the subset of generalized obligation functions where there is not externalities.
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A sharing function $\varrho$ is a map that associates with each pair $(P, P') \in P(N_0)$ where $P$ is 1-finer than $P'$, a vector $\varrho(P, P') \in \Delta(N)$ satisfying the following condition:

**Path independence condition**

Assume we have $P, P' \in P(N_0)$ such that $P$ is finer than $P'$, and 
\{\(P_1^1, P_2^1, \ldots, P_q^1\)\} and \{\(P_1^2, P_2^2, \ldots, P_q^2\)\} are such that 
\(P_1^1 = P_1^2 = P, P_q^1 = P_q^2 = P'\) and  
\(P_p^i\) is 1-finer than \(P_{p+1}^i\) for all \(i = 1, 2\) and \(p = 1, \ldots, q - 1\). Then,

\[
\sum_{p=1}^{q-1} \varrho_i \left( P_p^1, P_{p+1}^1 \right) = \sum_{p=1}^{q-1} \varrho_i \left( P_p^2, P_{p+1}^2 \right) \text{ for all } i \in N.
\]
Kruskal’s sharing rules

Example

- \( N = \{1, 2\}, \ P = \{\{0\}, \{1\}, \{2\}\}, \) and \( P' = \{\{0, 1, 2\}\} \)
- \( P_1^1 = P, \ P_1^2 = \{\{0, 1\}, \{2\}\}, \) and \( P_3^1 = \{\{0, 1, 2\}\} = P'. \)
- \( P_2^2 = P, \ P_2^2 = \{\{0\}, \{1, 2\}\}, \) and \( P_3^2 = P'. \)

By path independence,

\[ \varphi_1(P, P_2^1) + \varphi_1(P_1^1, P') = \varphi_1(P, P_2^2) + \varphi_1(P_2^2, P'). \]
Given the sharing function $\varphi$ we define $f^\varphi$ as follows:

$$f_i^\varphi(N_0, C) = \sum_{p=1}^{|N|} c_{ipjp} \left[ \varphi_i \left( P(g^{p-1}), P(g^p) \right) \right].$$

Interpretation: $P(g^{p-1})$ is 1-finer than $P(g^p)$. The cost of each arc added is divided among the agents taking into account the agents connected before adding the arc, $P(g^{p-1})$, and the agents connected after adding the arc, $P(g^p)$. It does not matter the way in which the agents are connected and the arc we add, whenever the arc connects the same components in $P(g^{p-1})$. 
First result

Theorem 1

\[ \{ f^\theta : \theta \text{ is a generalized obligation function} \} = \{ f^\varrho : \varrho \text{ is a sharing function} \} \]
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A rule $f$ satisfies **Restricted Additivity** (RA) if for all $mcstp \ (N_0, C)$ and $(N_0, C')$ satisfying that there exists an $mt \ t = \{(i^0, i)\}_{i \in N}$ in $(N_0, C)$, $(N_0, C')$, and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi(N)$ such that $c_{i^0_1i_1} \leq c_{i^0_2i_2} \leq \ldots \leq c_{i^0_{|N|}i_{|N|}}$ and $c'_{i^0_1i_1} \leq c'_{i^0_2i_2} \leq \ldots \leq c'_{i^0_{|N|}i_{|N|}}$, we have that

$$f(N_0, C + C') = f(N_0, C) + f(N_0, C').$$

We can solve little problems in order to solve the big one.
Why not the usual additivity property?

\[ f(N_0, C + C') = f(N_0, C) + f(N_0, C') \]

for all mcstp \((N_0, C)\) and \((N_0, C')\).
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\[ f(N_0, C + C') = f(N_0, C) + f(N_0, C') \]
for all mCSTP \((N_0, C)\) and \((N_0, C')\).

\[
\begin{align*}
  m(N_0, C) &= 2 \\
  m(N_0, C') &= 3 \\
  m(N_0, C + C') &= 6
\end{align*}
\]

So, there is no cost allocation rule simultaneously satisfying efficiency and additivity.
The rule $f$ satisfies **Strong Cost Monotonicity (SCM)** if given $(N_0, C)$ and $(N_0, C')$ such that $C \geq C'$, we have that

$$f(N_0, C) \geq f(N_0, C')$$

If some cost increases, no agent should pay less.
Bergantiños *et al* (2010) SCW obtain the following result:

**Lemma 1.** $f$ satisfies $RA$ and $SCM$ if and only if there exists a sharing function $\varphi$ such that $f = f^\varphi$.

**Corollary 1**

$f$ satisfies $RA$ and $SCM$ if and only if there exists a generalized obligation function $\theta$ such that $f = f^\theta$.

It is a trivial consequence of Theorem 1.
Characterization 1

- $f$ satisfies **Core Selection** (CS) if for all $mcstp\ (N_0, C)$ and all $S \subseteq N$, we have that $\sum_{i \in S} f_i(N_0, C) \leq m(S_0, C)$.

**CS** says that $f(N_0, C)$ belongs to the core of the problem.

**Theorem 2**

$f$ satisfies $RA$, $SCM$, and $CS$ if and only if $f$ is an obligation rule.
Characterization 2

- $f$ satisfies **Separability** \((SEP)\) if for all $S \subset N$ such that
  \[ m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C), \]
  \[ f_i(N_0, C) = \begin{cases} 
  f_i(S_0, C) & \text{when } i \in S \\
  f_i((N \setminus S)_0, C) & \text{when } i \in N \setminus S.
\end{cases} \]

Two subset of agents, $S$ and $N \setminus S$, can connect to the source separately or can connect jointly. If there are no savings when they connect jointly, \(SEP\) says that agents must pay the same in both circumstances.

**Theorem 3**

$f$ satisfies $RA$, $SCM$, and $SEP$ if and only if $f$ is an obligation rule.
Other characterizations of obligation rules

Lorenzo and Lorenzo-Freire (IJGT, 2009)
- RA
- Population Monotonicity

Bergantiños and Kar (GEB, 2010)
- SCM
- Population Monotonicity
- CPL (a linearity property on cones)
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Characterizations of the folk rule
(Felkamp et al., 1994)

- $i, j \in N$ are **symmetric** if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.
- $f$ satisfies **Symmetry** ($SYM$) if for all $mcstp (N_0, C)$ and all pair of symmetric agents $i, j \in N$,

$$f_i (N_0, C) = f_j (N_0, C) .$$

**Corollary 2**

- The folk rule is the unique rule satisfying $RA$, $SCM$, $CS$, and $SYM$.
- The folk rule is the unique rule satisfying $RA$, $SCM$, $SEP$, and $SYM$. 
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Thank you!