The problem

A finite number of individuals claim an indivisible object.
For each individual $i \in \{1, 2, \ldots, n\}$, $\theta_i \in \mathbb{R}_+$ is $i$’s valuation for the good.
An economy is a profile \( \theta_N = (\theta_i)_{i \in N} \in \mathbb{R}^n_+ \)
The model

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- Letting $f_i : \mathbb{R}^n_+ \rightarrow \{0,1\}$, an assignment is a profile $f_N(\theta_N) \equiv (f_i(\theta_N))_{i \in N}$
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- Letting \( t_i : \mathbb{R}_+^n \rightarrow \mathbb{R} \), a vector of transfers is denoted \( t(\theta_N) \equiv (t_i(\theta_N))_{i \in N} \)
- An allocation is an assignment coupled with a vector of transfers. It is feasible if \( \sum_{i \in N} f_i(\theta_N) \leq 1 \) and \( \sum_{i \in N} t_i(\theta_N) \leq 0 \).
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A mechanism $\varphi$ is a function that associates with each economy an allocation.
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A mechanism $\varphi$ is a function that associates with each economy an allocation.

Utility: $u_i(\varphi_i(\theta_N)) = \theta_i \times f_i(\theta_N) + t_i(\theta_N)$.
What properties should a mechanism satisfy?

1. **Strategy-Proofness:** For each \( \theta_N \in \mathbb{R}_+^n \) and each \( i \in N \)

\[
u_i(\phi_i(\theta_i, \theta_N'_{\setminus i})) \geq u_i(\phi_i(\theta'_i \theta_N'_{\setminus i}))
\]

for each \( \theta'_i \in \mathbb{R}_+ \), and each \( \theta_N'_{\setminus i} \in \mathbb{R}_+^{n-1} \).

Timos Athanasiou

Assigning an Indivisible Private Good: A Solomonic Solution
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   for each $\theta'_i \in \mathbb{R}_+$, and each $\theta'_N \setminus \{i\} \in \mathbb{R}_+^{n-1}$.

2. **Anonymity**: For each $\theta_N \in \mathbb{R}_+^n$, each $\pi \in \Pi$, and each $i \in N$

   \[ u_i(\phi_i(\theta_N)) = u_{\pi(i)}(\phi_{\pi(i)}(\theta_{\pi(i)}(N))) \].
What properties should a mechanism satisfy?

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3. **Assignment Efficiency**: For each \( \theta_N \in \mathbb{R}^n_+ \) there exists \( j \in N \) for whom \( f_j(\theta_N) = 1 \) and \( \theta_j \geq \theta_i \), for each \( i \in N \setminus \{j\} \)
We will say that $\varphi'$ Pareto dominates $\varphi$, if and only if,

1. for each $\theta_N \in \mathbb{R}_+$ and each $i \in N$, $u_i(\varphi'(\theta_N)) \geq u_i(\varphi(\theta_N))$, and
2. for some $\tilde{\theta}_N \in \mathbb{R}_+$ and some $j \in N$, $u_j(\varphi'(\tilde{\theta}_N)) > u_j(\varphi(\tilde{\theta}_N))$.

We will say that a Strategy-proof mechanism $\varphi$ is **Second-Best Efficient** if and only if there does not exist another Strategy-proof mechanism $\varphi'$ that Pareto dominates $\varphi$. 
Two Mechanisms

Assigning an Indivisible Private Good: A Solomonic Solution

Timos Athanasiou
The Implications of Anonymity

Weak Assignment Efficiency: For each $\theta_N \in \mathbb{R}^n_+$ and each $i \in N$ such that $f_i(\theta_N) = 0$, if for some $j \in N$, $f_j(\theta_N) = 1$, then $\theta_j \geq \theta_i$.

Proposition
If a mechanism $\varphi$ satisfies Strategy Proofness and Anonymity, then $\varphi$ satisfies Weak Assignment Efficiency
Proof

There exists some economy $\theta_N$ such that $f_j(\theta_N) = 1$ and for some $k \in \mathbb{N}$, $\theta_k > \theta_j$
Consider the economy \((\theta_k, \theta_{N \setminus \{j\}})\). By Strategy-Proofness,
\[
\varphi_j(\theta_N) = \varphi_j(\theta_k, \theta_{N \setminus \{j\}})
\]
By Anonymity, $u_j(\varphi_j(\theta_k, \theta_{N\setminus\{j\}})) = u_k(\varphi_k(\theta_k, \theta_{N\setminus\{j\}}))$
Let $\pi' \in \Pi$ be such that $\pi'(k) = j$, $\pi'(j) = k$ and $\pi'(i) = i$, for each $i \in N \setminus \{j, k\}$. By *Strategy-Proofness*, $\varphi_k(\theta_j, \theta_{N \setminus \{j\}}) = \varphi_k(\theta_{\pi'(N)})$. 

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Therefore, \( \varphi_j(\theta_N) \neq \varphi_k(\theta_{\pi'(N)}) \), which contradicts \textit{Anonymity}. 
Three conditions

1 **Condition A:** For each $\tilde{\theta}_N \in \mathbb{R}_+$ such that $\sum_{i \in N} f_i(\tilde{\theta}_N) = 0$ and each $j \in N$, there exist $\tilde{\theta}_j < x' < +\infty$ such that $f_j(x', \tilde{\theta}_N \setminus \{j\}) = 1$.

2 **Condition B:** For each $\theta_N \in \mathbb{R}_+$ and each $i \in N$, there does not exist $\epsilon > 0$ such that

$$\sum_{j \in N} t_j(x, \theta_N \setminus \{i\}) + \epsilon \leq 0, \text{ for each } x \geq 0.$$ 

3 **Condition C:** For each $\tilde{\theta}_N \in \mathbb{R}_+$ such that $\sum_{i \in N} f_i(\tilde{\theta}_N) = 0$ and each $j \in N$, if $y \equiv \inf \{x \geq \tilde{\theta}_j : f_j(x, \tilde{\theta}_N \setminus \{j\}) = 1\} < +\infty$, then there does not exist $\epsilon > 0$ such that

$$\sum_{i \in N} t_i(x', \tilde{\theta}_N \setminus \{j\}) + \epsilon \leq 0, \text{ for each } x' \geq y.$$
Conditions A and C do not imply Condition B.

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**Proposition**
For each pair of distinct mechanisms \( \varphi, \varphi' \) both satisfying *Strategy-Proofness* and *Weak Assignment Efficiency*, if \( \varphi \) satisfies Conditions A, Condition C and there exists \( \tilde{\theta}_N \in \mathbb{R}_+ \), with \( \tilde{\theta}_k \geq \tilde{\theta}_i \), for each \( i \in N \), and \( \delta > 0 \) such that for each \( \epsilon \in (0, \delta] \)
\[
\sum_{i \in N} f_i(\tilde{\theta}_k + \epsilon, \tilde{\theta}_{N \setminus \{k\}}) < \sum_{i \in N} f'_i(\tilde{\theta}_N),
\]
then \( \varphi' \) does not Pareto dominate \( \varphi \).
1 Within the class of mechanisms that satisfy Anonymity the set of Second-Best Efficient mechanisms contains mechanisms that destroy the good.

2 The set of Second-Best Efficient mechanisms is characterized
Extensions

1. The rationing problem: k homogeneous goods to n individuals (with k < n)
2. Matching with quasi-linear preferences