Convex games and bargaining sets

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Outline

1 Introduction
2 Preliminaries
3 Max-payoff vectors: a necessary condition
4 Characterization result
We study cooperative situations among agents (cooperative TU games) where marginal contributions of agents grow as coalitions players add also grow.

- $N = \{1, 2, \ldots, n\}$ is the set of players.
- $v(S)$ is the worth of coalition $S \subseteq N$.
- For all $i \in N$ and for all $S \subseteq T \subseteq N \setminus \{i\}$,

\[
v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).
\]

It is said the game is convex (Shapley, 1971)
We study cooperative situations among agents (cooperative TU games) where **marginal contributions** of agents grow as coalitions players add also grow.

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It is said the game is **convex** (Shapley, 1971)
Some characterizations of a convex game are:

- $v$ is **convex** $\iff$ **marginal worth vectors** are in the core
  - Shapley (1971) and Ichiisi (1981)

- $\iff$ the **core** and the **Weber set** coincide
  - Weber (1988)

- $\iff$ the cores of the game and subgames are **stable sets**
  - Einy and Shitovitz (1996)

- $\iff$ the **Weber set** is a subset of the **DM bargaining set** (balanced games)
  - Izquierdo & Rafels (2008)
Can we characterize the convexity of a game by comparing the core and the bargaining set?
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A cooperative TU-game a pair \((N, v)\) where
- \(N = \{1, 2, \ldots, n\}\) is the set of players and
- \(v\) is the characteristic function, \(S \mapsto v(S), v(\emptyset) = 0\).

A game is **0-monotonic** if for all \(S \subseteq T \subseteq N\),
\[
v(S) + \sum_{i \in T \setminus S} v(i) \leq v(T).
\]

A game is **superadditive** if for all \(S, T \subseteq N, S \cap T) \neq \emptyset\)
\[
v(S) + v(T) \leq v(S \cup T).
\]

A game \(v\) is **convex** if, for all \(i \in N\) and for all \(S \subseteq T \subseteq N \setminus \{i\}\)
\[
v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).
\]
Definitions

Core

The **set of preimputations** of a game $v$ is

$$I^*(v) = \{ x \in \mathbb{R}^N \mid x(N) = v(N) \}$$

The **set of imputations** of a game $v$ is

$$I(v) = \{ x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(i), \text{ for all } i \in N \}$$

The **core** of a game $v$ is

$$C(v) = \{ x \in I(v) \mid x(S) \geq v(S), \text{ for all } S \subseteq N \}$$
A payoff vector $x \in \mathbb{R}^N$ is in the **bargaining set** if for every objection to $x$ there is a counterobjection.

Several definitions:

- Davis and Maschler (1963)
- Mas-Colell (1989)
- Zhou (1994)
- Shimomura (1997)
- Granot (2010)
An **objection** to \( x \in \mathbb{R}^N \) is a coalition \( S \subseteq N \) and a payoff vector \( y \in \mathbb{R}^S \) such that

\[
y_i > x_i \text{ for all } i \in S \text{ and } y(S) = v(S)\]

A **counterobjection** to some objection \((S, y)\) is a coalition \( T \subseteq N \) and a payoff vector \( z \in \mathbb{R}^T \) such that

\[
z_i > y_i \text{ for all } i \in T \cap S, \quad z_i > x_i, \text{ for all } i \in T \setminus S \text{ and } z(T) = v(T)\]
## Definitions

### Bargaining sets

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</table>
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4 Characterization result
A **marginal worth vector** of the game \( v \) relative to \( \theta = (i_1, \ldots, i_n) \), \( m^\theta(v) \), is defined as:

\[
m^\theta_{i_k}(v) := v\left(\{i_1, \ldots, i_k\}\right) - v\left(\{i_1, \ldots, i_{k-1}\}\right), \quad \text{for all } k = 1, \ldots, n.
\]
A **marginal worth vector** of the game $v$ relative to $	heta = (i_1, \ldots, i_n)$, $m^\theta (v)$, is defined as:

$$m^\theta_{i_k} (v) := v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}), \quad \text{for all } k = 1, \ldots, n.$$
A **max-payoff** vector $x^\theta(v)$ of $v$ relative to $\theta = (i_1, \ldots, i_n)$ is defined by

\[
x^\theta_{i_k} := \max_{Q \subseteq P^\theta_{i_k}} \{ v(\{i_k\} \cup Q) - x^\theta(Q) \}, \quad \text{for all } k \in \{1, \ldots, n-1\},
\]

\[
x^\theta_{i_n} := v(N) - x^\theta(N \setminus \{i_n\}).
\]

\[
x^\theta_{i_1}(v) = v(\{i_1\}),
\]

\[
x^\theta_{i_2}(v) = \max\{ v(\{i_2\}), v(\{i_1, i_2\}) - x^\theta_{i_1}(v) \},
\]

\[
x^\theta_{i_3}(v) = \max\{ v(\{i_3\}), v(\{i_1, i_3\}) - x^\theta_{i_1}(v), v(\{i_2, i_3\}) - x^\theta_{i_2}(v), v(\{i_1, i_2, i_3\}) - x^\theta_{i_1}(v) - x^\theta_{i_2}(v) \}
\]

\[
\vdots
\]

\[
x^\theta_{i_n}(v) = v(N) - x^\theta(N \setminus \{i_n\})
\]

J.M. Izquierdo & C. Rafels — Convex games and bargaining sets
A max-payoff vector $x^\theta(v)$ of $v$ relative to $\theta = (i_1, \ldots, i_n)$ is defined by

$$x^\theta_{i_k} := \max_{Q \subseteq P^\theta_{i_k}} \{ v(\{i_k\} \cup Q) - x^\theta(Q) \}, \quad \text{for all } k \in \{1, \ldots, n-1\},$$

$$x^\theta_{i_n} := v(N) - x^\theta(N \setminus \{i_n\}).$$

$$x^\theta_{i_1}(v) = v(\{i_1\}),$$

$$x^\theta_{i_2}(v) = \max \{ v(\{i_2\}), v(\{i_1, i_2\}) - x^\theta_{i_1}(v) \},$$

$$x^\theta_{i_3}(v) = \max \{ v(\{i_3\}), v(\{i_1, i_3\}) - x^\theta_{i_1}(v), v(\{i_2, i_3\}) - x^\theta_{i_2}(v), v(\{i_1, i_2, i_3\}) - x^\theta_{i_1}(v) - x^\theta_{i_2}(v) \}$$

$$\vdots$$

$$x^\theta_{i_n}(v) = v(N) - x^\theta(N \setminus \{i_n\})$$
Consider the $2 \times 2$ glove market defined by matrix

\[
\begin{pmatrix}
3 & 4 \\
1 & 1 & 1 \\
2 & 1 & 1
\end{pmatrix}
\]

- $v(\{i\}) = 0$
- $v(\{1, 3\}) = v(\{1, 4\}) = 1$
- $v(\{2, 3\}) = v(\{2, 4\}) = 1$
- $v(S) = 1$, if $|S| = 3$
- $v(N) = 2$

Take the ordering $\theta^* = (1, 3, 4, 2)$,

\[
\begin{pmatrix}
1 & 3 & 4 & 2 \\
m^{\theta^*}(v) & 0 & 1 & \textbf{0} & 1 \\
x^{\theta^*}(v) & 0 & 1 & \textbf{1} & 0
\end{pmatrix}
\]
Max-payoff vectors: a necessary condition

Properties of max-payoff vectors

Max-payoff vector. Property 1

Given $\theta = (i_1, i_2, \ldots, i_n)$ and $S \subseteq N$,

if $x^\theta(S) < v(S)$, then $i_n \in S$

Max-payoff vector. Property 2

If $v$ is convex $x^\theta(v) = m^\theta(v)$, for all ordering $\theta = (i_1, i_2, \ldots, i_n)$
Max-payoff vectors: a necessary condition

The Theorem

Theorem 1 (Izquierdo and Rafels, 2010a)

For any arbitrary balanced game \( v \in B^N \) we have:

1. If \( C(v) = Z_{Sh}(v) \), then \( x^\theta(v) \in C(v) \), for all \( \theta \in \Theta_N \)
2. If \( C(v) = MB_{Sh}(v) \), then \( x^\theta(v) \in C(v) \), for all \( \theta \in \Theta_N \).

Hint: If not, by Property 1, we can raise an objection to \( x^\theta(v) \) through a coalition of maximal excess \( S^* \) with \( i_n \in S^* \), giving as much as possible to player \( i_n \).

The condition is not sufficient: for the game associated to the 2 \( \times \) 2 glove market, \( x^\theta(v) \in C(v) \), for all \( \theta \in \Theta_N \), but the core is strictly included in the Shimonura bargaining sets.
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1. Introduction
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4. Characterization result
The **coincidence** of the core with the bargaining set

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<tr>
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**A non-convex game:**

- Core
- Davis-M BS
- Mas-Colell BS
- Zhou BS
- Zhou(Sh) BS
- Mas-Colell(Sh) BS
Theorem 2 (Izquierdo and Rafels, 2010b)

Let $v \in G^N$. Then,

1. $v$ is convex $\iff Z_{Sh}(v) = C(v)$ and $v$ is superadditive;
2. $v$ is convex $\iff M_{Bh}(v) = C(v)$ and $v$ is 0-monotonic.
Characterization
Bargaining sets and the core

Sketch of the proof

If $\nu$ is convex then $C(\nu) = \mathcal{Z}_{SH}(\nu) = \mathcal{MB}_{Sh}(\nu)$

- Let $x \in I(\nu) \setminus C(\nu)$ and let $S^* \subseteq N$ be a minimal coalition of the largest excess,
  $$S^* \in \operatorname{argmax}\{\nu(S) - x(S)\}$$
  where, if $S \subsetneq S^*$, $\nu(S) - x(S) < \nu(S^*) - x(S^*)$.

- Compute the excess game
  $$\hat{\nu}_x(S) := \max_{R \subseteq S}\{\nu(R) - x(R)\}, \text{ for all } S \subseteq N$$
  and take the Shapley value of $\hat{\nu}_x$, namely $\Phi(\hat{\nu}_x)$.

- Define the objection $(S^*, y)$ with $y_i = x_i + \Phi_i$, for all $i \in S^*$.

- This objection cannot be countered.
Characterization result

Bargaining sets and the core

sketch of the proof

If \( C(v) = MB_{SH}(v) \) and \( v \) is 0-monotonic \( \Rightarrow v \) is convex

- By Theorem 1 \( x^{\theta}(v) \in C(v) \), for all \( \theta \).
- Since the game is not convex there exists

\[
\theta^* = (i_1, \ldots, i_{k^*}-1, i_{k^*}, i_{k^*}+1, \ldots, i_n)
\]

such that

\[
x^{\theta^*}(v) \neq m^{\theta^*}(v), \text{ (Property 2)}
\]

\[
x_{i_{k^*}}^{\theta^*} > m_{i_{k^*}}^{\theta^*} \geq v(\{i_{k^*}\}), \text{ and } x_{i_k}^{\theta^*} = m_{i_k}^{\theta^*}, \ k = 1, \ldots, k^*-1.
\]

- For all \( S \subseteq \{i_1, \ldots, i_{k^*}\} \) such that \(|S| \leq k^* - 1\), the subgame \( v_S \) is convex
Characterization result

Bargaining sets and the core

We construct a vector \( x \) from \( x^\Theta(v) \)

The vector \( x \not\in C(v) \) but we prove \( x \in MB_{SH}(v) \).

Einy, E., Shitovitz, B., 1996. Convex Games and Stable Sets, Games Econ. Behav. 16(2), 192–201.

