Copula-based orderings of multivariate positive dependence

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Why an ordering of positive dependence?

- Vague notion of positive dependence: The extent to which “high” and “low” realizations in the different dimensions of a multivariate distribution occur together.
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- Other applications
  - Measurement of agreement between judges;
  - Measurement of assortativeness of (multidimensional) matching; ...
What do these applications have in common?

Main ingredients of the applications

1. Measurement of dependence between *many* dimensions
What do these applications have in common?

Main ingredients of the applications

1. Measurement of dependence between many dimensions

2. The *marginal* distributions of multivariate distribution can *change*
Let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ be two different $m$-dimensional discrete random vectors with $m \geq 2$ and the finite set $S$ as support.
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The set of these random vectors is denoted $\mathcal{X}$.
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We look at $\succ$, an asymmetric and transitive binary relation (weak ordering) that can be interpreted as "is more dependent".
Let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ be two different $m$-dimensional discrete random vectors with $m \geq 2$ and the finite set $S$ as support.

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$F_X(x_1, \ldots, x_m) = \Pr [X_1 \leq x_1 \text{ and } \ldots \text{ and } X_m \leq x_m]$
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$F_X(x_1, \ldots, x_m) = \Pr [X_1 \leq x_1 \text{ and } \ldots \text{ and } X_m \leq x_m]$ 

$\overline{F}_X(x_1, \ldots, x_m) = \Pr [X_1 > x_1 \text{ and } \ldots \text{ and } X_m > x_m]$. 
Let \( X = (X_1, \ldots, X_m) \) and \( Y = (Y_1, \ldots, Y_m) \) be two different \( m \)-dimensional discrete random vectors with \( m \geq 2 \) and the finite set \( S \) as support.

The set of these random vectors is denoted \( \mathcal{X} \).

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The univariate marginal distribution functions of \( F_X \) are denoted by \( F_1, \ldots, F_m \).
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The univariate marginal distribution functions of $F_X$ are denoted by $F_1, \ldots, F_m$.

The set of random vectors with corresponding marginal distribution functions $F_1, \ldots, F_m$ is referred to as the Fréchet set $\mathcal{F} (F_1, \ldots, F_m)$ or shortly $\mathcal{F}$. 
For an arbitrary function $U$ its first difference operator of dimension $j$ be defined by

$$\Delta_j^\delta U(x_1, \ldots, x_m) = U(x_1, \ldots, x_j + \delta, \ldots, x_m) - U(x_1, \ldots, x_j, \ldots, x_m).$$
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For notational convenience: Let $x_{j} = x_{j}$ and $x\bar{j} = x_{j} + \delta_{j}$ with $\delta_{j} > 0$ for $j = 1, \ldots, m$. 

Definition ($k$-increasing)

A function $U : \mathbb{R}^m \to \mathbb{R}$ is said to be $k$-increasing if it holds that

$$\Delta_{\delta_{1}}^{\delta_{j_{1}}} \ldots \Delta_{\delta_{k}}^{\delta_{j_{k}}} U(x_1, \ldots, x_m) > 0,$$

for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, $\delta_{1}, \ldots, \delta_{k} > 0$ and $1 \leq j_{1}, \ldots, j_{k} \leq m$.

Example: a function $U$ is 2-increasing if it holds that:

$$U(x_1, \ldots, x_j, \ldots, x_m) + U(x_1, \ldots, x_{j}, \ldots, x_m) > U(x_1, \ldots, x_{j}, \ldots, x_m) + U(x_1, \ldots, x_j, \ldots, x_m).$$

(also known as a supermodular or superadditive function). If it is sufficiently differentiable:

$$\frac{\partial^2 U(x)}{\partial x_j \partial x_{j'}} > 0.$$
For an arbitrary function $U$ its first difference operator of dimension $j$ be defined by
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\begin{align*}
U(x_1, \ldots, x_{j_1}, x_{j_2}, \ldots, x_m) + U(x_1, \ldots, x_{j_1}, x_{j_2}, \ldots, x_m) > \\
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Existing literature:

Notation

- For an arbitrary function $U$ its first difference operator of dimension $j$ be defined by
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- If it is sufficiently differentiable: $\frac{\partial^2 U(x)}{\partial x_{j_1} \partial x_{j_2}} > 0$. 

Existing literature:
The intuition

Dimension 2

1

\(\overline{x}_2\)

\(x_2\)

Dimension 1

0

\(x_1\)

\(\overline{x}_1\)

1

\[+\varepsilon\]

\([-\varepsilon]\)

\(\left(\overline{x}_j_1, \overline{x}_j_2\right)\)

\(\left(x_j_1, x_j_2\right)\)

\(\left(\overline{x}_j_1, x_j_2\right)\)

\(\left(x_j_1, \overline{x}_j_2\right)\)

[ Goes back to Hamada (1974) ]
Existing literature:
Formally

Definition (2-rearrangement)

Let $X$ and $Y$ be in $\mathcal{X}$. Consider a rectangle $B_2 = [\underline{x}_{j_1}, \overline{x}_{j_1}] \times [\underline{x}_{j_2}, \overline{x}_{j_2}]$ whose vertices are in $S$, with $\underline{x}_j < \overline{x}_j$ for all $j \in \{j_1, j_2\}$. If $Y$ can be obtained from $X$ by adding a positive probability mass $\varepsilon$ to all vertices of the rectangle $B_2$ with an even number of components $x_j = \underline{x}_j$ and subtracting $\varepsilon$ from all vertices of the rectangle $B_2$ with an odd number of components $x_j = \overline{x}_j$, then $Y$ is obtained from $X$ by a positive 2-rearrangement.
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**Axiom (2-dependence principle (2-DEP))**

Let $X$ and $Y$ be in $\mathcal{X}$. If $Y$ can be obtained from $X$ by a finite sequence of positive 2-rearrangements, then $X \prec Y$. 
Existing literature:

An important result

Proposition (Epstein and Tanny (1980))

Let $X$ and $Y$ be in $\mathcal{F}$ with support $S$ and suppose $m = 2$. The dependence ordering $\prec$ on $\mathcal{F}$ satisfies 2DEP if and only if $X \prec Y$ is equivalent to:

1. $\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m)$ for all 2-increasing utility functions $U$,
2. $F_X(x_1, \ldots, x_m) \leq F_Y(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $J(S)$,
3. $\bar{F}_X(x_1, \ldots, x_m) \leq \bar{F}_Y(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $M(S)$.

Strong and useful result (it combines three perspectives)
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- However, there are also two “inconveniences”:
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- Strong and useful result (it combines three perspectives)
- However, there are also two “inconveniences”:
  - Only bivariate random vectors
  - Only random vectors with the same marginal distributions
Structure of the talk

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<thead>
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<tbody>
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1. Existing Literature: Epstein and Tanny (1980)
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2. Step 1: Beyond the bivariate case
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3. **Step 2**: Different marginal distributions
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4. Conclusion
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2. **Step 1: Beyond the bivariate case**
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4. Conclusion
Step 1. Beyond the bivariate case:
The supermodular ordering

- Natural approach: relax the premise

**Proposition (supermodular dependence ordering)**

Let X and Y be in \( \mathcal{F} \) with support S and suppose \( m \geq 2 \). The dependence ordering \( \prec \) on \( \mathcal{F} \) satisfies 2DEP if and only if \( X \prec Y \) is equivalent to

\[
\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m) \text{ for all } 2-\text{increasing utility functions } U.
\]
Step 1. Beyond the bivariate case:
The supermodular ordering

An example of a rearrangement that leads to $F_X(x_1, \ldots, x_m) \leq F_Y(x_1, \ldots, x_m)$, but can never be reached by positive 2-rearrangements:

$$
\begin{array}{c|c|c}
+\varepsilon & -\varepsilon \\
\hline
(x_{j1}, x_{j2}, x_{j3}) & (x_{j1}, x_{j2}, \bar{x}_{j3}) \\
(x_{j1}, \bar{x}_{j2}, \bar{x}_{j3}) & (x_{j1}, \bar{x}_{j2}, x_{j3}) \\
(\bar{x}_{j1}, x_{j2}, \bar{x}_{j3}) & (\bar{x}_{j1}, x_{j2}, x_{j3}) \\
(\bar{x}_{j1}, \bar{x}_{j2}, x_{j3}) & (\bar{x}_{j1}, \bar{x}_{j2}, \bar{x}_{j3})
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$$
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$$

- Two problems:

How to test this? (See Athey, 2000)

Why only confining attention to rearrangements involving 2 dimensions?
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- Natural approach: relax the premise

**Proposition (supermodular dependence ordering)**

Let $X$ and $Y$ be in $\mathcal{F}$ and suppose $m \geq 2$. The dependence preorder $\prec$ on $\mathcal{F}$ satisfies 2DEP if and only if $X \prec Y$ is equivalent to

$$
\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m) \quad \text{for all increasing utility functions } U.
$$

- Two problems:
  - How to test this? (See Athey, 2000)
  - Why only confining attention to rearrangements involving 2 dimensions?
Step 1. Beyond the bivariate case
A more general type of rearrangements

Definition (positive $k$-rearrangement.)

Let $X$ and $Y$ be in $\mathcal{X}$ and suppose $m \geq k \geq 2$. Consider a hyperbox $B_k = [x_{j_1}, \bar{x}_{j_1}] \times \cdots \times [x_{j_k}, \bar{x}_{j_k}]$ whose vertices are in $S$, with $x_j < \bar{x}_j$ for all $j \in \{j_1, \ldots, j_k\}$. If $Y$ can be obtained from $X$ by adding positive probability mass $\varepsilon$ to all vertices of the rectangle $B_k$ with an even number of components $x_j = x_j$ and subtracting $\varepsilon$ from all vertices of the rectangle $B_k$ with an odd number of components $x_j = \bar{x}_j$, then $Y$ is obtained from $X$ by a positive $k$-rearrangement.

- Notational convention:
  - when $k$ is even (odd), we refer to an even (odd) rearrangement
Step 1. Beyond the bivariate case
A more general type of rearrangements

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- Notational convention:
  - when $k$ is even (odd), we refer to an even (odd) rearrangement
  - when $\varepsilon$ is positive (negative), we refer to a positive (negative) rearrangement.
Step 1. Beyond the bivariate case
A more general type of rearrangements

- An example of a positive 4 rearrangement:

<table>
<thead>
<tr>
<th>+ε</th>
<th>-ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>($X_1$, $X_2$, $X_3$, $X_4$)</td>
<td>($X_1$, $X_2$, $X_3$, $X_4$)</td>
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- Vague notion of positive dependence: The extent to which “high” and “low” realizations in the different dimensions of a multivariate distribution occur together
Step 1. Beyond the bivariate case
A more general type of rearrangements

- An example of a positive 4 rearrangement:

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<tr>
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- Vague notion of positive dependence: The extent to which “high” and “low” realizations in the different dimensions of a multivariate distribution occur together.
- Positive 4 (even) rearrangements lead to more dependence.
Step 1. Beyond the bivariate case
A more general type of rearrangements

- An example of a positive 3 rearrangement:

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<tr>
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Step 1. Beyond the bivariate case
A more general type of rearrangements

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- On the other hand: It is not obvious that positive 3 (odd) rearrangements lead to more dependence.
Step 1. Beyond the bivariate case
A more general type of rearrangements

**Axiom (k-dependence principle (k-DEP))**

Let $X$ and $Y$ be in $\mathcal{X}$ and suppose $m \geq k \geq 2$. If $Y$ can be obtained from $X$ by a finite sequence of positive $k$-rearrangements, then $X \preceq Y$. 
Step 1. Beyond the bivariate case
A more general type of rearrangements

Axiom (k-dependence principle (k-DEP))
Let $X$ and $Y$ be in $\mathcal{X}$ and suppose $m \geq k \geq 2$. If $Y$ can be obtained from $X$ by a finite sequence of positive $k$-rearrangements, then $X \prec Y$.

Axiom (k-dependence principle* (kDEP*))
Let $X$ and $Y$ be in $\mathcal{X}$ and suppose $m \geq k \geq 2$. If $Y$ can be obtained from $X$ by a finite sequence of positive even-rearrangement or negative odd-rearrangements, then $X \prec Y$. 
Step 1. Beyond the bivariate case:
Orthant dependence orderings

Proposition (orthant dependence orderings)

Let $X$ and $Y$ be in $\mathcal{F}$ with support $S$ and suppose $m \geq 2$.
The upper orthant dependence ordering $\prec_{UO}$ on $\mathcal{F}$ satisfies $k$DEP if and only if $X \prec_{UO} Y$ is equivalent to:

1. $\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m)$, for all $k$-increasing utility functions $U$,

2. $F_X(x_1, \ldots, x_m) \leq F_Y(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $M(S)$.

The lower orthant dependence ordering $\prec_{LO}$ on $\mathcal{F}$ satisfies $k$DEP* if and only if $X \prec_{LO} Y$ is equivalent to:

1. $\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m)$, for all even-increasing and odd-decreasing utility functions $U$,

2. $F_X(x_1, \ldots, x_m) \leq F_Y(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $J(S)$. 
Step 1. Beyond the bivariate case: Concordance dependence ordering

Proposition (concordance dependence ordering)

Let $X$ and $Y$ be in $\mathcal{F}$ and suppose $m \geq 2$. The concordance dependence ordering $\prec_C$ on $\mathcal{F}$ satisfies $kDEP$ and $kDEP^*$ if and only if $X \prec_C Y$ is equivalent to:

1. $\int U(x_1, \ldots, x_m) dF_X(x_1, \ldots, x_m) \leq \int U(x_1, \ldots, x_m) dF_Y(x_1, \ldots, x_m)$, for all even-increasing and odd-increasing or odd-decreasing utility functions $U$,

2. $F_X(x_1, \ldots, x_m) \leq F_Y(x_1, \ldots, x_m)$ and $\bar{F}_X(x_1, \ldots, x_m) \leq \bar{F}_Y(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $M(S) \cup J(S)$. 

Koen Decancq (KUL and CORE)
## Structure of the talk

<table>
<thead>
<tr>
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<th>Same Margins</th>
<th>Different Margins</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bivariate</strong></td>
<td>Epstein and Tanny (1980)</td>
<td></td>
</tr>
<tr>
<td><strong>Multivariate</strong></td>
<td>Step 1.</td>
<td>Step 2.</td>
</tr>
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</table>

1. Existing Literature: Epstein and Tanny (1980)
2. Step 1: Beyond the bivariate case
3. **Step 2: Different marginal distributions**
4. Conclusion
Step 2. Different Margins:
The copula as a useful tool

- Extend ordering by an invariance principle (that defines equivalence classes):
Step 2. DifferentMargins:
The copula as a useful tool

- Extend ordering by an invariance principle (that defines equivalence classes):

**Axiom (Scale Invariance principle (INV))**

Let $X$ and $Y$ be in $\mathcal{X}$ and let $T_1, \ldots, T_m$ be strictly increasing transformation functions. If $Y = (T_1(X_1), \ldots, T_m(X_m))$, then $X \sim Y$. 

Schweizer and Walde (1981): "...it is precisely the copula which captures those properties of the joint distribution which are invariant under strictly increasing transformations".
Step 2. Different Margins:
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- Extend ordering by an invariance principle (that defines equivalence classes):

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**Definition (copula function)**

An $m$-dimensional copula function $C$ is an $m$-dimensional distribution function whose one-dimensional marginal distribution functions follow a standard uniform distribution function.
Step 2. Different Margins:
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### Axiom (Scale Invariance principle (INV))

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### Definition (copula function)

An $m$-dimensional copula function $C$ is an $m$-dimensional distribution function whose one-dimensional marginal distribution functions follow a standard uniform distribution function.

- Similarly we define $\overline{C}$
Step 2. Different Margins:
Why is the copula a useful and popular tool?

Theorem (Sklar, 1959)
Let $X$ be in $\mathcal{X}$ with joint distribution function $F_X$ and marginal distribution functions $F_1, \ldots, F_m$. Then there exists a copula function $C_X$ such that for all $x$ in $\mathbb{R}^m$:

$$F_X(x_1, \ldots, x_m) = C_X(F_1(x_1), \ldots, F_m(x_m)).$$  \hspace{1cm} (1)

Moreover, $C_X$ is uniquely determined on $\text{Range}(F_1) \times \cdots \times \text{Range}(F_m)$. 

Examples
the independence copula:
$C_\Phi(p_1, \ldots, p_m) = p_1 \cdots p_m$
the comonotonic copula:
$C_\tau(p_1, \ldots, p_m) = \min(p_1, \ldots, p_m)$
Step 2. Different Margins:
Why is the copula a useful and popular tool?

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- the independence copula: $C_\perp(p_1, \ldots, p_m) = p_1 \times \cdots \times p_m$
Step 2. Different Margins:
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Examples

- the independence copula: $C_\perp(p_1, \ldots, p_m) = p_1 \times \cdots \times p_m$
- the comonotonic copula: $C_+(p_1, \ldots, p_m) = \min \{p_1, \ldots, p_m\}$
Step 2. Different Margins:
Copula-based concordance dependence ordering

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Let $X$ and $Y$ be in $\mathcal{F}$ with support $S$ and suppose $m \geq 2$. The concordance dependence ordering $\prec_C$ on $\mathcal{F}$ satisfies INV, $k\text{DEP}$ and $k\text{DEP}^*$ if and only if $X \prec_C Y$ is equivalent to:

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Koen Decancq (KUL and CORE)
Structure of the talk

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Conclusion:
What did we do with the ingredients?
Conclusion:
What did we do with the ingredients?

1. Measurement of dependence between many dimensions:
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1 Measurement of dependence between many dimensions:
\[ \Rightarrow \text{Multivariate dependence increasing rearrangements} \]
Conclusion:
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2. The marginal distributions of multivariate distribution can change
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What did we do with the ingredients?

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2. The marginal distributions of multivariate distribution can change
   ⇒ *The copula as a useful tool*
Conclusion:
What can be done/has to be done with the ingredients?

1. Measurement of dependence between many dimensions:
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1. Measurement of dependence between many dimensions:
   \[ \Rightarrow \text{Rearrangements behind other dependence orderings?} \]
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   ⇒ Is the invariance principle not too strong?
   ⇒ What if we impose a linear invariance principle?
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   ⇒ *Is the invariance principle not too strong?*
   ⇒ *What if we impose a linear invariance principle?*

Applications!