A Two-Factor Asset Pricing Model based on the Fat Tail Distribution of Firms Sizes

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✓ Asset pricing is a major component of economic theory and practice.
✓ The International Financing Reporting Standards (IFRS) formerly the International Accounting Standard (IAS) requires that firms’ liabilities be valued at market value.
✓ Asset pricing is involved in
  • investment analysis,
  • capital budgeting,
  • merger and acquisition transactions,
  • financial reporting,
  • tax liability and litigation, ….
✓ Price is set by supply-demand (equilibrium), consumption preference, no arbitrage
✓ Present value of future dividends (time-preference and discount factor)
✓ Behavior and "convention", …
The EMH’s concept of informational efficiency has a Zen-like, counter-intuitive flavour to it:

The more efficient the market, the more random the sequence of price changes generated by such a market.

The most efficient market of all is one in which price changes are completely random and unpredictable.
Asset pricing models

General prediction:
- Only non-diversifiable risks are remunerated
- Excess returns ~ load on factors;

- The CAPM \[ E[r_i - r_0] = \beta_i \times E[r_m - r_0] \]
  - Assumption: equilibrium

- The APT \[ r_i = \alpha + \beta_1 \times f_1 + \cdots + \beta_p \times f_p + \varepsilon_i \]
  \[ E[r_i - r_0] = \beta_1 \times \pi_1 + \cdots + \beta_p \times \pi_p \]
  - Assumption: no arbitrage opportunity

- Compatibility:
  - each asset as an infinitesimal weight the economy
  - mean-variance efficiency of the replicating portfolios
The pricing anomalies

- **Small firm effect** (Banz 1981)
- **Book-to-market** (Stattman 1980, Roseberg, Reid and Lanstein 1985, Daniel & Tittman 1997)
- **Reversal of long term returns** (DeBondt and Thaler 1985, 1987)
- **Continuation of short-term trends** (Jegadeesh and Titman 1993)
- **Preference for skewness** (Rubinstein 1973, Harvey and Siddique 2000)
- **Excess volatility** (Shiller, 1982)
- ...
- **Fama and French three factor model** (1993, 1995)
Small-minus-Big vs. Book-to-Market

Dollars (log scale)

High-minus low book-to-market

Small minus big

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Due to the fat tail nature of the distribution of firm sizes, the market portfolio is not well-diversified:

\[ H_N = \| \tilde{\mathbf{w}}_m \|^2 = \sum_{i=1}^{N} w_{m,i}^2 \rightarrow 0 \]

There exists a diversification premium related to the non-diversified nature of the market portfolio.

The internal consistency of linear factor models allows to account very naturally for the existence of a diversification factor.
Executive Summary

- The diversification factor (ICC or Zipf factor) can be closely related to the Size factor (SMB) introduced by Factor and French,

- To some extent, the diversification Zipf factor is also related to the book-to-market (HML) effect,

- The Fama-French three factor model does not provide a significant improvement, neither in terms of $R^2$ nor in terms of $\alpha$, with respect to our two factor model.
Our result is based on (i) the “internal consistency” condition that, in a complete market, the market portfolio is constituted of the assets whose returns it is supposed to explain and (ii) the distribution of the capitalization of firms is sufficiently heavy-tailed.

Ingredient (i) leads mechanically to correlations between return residuals which are equivalent to the existence of a new “internal consistency” factor.

By the generalized central limit theorem, ingredient (ii) ensures that the internal consistency factor does not disappear even for infinite economies and may produce significant undiversifiable risks for arbitrary well-diversified portfolios. This new risk should be remunerated and therefore be a new factor.

The new self-consistency Zipf factor provides a rationalization of the SMB (Small Minus Big) factor and of the HML (High-minus-Low Book-to-Market) factor introduced by Fama and French (1993).
Power Law Distributions in Nature and Society

- Power Laws appear in many systems
- Probability Distribution Function given by
  \[ p(x) \sim 1/ x^{1+\mu} \]
- Power laws can be explained by several underlying mechanisms
- One can deduce special features of these mechanisms, for instance by inspecting local deviation from power laws
Heavy Tails

- Probability density function \( f(x) \)

\[
\Pr\{a \leq x \leq b\} = \int_a^b f(x)\,dx
\]

- The distribution is called heavy tailed if it has infinite second moment:

\[
\int_{-\infty}^{\infty} x^2 f(x)\,dx = \infty
\]

- For power-law distributions (Pareto distributions):

\[
F(x) := \Pr\{\xi \leq x\} = 1 - x^{-\beta}, x \geq 1
\]

\[
f(x) := \frac{dF(x)}{dx} = \beta x^{-\beta - 1}, x \geq 1
\]

\[
\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_1^\infty x^2 \beta x^{-\beta - 1}\,dx = \frac{\beta}{2 - \beta} x^{2 - \beta}\bigg|_1^{\infty} = \infty \Rightarrow \beta < 2
\]

- Note also that if \( \beta \leq 1 \) \[
\int_1^\infty x\beta x^{-\beta - 1}\,dx = \frac{\beta}{1 - \beta} x^{1 - \beta}\bigg|_1^{\infty} = \infty
\]

\text{(Infinite expectation)}
Heavy tails in pdf of earthquakes

\[ \log N = a - \frac{2}{3} \log M_D \]

Harvard catalog

Heavy tails in pdf of seismic rates

SCEC, 1985-2003, m\geq2, grid of 5x5 km, time step=1 day

(Saichev and Sornette, 2005)

Heavy tails in ruptures

(CNES, France)

Heavy tails in pdf of rock falls, Landslides, mountain collapses

(a) Areas of 11,000 landslides triggered by January 17, 1994 Northridge, California earthquake

\[ \log(-\frac{dN_{CL}}{dA_L}) = -2.28 \log(A_L) + 0.13, r^2 = 0.995 \]

Turcotte (1999)
Heavy tails in pdf of forest fires

Fig. 2. Noncumulative frequency-area distributions for actual forest fires and wildfires in the United States and Australia: (A) 4274 fires on U.S. Fish and Wildlife Service lands (1988–1995) (9), (B) 120 fires in the western United States (1850–1960) (10), (C) 164 fires in Alaskan boreal forests (1990–1991) (11), and (D) 298 fires in the ACT (1985–1991) (12). For each data set, the noncumulative number of fires per year (–dC/dA) with area (A) is given as a function of A (13). In each case, a reasonably good correlation over many decades of A, is obtained by using the power-law relation ([Eq. 1] with α = 1.21 to 1.40) -- α is the slope of the best-fit line in log-log space and is shown for each data set.


Heavy tails in pdf of Solar flares

(Newman, 2005)

Heavy tails in pdf of Hurricane losses

Damage values for top 30 damaging hurricanes normalized to 1995 dollars by inflation, personal property increases and coastal county population change

Y = M0*X^M1

M0  57911
M1 -0.80871
R  0.97899

Heavy tails in pdf of rain events

(Neuman, 2005)

Peters et al. (2002)
After-tax present value in millions of 1990 dollars

Heavy-tail of price financial returns

Heavy-tail of pharmaceutical sales

Heavy-tail of crash losses (drawdowns)

Heavy-tail of movie sales
Heavy-tail of pdf of book sales

Heavy-tail of pdf of health care costs

Heavy-tail of pdf of terrorist intensity

Heavy-tail of pdf of war sizes
Numbers of occurrences of words in the novel Moby Dick by Hermann Melville.

Numbers of citations to scientific papers published in 1981, from time of publication until June 1997.

Numbers of hits on web sites by 60,000 users of the America Online Internet service for the day of 1 December 1997.

Numbers of copies of bestselling books sold in the US between 1895 and 1965.

Number of calls received by AT&T telephone customers in the US for a single day.

Diameter of craters on the moon. Vertical axis is measured per square kilometre.

Peak gamma-ray intensity of solar flares in counts per second, measured from Earth orbit between February 1980 and November 1989.

Intensity of wars from 1816 to 1980, measured as battle deaths per 10 000 of the population of the participating countries.

Frequency of occurrence of family names in the US in the year 1990.

MECHANISMS FOR POWER LAWS

1. percolation, fragmentation and other related processes,

2. directed percolation and its universality class of so-called “contact processes”,

3. cracking noise and avalanches resulting from the competition between frozen disorder and local interactions, as exemplified in the random field Ising model, where avalanches result from hysteretic loops [34],

4. random walks and their properties associated with their first passage statistics [35] in homogenous as well as in random landscapes,

5. flashing annihilation in Verhulst kinetics [36],

6. sweeping of a control parameter towards an instability [25, 37],

7. proportional growth by multiplicative noise with constraints (the Kesten process [38] and its generalization for instance in terms of generalized Lotka-Volterra processes [39], whose ancestry can be traced to Simon and Yule,

8. competition between multiplicative noise and birth-death processes [40],
9. growth by preferential attachment [32],

10. exponential deterministic growth with random times of observations (which gives the Zipf law) [41],

11. constrained optimization with power law constraints (HOT for highly optimized tolerant),

12. control algorithms, which employ optimal parameter estimation based on past observations, have been shown to generate broad power law distributions of fluctuations and of their corresponding corrections in the control process [42, 43],

13. on-off intermittency as a mechanism for power law pdf of laminar phases [44, 45],

14. self-organized criticality which comes in many flavors:
   • cellular automata sandpiles with and without conservation laws,
   • systems made of coupled elements with threshold dynamics,
   • critical de-synchronization of coupled oscillators of relaxation,
   • nonlinear feedback of the order parameter onto the control parameter
   • generic scale invariance,
   • mapping onto a critical point,
   • extremal dynamics.

Zipf’s law

- Particular case of power law with $\mu = 1$
- Describes the inverse proportionality between the variable and its rank
- Borderline regime for the mean of the rv
- Zipf’s law has been documented for
  - distribution of word frequency in natural languages (G.K. Zipf, 1949)
  - distribution of city sizes (X. Gabaix, 1999)
  - Internet traffic & web access statistics (L.A. Adamic and B.A. Huberman, 2000, etc...)
  - etc... (L.A. Adamic and B.A. HUBerman, Glottometrics, 2002)
  - distribution of number of species per genera
  - Open-source software package in-degree connectivity (Maillart et al., 2008)
Zipf’s law

City sizes

Firm sizes

X. Gabaix, Quart. J. of Economics, 1999

Adamic and Huberman (2002)

Distribution of Packages Dependencies in OSS

- “centrality” of a given package
  - # of other packages that it
  - in-directed links (thereafter “links”)
- Distribution of links obeys a Zipf’s law
  - over 4 orders of magnitude
  - stable over time (2005-2008)

T. Maillart, D. Sornette, S. Spaeth and G. von Krogh,
Empirical Tests of Zipf’s law Mechanism In Open Source Linux Distribution,

same for RED HAT LINUX
(Challet & Lombardoni,
Proportional Growth and Zipf’s Law

• Important links btw. Zipf’s law and Stochastic Growth
  – Yule’s theory of the power law distribution of the number of species (1924)
  – Champernowne’s theory of stochastic recurrence equations (1953)

• Gibrat of proportionate effect (Librairie du Recueil, 1931)

• H.A Simon (in Biometrika, 1955)
  – simple mechanism for Zipf’s law based on Gibrat,
  – implemented a stochastic growth model with new entrants

• Recently rediscovered
  – “Preferential Attachment” (Barabasi et Albert, 1999)
Zipf’s law for firms: relevance of birth and death processes

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A power law with unit tail index $\mu=1$

$$\Pr [S \geq s] = \frac{1}{s^\mu} \cdot 1_{s \geq 1}$$


**Figure 1**: PDF of U.S. firm sizes, 1997 Economic Census data (Axtell 2006)

**Figure VII**: Cumulative distribution of the size (assets under management) of the top mutual funds in 1999. Source: Center for Research on Security Prices (Gabaix et al. 2006)
Distribution of US University endowments

\[ \ln(\text{Rank}) = \alpha \ln(\text{Endowment}) + \beta \]
\[ \alpha \approx -1 \]

Source: US Colleges and Universities with endowments greater than $1 billion in 2004
A power law with unit tail index


- Robustness vis-à-vis the proxy of the firm size: assets, market capitalizations, number of employees, profits, revenues, sales, value added...

- Several models: the law of proportional effect, economies of scale and costs reduction, the distribution of managerial talents and efficient allocation of productivity factors across managers, the partition of the set of workers...
Consequence on the concentration of the market portfolio

- The market portfolio: value-weighted portfolio of all the assets traded on the market

- Vector of composition: \( \vec{w}_m = \left( w_{m,1}, \ldots, w_{m,N} \right) \)

- Definition: A portfolio is well-diversified if

\[
H_N = \left\| \vec{w}_m \right\|^2 = \sum_{i=1}^{N} w_{m,i}^2 \xrightarrow{N \to \infty} 0
\]
Consequence on the concentration of the market portfolio

- Consider an economy of \( N \) firms, whose sizes \( S_i, \quad i = 1, \ldots, N \), are drawn from a Pareto law with tail index \( \mu \).

- Let \( w_{1,N} = \frac{\max S_i}{\sum_{i=1}^N S_i} \),

we have:

\[
\Pr [S \geq s] = \frac{1}{s^\mu} \cdot 1_{s \geq 1}
\]

\[
\lim_{N \to \infty} \mathbb{E}[w_{1,N}] = 0, \quad \text{as } \mu \geq 1
\]

\[
\lim_{N \to \infty} \mathbb{E}[1/w_{1,N}] = \frac{1}{1-\mu}, \quad \text{as } \mu < 1
\]
Consequence on the concentration of the market portfolio

Example 1: let the sizes, sorted in descending order, of the $N$ firms be given by

$$S_{i,N} = \left( \frac{i}{N} \right)^{-1/\mu}$$

Then:

$$w_{m,1} \rightarrow 0, \quad \text{if } \mu \geq 1,$$

$$w_{m,1} \rightarrow \frac{1}{\zeta(1/\mu)}, \quad \text{if } \mu < 1,$$

where $\zeta$ denotes the Riemann zeta function $\zeta(z) = \sum_{i=1}^{\infty} i^{-z}$.
Consequence on the concentration of the market portfolio

- **Example 1**: let the sizes, sorted in descending order, of the $N$ firms be given by $S_{i,N} = \left( \frac{i}{N} \right)^{-1/\mu}$

Then

$$H_N = \left\{ \begin{array}{ll}
\frac{1}{1-\frac{(1-\mu)^2}{\ln N + \gamma}} \cdot \frac{1}{N} + O \left( N^{-2/\mu-2} \right), & \mu > 2, \\
\frac{1}{4N} + O \left( N^{-3/2} \ln N \right), & \mu = 2, \\
\left( \frac{1-\mu}{\mu} \right)^2 \zeta(2/\mu) \cdot \frac{1}{N^{2-2/\mu}} + O \left( N^{3(1/\mu-1)} \right), & 1 < \mu < 2, \\
\frac{\pi^2}{6 (\gamma + \ln N)^2} + O \left( N^{-1}(\gamma + \ln N)^{-2} \right), & \mu = 1, \\
\frac{\zeta(2/\mu)}{\zeta(1/\mu)^2} + O \left( N^{1-1/\mu} \right), & \mu < 1.
\end{array} \right.$$
Consequence on the concentration of the market portfolio

- **Example 1:** let the sizes, sorted in descending order, of the $N$ firms be given by

\[ S_{i,N} = \left( \frac{i}{N} \right)^{-1/\mu} \]

Plain line: $N=\infty$; Dotted line: $N=1,000$; Dash-dotted line: $N=10,000$. 

\[ W_{m,1} \]

\[ 1/H \]

\[ \mu \]

\[ \mu \]
Consequence on the concentration of the market portfolio

- Example 2: let the firm sizes be randomly drawn from a power law distribution of size with tail index $\mu$, i.e.

$$s^\mu \cdot \Pr[S > s] \to c \text{ as } s \to \infty,$$

$$H_N = \begin{cases} 
\frac{1}{N} \frac{E[S^2]}{E[S]^2} + o_p(1/N), \\
\frac{c}{E[S]^2} \frac{\ln N}{N} + o_p \left( \frac{1}{N \ln N} \right), \\
\left[ \frac{\pi c}{2 \Gamma \left( \frac{\mu}{2} \right) \sin \frac{\mu \pi}{4} \left( N^2 - 2/\mu \right)^{1/\mu} \xi_N + o_p \left( \frac{1}{N^{2-2/\mu}} \right) \right]^{2/\mu} \xi_N + o_p \left( \frac{1}{N^{2-2/\mu}} \right), \\
\frac{\pi}{2 \ln^2 N} \cdot \xi_N + o_p \left( \frac{1}{\ln^3 N} \right), \\
\frac{4}{\pi^{1/\mu}} \left[ \Gamma \left( \frac{1+\mu}{2} \right) \cos \frac{\pi \mu}{4} \right]^{2/\mu} \xi_N, \\
\frac{1}{N^{2-2/\mu}} \cdot \xi_N 
\end{cases}$$

provided that $E[S^2] < \infty$,
Consequence on the concentration of the market portfolio

- Example 2: case $\mu=1$,

$$H_N \sim \frac{\pi}{2 \cdot (\ln N)^2} \cdot \xi_N,$$

$\xi_N$ is a sequence of positive random variables with stable limit law $S(1/2,1)$, i.e., the Levy law

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot x^{-3/2} e^{-\frac{1}{2x}}$$

Density of the Levy Law
Consequence on the concentration of the market portfolio

- Example 2: case $\mu=1$, 

$$H_N \simeq \frac{\pi}{2 \cdot (\ln N)^2} \cdot \xi_N,$$

with $\xi_N \simeq 2.198$, a typical value of $H_N$ is 4-5% for a market where 7000 to 8000 assets are traded.

$H_N \simeq 4-5\%$ means that there are only about 20-25 independent lines in a typical portfolio supposedly well-diversified on 7000 - 8000 assets.
Fraction of total capitalization of F-quantile of upper firm sizes is $F^{(\mu-1)/\mu}$.

- For power laws with $\mu=1.4$, 20% of the largest firms account for 80% of the total capitalization.

- For power laws with $\mu=1.01$, 0.1% of the largest firms account for 95% of the total capitalization.
Consequence on the concentration of the market portfolio

- Two questions:
  - how can the market portfolio alone explain the expected return on any asset, irrespective of its size, as predicted by the CAPM?
  - is it actually optimal for a rational investor to put her money in this risky portfolio alone, as suggested by the theorem of separation in two funds?
Consequence on the concentration of the market portfolio

- Our claims:
  - the lack of diversification of the market portfolio is responsible, to a large extent, for the failure of the CAPM to explain the cross-section of stock returns,
  - In addition to the market premium, investors require a concentration premium.
- Departure from the “traditional” explanations in terms of macro-economic factors, firm-specific factors, or behavioral factors.
Consequence on the concentration of the market portfolio

A justification:

- Most of these factors provide a significant improvement in explaining the cross-section of asset returns.
- BUT, they do not provide a clear identification of the most prominent ones.
- Our approach focuses on the undisputable fact that the market portfolio is highly concentrated on a small number of very large companies and therefore cannot account for the behavior of the smallest ones.
Consider an economy with $N$ firms whose returns on stock prices are determined according to the following equation

$$
\vec{r} = \vec{\alpha} + \vec{\beta}_m \cdot [r_m - \mathbb{E}[r_m]] + B\vec{\phi} + \vec{\varepsilon},
$$

- $\vec{r}$ is the random $N \times 1$ vector of asset returns;
- $\vec{\alpha} = \mathbb{E}[\vec{r}]$ is the $N \times 1$ vector of asset return mean values. We do not make any assumption neither on the ex-ante mean-variance efficiency of the market portfolio, nor on the absence of arbitrage opportunity, so that $\vec{\alpha}$ is not, a priori, specified;
- $r_m$ is the random return on the market portfolio;
- $\vec{\beta}_m$ is the $N \times 1$ vector of the factor loadings of the market factor;
- $\vec{\phi}$ is the random $N \times 1$ vector of risk factors $\phi_i$ which are assumed to have zero mean ($\mathbb{E}[\phi_i] = 0$), unit variance, are uncorrelated with each other and with $r_m$;
- $B$ is the $N \times q$ matrix of factor loadings;
- $\vec{\varepsilon}$ is the random $N \times 1$ vector of disturbance terms with zero average $\mathbb{E}[\vec{\varepsilon}] = 0$ and covariance matrix $\Omega = \mathbb{E}[\vec{\varepsilon} \cdot \vec{\varepsilon}]$. The disturbance terms are assumed to be uncorrelated with the market return $r_m$ and the factors $\phi_i$.  

Accounting for the fact that

\[ r_m = \bar{w}_m' \cdot \vec{r}. \]

we get

\[
\left[ \bar{w}_t' \cdot \vec{\beta} - 1 \right] \cdot (r_m - \mathbb{E}[r_m]) + w_m' B \vec{\phi} + \bar{w}_m' \cdot \vec{\varepsilon}_t = 0
\]

which allows concluding that

The disturbances terms are correlated

\[ \bar{w}_m' \cdot \vec{\varepsilon}_t = 0 \quad \text{almost surely,} \]

\[ \bar{w}_m' \cdot \vec{\beta} = 1 \quad \text{and} \quad \bar{w}_m' B = 0. \]
Correlation structure of the disturbance terms

- The fact that the disturbance terms are correlated means that there exists at least one common “factor” $f$ to the $\varepsilon$’s:

$$\varepsilon' = \gamma' \cdot f + \eta' ,$$

where $\gamma '$ is the vector of factor loadings.

- Actually, $f$ is not a factor in so far as it cannot be uncorrelated with $\eta$ due to the internal consistency relation $\bar{w}_m' \cdot \bar{\varepsilon}_t = 0$, which yields

$$f = - \frac{\bar{w}_m' \eta}{\bar{w}_m' \gamma'},$$

provided that $\bar{w}_m' \gamma' \neq 0$. (otherwise, we should have $\bar{w}_m' \eta = 0$.)
Correlation structure of the disturbance terms

- The market model becomes

\[ \tilde{r}_t = \bar{\alpha} + \bar{\beta} \cdot r_m(t) + \bar{\gamma} \cdot f + \tilde{\eta}, \]

with:

- \( \text{Cov} \ (r_m, f) = 0, \)
- \( \text{Cov} \ (r_m, \tilde{\eta}) = 0, \)
- \( \text{Var} \ \tilde{\eta} = \Delta, \)
- \( \text{Var} \ f = \frac{\bar{w}_m' \Delta \bar{w}_m}{(\bar{w}_m' \bar{\gamma})^2} \)
- \( \text{Cov} \ (f, \tilde{\eta}) = -\frac{1}{\bar{w}_m' \bar{\gamma}}. \)

(\( \Delta \) can be chosen as a diagonal matrix)
Asymptotic variance of the equally-weighted portfolio

\[
\text{Var } r_e = \begin{cases} 
\beta_e^2 \cdot \text{Var } r_m + O_p(1/N), \\
\beta_e^2 \cdot \text{Var } r_m + \frac{c \cdot \bar{\Delta} \ln N}{E[S]^2} + o_p(\ln N/N), \\
\beta_e^2 \cdot \text{Var } r_m + \left( \frac{\pi c E[\Delta^{1/2}]}{2 \Gamma \left( \frac{\mu}{2} \right) \sin \frac{\mu \pi}{4} \sqrt{\xi N}} \right)^{2/\mu} \frac{1}{E[S]^2} \cdot \frac{1}{N^{2-2/\mu}} \cdot \xi N + o_p \left( \frac{1}{N^{2-2/\mu}} \right) \\
\beta_e^2 \cdot \text{Var } r_m + \frac{\pi E[\Delta^{1/2}]}{2} \frac{E[\gamma]^2}{E[|\gamma|^2]} \frac{1}{\ln^2 N} \cdot \xi N + o_p \left( \frac{1}{\ln^2 N} \right), \\
\beta_e^2 \cdot \text{Var } r_m + E \left[ \Delta^{\mu/2} \right]^{2/\mu} \frac{E[\gamma]^2}{E[|\gamma|^2]} \frac{4}{\pi^1/\mu} \left[ \Gamma \left( \frac{1 + \mu}{2} \right) \cos \frac{\pi \mu}{4} \right]^{2/\mu} \frac{\xi N}{\xi N^2} + o_p(1), \\
\end{cases}
\]

provided that \( E[S^2] < \infty \),

\[ \mu = 2 \]

\[ \mu \in (1, 2) \]

\[ \mu = 1 \]

\[ \mu \in (0, 1) \]

Specific market risk

Non-diversified risk

Additional contribution:

\[
\text{Var } r_e \overset{\text{law}}{=} \beta_e^2 \cdot \text{Var } r_m + K_\mu \cdot H_N
\]

\[ K_\mu / N_{\text{eff}} \]
Average R² of the regression of the returns of 20 equally weighted portfolios

\[ \mathbf{P}(S) \sim 1/S^{1+\mu} \quad \mu=1 \]

\[ \mathbf{\bar{r}}_t = \bar{\alpha} + \beta \cdot \mathbf{r}_m(t) + \gamma \cdot \mathbf{f} + \mathbf{\eta} \]

\[
\text{Var } r_e = \beta_e^2 \cdot \text{Var } \mathbf{r}_m + \gamma_N^2 \cdot \frac{\sum_{i=1}^{N} S_i^2 \Delta_{ii}}{\left(\sum_{i=1}^{N} S_i \gamma_i\right)^2} + O_p(1/N).
\]

<table>
<thead>
<tr>
<th>N=1000</th>
<th>N=10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu=2)</td>
<td>94%</td>
</tr>
<tr>
<td>(\mu=1)</td>
<td>80%</td>
</tr>
<tr>
<td>(\mu=0.5)</td>
<td>56%</td>
</tr>
</tbody>
</table>

D. Sornette – ETH Zurich – http://www.er.ethz.ch/
The market model is:

\[ r_i = \alpha + \beta_1 \times f_1 + \ldots + \beta_p \times f_p + \varepsilon_i \]

Therefore, the APT applies and tell us that

\[
\mathbb{E}[r_i - r_0] = \beta_1 \times \pi_1 + \ldots + \beta_p \times \pi_p
\]

- The market model is:

\[
\tilde{r}_t = \tilde{\alpha} + \tilde{\beta} \cdot r_m(t) + \tilde{\gamma} \cdot f + \tilde{\eta},
\]

- Therefore, the APT applies and tell us that

\[
\mathbb{E}[r_i - r_0] = \beta_i \cdot \mathbb{E}[r_m - r_0] + (\gamma_i - \gamma_m \cdot \beta_i) \cdot \mathbb{E}[r_{icc} - r_0]
\]

where \( r_{icc} \) is the return on the equally-weighed portfolio \( r_e \) minus the return on the market portfolio \( r_m \), which is used as a proxy for \( f \).
Empirical consequences

- Multi-factor time series regression:

\[ r_{i,t} - r_0 = \alpha_i + \beta_i \cdot [r_m(t) - r_0] + \beta_i^{ICC} \cdot r_{icc}(t) + \beta_i^{SMB} \cdot r_{smb}(t) + \beta_i^{HML} \cdot r_{hml}(t) + \varepsilon_i(t) \]

with \( r_{smb} \) and \( r_{hml} \), the two Fama & French factors

- If our specification is correct:

\[ \alpha_i = \beta^{SMB} = \beta^{HLM} = 0 \]
Parameter estimates of the linear regression of the excess returns on 25 equally-weighed portfolios (sorted by quintiles of the distribution of size – Small, 2, 3, 4 and Big – and by quintiles of the distribution of Book equity to Market equity ratio – Low, 2, 3, 4 and High) regressed on the excess return on the market portfolio, on the two Fama-French factors SMB and HML and on the proxy for the additional risk factor due to the internal consistency constraint given by the difference between the return on the equally-weighted portfolio and the return on the market portfolio.

948 months
$R^2$ of the linear regression of the excess returns of 25 equally-weighed portfolios (sorted by quintiles of the distribution of size – Small, 2, 3, 4 and Big – and by quintiles of the distribution of Book equity to Market equity ratio – Low, 2, 3, 4 and High) on the market portfolio (Rm), on the market portfolio and the factor ICC (ICC), on the market portfolio and the size factor (SMB), on the market portfolio and the book to market factor (HML), on the market portfolio and the two Fama & French factors (HML + SMB), on the market portfolio, the factor ICC and the size factor (ICC + SMB), on the market portfolio, the factor ICC and the book to market factor (ICC + HML) and, finally on all these four factors (Market, ICC, SMB and HML). Figures in boldface represent the maximum value of the $R^2$ within the group of regression with two factors (columns ICC, SMB and HML) and with three factors (columns HML + SMB, ICC + SMB and ICC + HML). The two last rows reports Gibbons et al. (1989) test statistics and $p$-values.

<table>
<thead>
<tr>
<th></th>
<th>Rm</th>
<th>ICC</th>
<th>SMB</th>
<th>HML</th>
<th>HML + SMB</th>
<th>ICC + SMB</th>
<th>ICC + HML</th>
<th>All four factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>76.1%</td>
<td>87.3%</td>
<td>84.7%</td>
<td>82.2%</td>
<td>90.6%</td>
<td>88.6%</td>
<td>90.8%</td>
<td>91.4%</td>
</tr>
<tr>
<td></td>
<td>(72.6%,79.7%)</td>
<td>(85.3%,89.3%)</td>
<td>(82.4%,87.2%)</td>
<td>(79.4%,85.3%)</td>
<td>(89.1%,92.2%)</td>
<td>(86.7%,90.6%)</td>
<td>(89.5%,92.2%)</td>
<td>(90.2%,92.8%)</td>
</tr>
<tr>
<td>GRS</td>
<td>4.37</td>
<td>4.11</td>
<td>4.41</td>
<td>4.02</td>
<td>4.07</td>
<td>4.19</td>
<td>3.92</td>
<td>4.06</td>
</tr>
<tr>
<td>p-value</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Parameter estimates of the linear regression of the excess returns on 10 equally-weighted industry portfolios regressed on the excess return on the market portfolio, on the two Fama-French factors SMB and HML and on the proxy for the additional risk factor due to the internal consistency constraint given by the difference between the return on the equally-weighted portfolio and the return on the market portfolio.

Time span: Jan. 1927 – Dec 2005; 948 months

<table>
<thead>
<tr>
<th>Industry Portfolio</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \beta^{\text{SMB}} )</th>
<th>( \beta^{\text{HML}} )</th>
<th>( \beta^{\text{ICC}} )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumer Non Durables</td>
<td>-0.0003</td>
<td>0.84</td>
<td>0.08</td>
<td>0.10</td>
<td>0.77</td>
<td>94%</td>
</tr>
<tr>
<td>Consumer Durables</td>
<td>-0.0024</td>
<td>1.12</td>
<td>0.21</td>
<td>0.07</td>
<td>0.97</td>
<td>92%</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>-0.0004</td>
<td>1.07</td>
<td>0.12</td>
<td>0.17</td>
<td>0.76</td>
<td>97%</td>
</tr>
<tr>
<td>Energy</td>
<td>0.0019</td>
<td>0.95</td>
<td>0.13</td>
<td>0.34</td>
<td>0.55</td>
<td>69%</td>
</tr>
<tr>
<td>Business Equipment</td>
<td>0.0016</td>
<td>1.22</td>
<td>-0.29</td>
<td>-0.65</td>
<td>1.52</td>
<td>92%</td>
</tr>
<tr>
<td>Telecom</td>
<td>0.0030</td>
<td>0.92</td>
<td>-0.30</td>
<td>-0.54</td>
<td>0.98</td>
<td>73%</td>
</tr>
<tr>
<td>Shops</td>
<td>0.0000</td>
<td>0.91</td>
<td>0.11</td>
<td>-0.11</td>
<td>0.93</td>
<td>90%</td>
</tr>
<tr>
<td>Health</td>
<td>0.0037</td>
<td>0.91</td>
<td>-0.04</td>
<td>-0.54</td>
<td>0.92</td>
<td>80%</td>
</tr>
<tr>
<td>Utilities</td>
<td>0.0006</td>
<td>0.85</td>
<td>0.21</td>
<td>0.55</td>
<td>-0.06</td>
<td>66%</td>
</tr>
<tr>
<td>Others</td>
<td>-0.0008</td>
<td>0.95</td>
<td>0.07</td>
<td>0.39</td>
<td>0.93</td>
<td>95%</td>
</tr>
</tbody>
</table>

Reject \( \beta=0 \) at the 1% level
Reject \( \beta=0 \) at the 5% level
$R^2$ of the linear regression of the excess returns of 10 equally-weighed industry portfolios on the market portfolio (Rm), on the market portfolio and the factor ICC (ICC), on the market portfolio and the size factor (SMB), on the market portfolio and the book to market factor (HML), on the market portfolio and the two Fama & French factors (HML + SMB), on the market portfolio, the factor ICC and the size factor (ICC + SMB), on the market portfolio, the factor ICC and the book to market factor (ICC + HML) and, finally on all these four factors (Market, ICC, SMB and HML). Figures in boldface represent the maximum value of the $R^2$ within the group of regression with two factors (columns ICC, SMB and HML) and with three factors (columns HML + SMB, ICC + SMB and ICC + HML). The two last rows reports Gibbons et al. (1989) test statistics and $p$-values.

### 10 equally-weighted industry portfolios

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
<th>Group 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAPM</td>
<td>ICC</td>
<td>Cap</td>
<td>Value</td>
</tr>
<tr>
<td>Consumer Non Durables</td>
<td>75.9%</td>
<td><strong>94.1%</strong></td>
<td>88.4%</td>
</tr>
<tr>
<td>Consumer Durables</td>
<td>74.4%</td>
<td><strong>92.3%</strong></td>
<td>87.9%</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>82.2%</td>
<td><strong>96.7%</strong></td>
<td>92.0%</td>
</tr>
<tr>
<td>Energy</td>
<td>58.3%</td>
<td><strong>67.8%</strong></td>
<td>63.7%</td>
</tr>
<tr>
<td>Business Equipment</td>
<td>74.5%</td>
<td><strong>87.4%</strong></td>
<td>86.2%</td>
</tr>
<tr>
<td>Telecom</td>
<td>62.7%</td>
<td><strong>68.2%</strong></td>
<td>68.1%</td>
</tr>
<tr>
<td>Shops</td>
<td>71.8%</td>
<td><strong>90.1%</strong></td>
<td>86.7%</td>
</tr>
<tr>
<td>Health</td>
<td>65.1%</td>
<td>74.5%</td>
<td><strong>75.9%</strong></td>
</tr>
<tr>
<td>Utilities</td>
<td>58.3%</td>
<td><strong>60.8%</strong></td>
<td>58.9%</td>
</tr>
<tr>
<td>Others</td>
<td>71.9%</td>
<td><strong>92.8%</strong></td>
<td>83.6%</td>
</tr>
<tr>
<td>Average</td>
<td>69.5%</td>
<td><strong>82.4%</strong></td>
<td>79.1%</td>
</tr>
</tbody>
</table>
Relation between ICC and the Fama & French two Factor

- In the presence of $r_{ICC}$, the relevance of the two Fama & French factors does not disappear but is weakened.

- The size effect: by construction ICC and SMB are close; indeed the ICC factor is long in the equally-weighted portfolio and short in the market portfolio, it is therefore long on the small caps and short on the large caps.
Relation between ICC and the Fama & French two Factor

- The book-to-market effect:
  - Empirical evidence: high book-to-market stocks have significantly lower beta’s with respect to the market portfolio compared with low book-to-market stocks.
  - According to our model, the market premium related to the lack of diversification of the market portfolio is

\[
(\gamma_i - \gamma_m \cdot \beta_i) \cdot E[r_{icc} - r_0]
\]

=> *Ceteris paribus*, the internal consistency constraint leads to a higher expected rate of return for stock with a low beta if the term $\gamma_m$ is positive.
Due to the fat tail nature of the distribution of firm size, the market portfolio is not well-diversified:

\[ H_N = \left\| \vec{w}_m \right\|^2 = \sum_{i=1}^{N} w_{m,i}^2 \rightarrow 0 \]

There exist a diversification premium related to the non-diversified nature of the market portfolio,

The internal consistency of linear factor models allows accounting very naturally for the existence of a diversification factor,
The diversification factor (ICC or Zipf factor) can be closely related to the Size factor (SMB) introduced by Factor and French.

To some extent, the diversification factor is also related to the book-to-market (HML) effect.

The Fama-French three factor model does not provide a significant improvement, neither in terms of $R^2$ nor in terms of $\alpha$, with respect to our two factor model (based on the undisputable fact that the market portfolio is highly concentrated on a small number of very large companies).