Impossibility Theorem in Proportional Representation Problem

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Abstract. The study examines general axiomatics of Balinski and Young and analyzes existed proportional representation methods using this approach. The second part of the paper provides new axiomatics based on rational choice models. New system of axioms is applied to study known proportional representation systems. It is shown that there is no proportional representation method satisfying a minimal set of the axioms (monotonicity and neutrality).

Keywords: Proportional representation, axiomatics, impossibility theorem.

PACS: 89.65.Ef

INTRODUCTION

The most extensive studies of proportional representation systems were held in the United States. This is due to two-hundred years history of using different methods of seats distribution in the House of Representatives. Despite the fact that Balinski and Young [1, 2, 3, 4, 5] examined the population size, we rewrite the properties in terms of number of votes and extend their formulation.

Formulation Of Proportional Representation Problem In Terms Of Rational Choice

The parliament is elected by party lists system. Every voter from set N (|N| = n) is characterized by preferences represented by linear order P on a set of parties A (|A| = k). Procedure should determine the representation of each party completing S seats in parliament

\[ F : P^n \rightarrow A^S. \]

The final choice is a set of S alternatives. We assume that \( S > |A| = k \).

The set of voters, who prefers alternative \( X \) to alternative \( Y \) denoted as

\[ V(x, y, \bar{P}) = \{ i \in N \mid (x, y) \in P_i \}. \]

The procedure of proportional representation is characterized by the choice function

\[ C(\bar{P}, A, S) = \{ y \mid y \in F(\bar{P}, A, S) \}. \]

Sometimes it is convenient to characterize the choice as a vector \((s_1, s_2, \ldots, s_k)\), where \( s_j \) - number of seats of the party j, the sum is also equal to S. Accordingly, the vector \((v_1, v_2, \ldots, v_k)\) will be the distribution of votes for the party.
Properties

1. Symmetry.
Distribution of seats does not depend on any characteristics of parties.

2. Uniformity.
Distribution of seats will not change with a proportional increase in the number of votes.

3. Proportionality.
If the problem has an exact solution in integers, it must be a distribution.

For each distribution there is a convergent sequence shares of votes, giving the same distribution of seats.

5. Pairwise fairness.
For each pair of parties $i$ and $j$, $|v_i - s_i| + |v_j - s_j|$ can not be decreased moving one seat from one party to another.

When two parties merge, their representation should not differ by more than one seat.
$$s_x + s_y - 1 \leq s_{x \cup y} \leq s_x + s_y + 1.$$ 

7. Monotonicity of the number of seats.
$$C\left(\tilde{P}, A, S\right) \subseteq C\left(\tilde{P}, A, S + 1\right)$$

8. Monotonicity of the number of votes.
If the number of votes the party $i$ has increased, while in the other parties remained unchanged, the representation of the party should not be reduced.

9. Quota property.
Number of seats should deviate from the quota by not more than one seat.

10. Consistency.

Representation of parties with the same set of votes should not differ by more than one seat.

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<th>TABLE 1. Properties of proportional representation systems.</th>
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Lemma 1

Let $V(x, y, \tilde{P}) \subseteq V(t, z, \tilde{P})$, $x, y, t, z \in A$, procedure satisfies monotonicity, neutrality, then $s_i \geq s_x$ and $s_y \geq s_z$.

Proof.
Consider the profile $\mathcal{P}^n$ in which the alternatives $x, y$ stand in the place of alternatives $t, z$, respectively, then $V(x, y, \bar{P}) \subseteq V(x, y, \bar{P}')$. From the monotonicity property follows $s'_x \leq s'_x$ and $s'_y \geq s'_y$.

According to the neutrality solution should be maintained regardless of the names of alternatives, then $s'_y = s'_x$ and $s'_z = s'_y$. Hence $s_x \leq s_y$ and $s_y \geq s_z$. ■

**Impossibility theorem**

For $n \geq 3 \; k \geq 3$ there is no procedure simultaneously satisfying the properties of monotonicity, neutrality. Proof.

Consider the profile for $n = 3, \ A = \{x, y, z\}$

\[
\begin{array}{ccc}
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

$V(x, y, \bar{P}) \subseteq V(y, z, \bar{P})$ By Lemma 1 it follows $s'_y \geq s'_x$ and $s'_z \geq s'_z$.

Similarly, we find $s'_z \geq s'_y$ and $s'_x \geq s'_x$.

So $s'_z \geq s'_y \geq s'_x \geq s'_z$ leads to $s'_z = s'_y = s'_x$.

This condition is not feasible at $S$ is not a multiple of 3. ■

**ACKNOWLEDGMENTS**

The author is much indebted to Professor Fuad Aleskerov for the invaluable advice during the work. The paper was partially supported by the Scientific Foundation of the State University-Higher School of Economics under grant №10-04-0030 and Decision Choice and Analysis Laboratory.

**REFERENCES**