Marketing via Friends: Strategic Diffusion of Information in Social Networks with Homophily

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Abstract

The paper studies the impact of homophily on the optimal strategies of a monopolist, whose marketing campaign of new product relies on a word of mouth communication. Homophily is a tendency of people to interact more with those who are similar to them. In the model there are two types of consumers embedded into a social network, which differ in friendship preferences and desirable design of product. Consumers can learn about the product directly from an advertisement or from their neighbors. The monopolist chooses the product design and price to influence a pattern of communication among consumers. We find a number of results: (i) for low levels of homophily the product attractive to both types of consumers is preferred to specialized products; (ii) the price elasticity is increasing in homophily; (iii) an increase in the homophily benefits both the monopolist and consumers; and (iv) the product attractive to both types may be optimal even if the monopolist obtains profits only from sales to one type of consumers.

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1 Introduction

In the last decade word of mouth (WOM hereafter) and viral marketing have received a considerable amount of attention from mass-media and the scientific community as efficient

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marketing tools (see for instance Campbell, 2009, Goyal and Galeotti, 2007, Leskovec et al. 2007, and Iribarren and Moro, 2007). The idea that a company can recruit consumers to advertise its products for free is really exciting. The WOM marketing takes an advantage of the natural human's inclination to spread information. A recent study by Reichheld (2003) shows that willingness of consumers to recommend a company to their friends not just augments sales, but by far is the best predictor of a company's growth.

Apart from being efficient when it works, performance of a WOM campaign is quite uncertain. A report by Riley and Wigder (2007) from Jupiter Research reveals that only 15% of viral campaigns are considered to be successful, moreover 55% of companies planned to reduce the use of this tactics next year. This raises the question why companies that face the same network of consumers show so different performance in terms of success of WOM campaigns? In the paper we argue that volatile behavior of WOM marketing campaigns at least partially can be explained by a phenomenon known as "homophily". Homophily is a tendency of people to interact more with those who are similar to them, which has been documented at least since Aristotle's time.[1,2]

Our paper contributes to the WOM literature in two dimensions. First and most importantly, the paper introduces homophily into the network upon which WOM spreads and studies its impact on the optimal strategy of the monopolist. The notion of homophily enriches network structure by specifying a probability of friendship relationships among groups of consumers. Second, the paper extends the monopolist's problem by including product design that affects further WOM communication. To the author's best knowledge product design has not been the subject of academic research in WOM framework.

The description of the model is following. There is a monopolist that introduces a new product to an initially uninformed population of heterogeneous consumers of two types. Consumers are embedded into a social network, which is represented by a random graph with an arbitrary degree distribution. Across types, consumers differ in friendship preferences and desirable design of the product. Within types, consumers differ in a willingness to pay for the product. We model consumers friendship preferences by a linking bias towards types, which represents homophily of the society. Consumers communicate with their friends and learn about the existence of the product from neighbors who already have acquired it. The monopolist knows the degree distribution and homophily level of the society and strategically chooses the price and design of the product. To induce sales the monopolist advertises the product directly to an infinitesimal part of the population and the rest of the population is expected to find out about the product through WOM communication.

[1]In Aristotles Rhetoric and Nichomachean Ethics, he noted that people "love those who are like themselves" (Aristotle 1934, p. 1371).

[2]The term homophily appeared in the sociological literature for the first time in Lazarsfeld and Mertons (1954) who also quoted the proverbial expression - "birds of a feather flock together," which has summarized homophily ever since.

Our analysis begins by examining a necessary conditions on a network structure such that WOM can spread over a significant proportion of the population[3]. This was a case of such remarkable examples of WOM campaigns as diffusion of Hotmail accounts[4] and the advertising campaign of tiny budget movie "The Blair Witch Project"[5]. We find that in the case of sparse networks a sufficiently high level of homophily is a necessary for a successful WOM campaign. High levels of homophily imply that preferences of connected consumers are correlated, which allows the monopolist to develop the product attractive for longer chains of connected consumers.

Next, we turn to the optimal design of the product. The commonly employed assumption in the diffusion literature is that a message to be spread in a network is given, and main focus is upon the effect of network structure on its propagation (for a survey see Geroski, 2000). In contrast, we assume an active role of the monopolist. In the model the monopolist designs a message to the network by choosing the price and characteristic of the product. In our base-line model we find that for sufficiently high levels of homophily, when people mostly interact with those who are similar to them, specialized products designed to target needs of one type of consumers are optimal. However, for sufficiently low levels of homophily, the product attractive for both types of consumers is preferred to specialized design even if there is no cost of producing more than one product. The latter happens, since the majority of links connect consumers of different types and to insure spreading the product should be attractive to both types.

The sociological literature on homophily adopts a view that diversity of individual's contacts is a socially desirable property per se (e.g. Moody, 2001). Although this assertion could be supported by evidence, no rigorous analysis has been made. Perhaps surprisingly, in our model social welfare is increasing in the level of homophily. The result comes from informational and monetary benefits for consumers generated by an increase in the level of homophily. Informational benefits consist in a higher awareness of consumers about the product. Monetary benefits come from a lower price charged by the monopolist, which converts a higher awareness of the product into a higher volume of sales.

There is a popular idea in business and academic literature that focusing advertisement efforts on a group of consumers is the efficient strategy. We show that it is indeed true - an advertisement strategy of targeting consumers of one group is optimal. However, the same does not always hold for the product design even when advertisement is targeted to one group. In the case when the society exhibits low levels of homophily the optimality of product specialization depends on the density of a social network. If the density is low then the expected demand triggered per advertisement is small and the monopolist specializes on a group of consumers targeted by the advertisement. If the network density

[3]In the network literature this phenomenon is known as a global cascade

[4]Hotmail spent a mere 50,000 dollars on traditional marketing and still grew from zero to 12 million users in 18 months.

[5]A movie released in 1999 with principal photography budget ranging from \$20,000 to \$25,000.

is sufficiently high then it is optimal for the monopolist to choose compromise design. This strategy sacrifices some initial adopters from the targeted group, but insures that initial acquisition leads to higher level of WOM communication.

A term “freakonomics” has firmly entered to vocabularies of many economists. The popular book of the same name[6] with over 3 million copies sold worldwide gained popularity not only among a general public, but became well known in the academic community (e.g. DiNardo, 2006, DiNardo 2007, Rubinstein, 2006). The case of “Freakonomics” is not unique. One can recall such examples as “Linked: The New Science of Networks” on networks by Barabási, “The Selfish Gene” on evolution by Richard Dawkins etc. All these books provoked numerous discussions in the academic circles, while the primary audience was the general public. Influenced by this phenomenon, we consider the monopolist that is interested only in one type of consumer (for instance the academic community). We show that designing a product attractive to both types of consumers may be the optimal strategy, even though monopolist benefits only from one type.

1.1 Related Literature

In this section we relate our paper to two streams of the networks literature. The first one studies strategic diffusion of information in networks with underlying assumption that nodes are matched randomly (see, for instance, Campbell, 2009, Goyal and Galeotti, 2007, Galeotti and Mattozzi, 2008). The second stream of the literature studies the impact of homophily on various processes unfolding on networks (for instance Golub and Jackson, 2009, Van der Leij et al., 2009, Buhai and Van der Leij, 2008 and Valat, 2009). Our paper bridges these two streams in a simple model, which studies the impact of social structure, given by the homophily, on the diffusion of information. The model allows us to yield a number of insights into how homophily shapes optimal strategies of the monopolist and influences a social welfare.

The two recent papers studying strategic diffusion of information that are most related to our research are Campbell (2009) and Goyal and Galeotti (2007). Campbell (2009) studies the optimal pricing and advertisement strategies of a monopolist when consumers are engaged in WOM communication. Goyal and Galeotti (2007) study general model of the strategic diffusion, where they distinguish between level of interaction and content of interaction. In their paper, the level of interaction is characterized by a degree distribution, while content of interaction is a way in which actions of others affect individual incentives. There are two key differences between these papers and our paper. First, we extend existing models by relaxing the random matching assumption and study how such intrinsic property of a network as who is connected to whom affects strategic diffusion of information. Second, to our best knowledge, we are the first to consider the

[6] The full title of the book: “Freakonomics: A Rogue Economist Explores the Hidden Side of Everything”.

optimal design of the product in the presence of WOM communication.

The recent paper from the second stream of literature Golub and Jackson (2009) studies how different mechanisms of communication operating through a network are affected by homophily of the society. The principal difference of our paper is that in our setup the monopolist (the sender of a message) takes an active role and influences WOM spreading by choosing the design of the product.

Within a broader literature that considers epidemic diffusion (Newman, 2002; Sander et al., 2002) our paper contributes to the analysis of multi-type networks with homophily by extending the Newman's generating functions approach. In particular, we consider the case when node is operational (is able to transmit information further in the network) with some probability, which depends on a type of the node.

The rest of the paper is organized as follows. Section 2 presents a stylized model of the strategic diffusion. In section 3 we derive the expected size of cascade of sales per advertisement. Section 4 presents the main results on the optimal price and design of the product and considers welfare implications of homophily. Section 5 examines the optimal product design and advertisement strategy when the monopolist can target advertisement by types of consumers. Section 6 considers robustness of the obtained results to a variation in the shape of preference frontier. Section 7 studies the optimal strategy of the monopolist that is interested only in one group of consumers. Section 8 considers the case of a global cascade of sales. Finally, Section 9 outlines avenues for future research and concludes.

2 Model

In this section we formally present the model, which consists of three main blocks: network structure, consumer preferences and monopolist problem.

2.1 Network Structure

There is a continuum of consumers of two types A and B, which are embedded into undirected social network. Consumers of type A constitute measure γ of the population and the rest are consumers of type B. We focus on the case with consumers of two types because it provides basic intuitions and insights, while keeping an analysis transparent[7].

The network is represented by a random graph characterized by a degree distribution $p(k)$ and probabilities of ties among types of consumers (ρ^A, ρ^B). The parameter ρ^i is the probability that a randomly chosen link of a consumer of type i leads to a consumer of the same type and with complementary probability to consumer of another type. Thus links of a consumer can be partitioned into two sets: links to consumers of the same type

[7]For example, the case of consumers of three types with the third type that is not interested in the product is the same as the case of two types with a corrected degree.

and links to consumers of another type. The probability that a consumer of type i with k links has $j \leq k$ links to consumers of the same type is given by the binomial expression:

$$Pr\left(J = j | k, \rho^i\right) = \frac{k!}{j!(k-j)!}(\rho^i)^j(1-\rho^i)^{k-j} \tag{1}$$

The expected number of links connecting a type i consumer to consumers of the same type is given by:

$$\mathbb{E}\left(J | k, \rho^i\right) = \sum_{j=0}^{k}\left[j \times Pr\left(J = j | k, \rho^i\right)\right] = k\rho^i$$

A randomly selected consumer of type A with probability $p(k)$ has k links, $k(1-\rho^A)$ of which connect her to consumers of type B. Taking the expectation we find that on average consumer of type A has $z_1(1-\rho^A)$ links to consumers of type B, where z_1 is expected number of first neighbors. Multiplying obtained expression by the measure of consumers of type A in the population we obtain total number of links of type AB, which is $\gamma z_1(1-\rho^A)$. By analogy a number of links of type BA is equal to $(1-\gamma)z_1(1-\rho^B)$. Using the fact that the graph is undirected and number of links of type AB should be equal to the number of links of type BA we arrive to the equality $\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$. Solving for ρ^B we obtain:

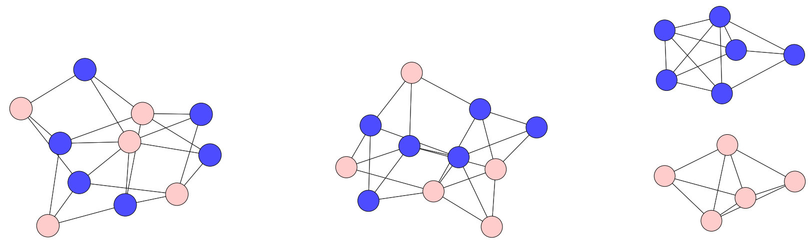
$$\rho^B = 1 - \frac{\gamma}{1-\gamma}(1-\rho^A) \tag{2}$$

Therefore, without loss of generality, in the case of two types of consumers we have just one parameter $\rho = \rho^A$ that characterizes linking preferences of all consumers. The parameter ρ represents a level of homophily of the society, since it specifies the probability of friendship relationships among consumers of the same type, for both types.

It is important to underline that friendship relationships among consumers are formed on the basis of many parameters such as geographical proximity, common interests and so on. A network formation process itself is beyond a scope of this paper. In the analysis we assume that the network of social contacts is exogenously given, and is the same for all products in question. The spreading of WOM in the network for different types of products varies due to differences in homophily level that the society exhibits towards these products.

Figure 1 illustrates 3 different networks with the same degree distribution where all nodes preserve the same connectivity. The only parameter that changes is the level of homophily of the society. As one can observe, depending on ρ networks range from perfectly mixed to two separated graphs, where consumers of type A are completely disjoint from consumers of type B.

To avoid an ambiguity we introduce some key definitions concerning a measurement of homophily level. A benchmark case that we will use extensively is the case when links



(a) (b) (c)

Figure 1: All three graphs have nodes with the same number of neighbors, however they differ in the homophily level. In (a) consumers are linked only to consumers of another type, $\rho = 0$; in (b) we have random mixing of consumers, $\rho = 0.5$; in (c) consumers exhibit extreme homophily, $\rho = 1$.

among consumers are formed with the uniform probability independently of a type.

Definition 1. *The friendship ties in the society are randomly matched if $\rho = \gamma$.*

We can think about a network of detergent consumers as an example of a network with random matching. A plausible assumption would be that preference towards a liquid or powdered detergent is not important itself for forming ties among consumers and thus we may think that consumers of detergent are matched randomly.

In the sociological literature, a tendency of friendship to be biased towards own type beyond the relative proportion in the population is referred to as "inbreeding homophily" (see, for example Coleman, 1958, Marsden, 1987 and McPherson et al., 2001). In this case the proportion of links going to consumers of the same type is higher than otherwise would be implied by random matching.

Definition 2. *The society exhibits the inbreeding homophily if $\rho > \gamma$*

There are also networks in which a situation can be reversed and social ties are biased towards different-type relationships (e.g. network of sexual contacts).

Definition 3. *The society exhibits a heterophily if $\rho < \gamma$*

To illustrate ideas let us consider examples of random matching and network which exhibits homophily. If consumers are matched randomly with a uniform probability then consumer of type A has on average the proportion $\rho = \gamma$ of neighbors of the same type. At the same time the expression (2) implies that the average proportion of neighbors of consumer of type B of the same type is $\rho^B = (1 - \gamma)$, which equals to proportion of consumers of type B in the population. In the case when consumers of type A are linked more often among themselves as compared to the case of random matching, $\rho > \gamma$, by expression (2) the same applies to the consumers of type B, since $\rho^B > (1 - \gamma)$.

2.2 Consumer Preferences

Consumers, in addition of having linking preferences, differ in two other respects. First, across types consumers differ in preferences towards a product design. Consumers of type A prefer one characteristic of the product, while consumers of type B are interested in the opposite features. Second, within types consumers differ in a reservation price $\bar{P}_j$ they are willing to pay for the product and the minimal level of desirable characteristic $\bar{w}_j$, which induces them to buy the product.

More formally, in the model two variables affect the decision of consumers: the price $P \in [0,1]$ and characteristic of product $w \in [0,1]$. For concreteness, a consumer j of type A buys the product if characteristic is higher than the threshold level $w \geq \bar{w}_j$ and the price is lower than the reservation price $P \leq \bar{P}_j$. In contrast, a consumer l of type B buys the product if $w \leq \bar{w}_l$ and $P \leq \bar{P}_l$.

We assume that within a type the reservation price and characteristic threshold are distributed according to $f^i(\bar{w}, \bar{P})$ probability density function. Hence, a randomly chosen consumer j of type A, which is aware of the product with a characteristic w and price P buys it with the probability:

$$q^A = Pr(w \geq \bar{w}_j \cap P \leq \bar{P}_j) = \int_0^w \int_P^1 f^A(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

And similarly a randomly selected consumer l of type B, which knows about the product buys it with the probability:

$$q^B = Pr(w \leq \bar{w}_l \cap P \leq \bar{P}_l) = \int_w^1 \int_P^1 f^B(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

To simplify the analysis for the major part of it we assume that the threshold characteristic and threshold price distributed independently and identically according to the uniform distribution $U[0,1]$ for both types. This implies that $f^A(\bar{w}, \bar{P}) = f^B(\bar{w}, \bar{P}) = 1$ and probabilities to buy the product are given by the following expressions:

$$\begin{cases} q^A &= (1-P)w \\ q^B &= (1-P)(1-w) \end{cases}$$

For given price the system describes a preference frontier, depicted in Figure 2, which encompasses all admissible pairs of probabilities for two types of consumers to buy the product. By choosing the product design the monopolist identifies a probability pair (q^A, q^B) and fixes network of potential buyers. The network of potential buyers consists of all consumers that would buy the product if they know about it. An increase in P shifts frontier inwards, simultaneously decreasing probabilities to buy the product for two types.

In the paper we will encounter two special types of the product design.

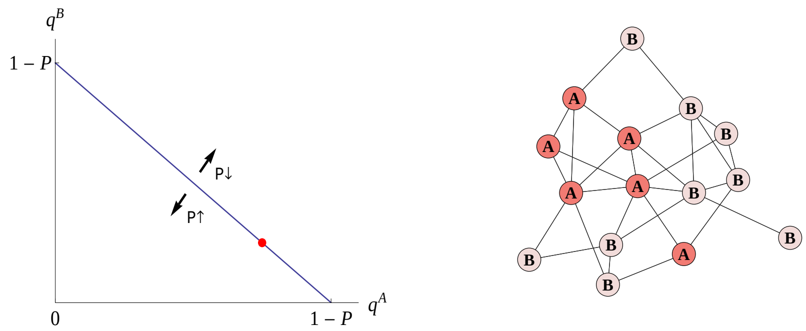


Figure 2: On the left hand side preferences frontier with characteristic of the product being marked by circle. On the right hand side implied social network, with probability to buy the product shown by intensity of the color.

Definition 4. *The design is called symmetric if the product characteristic w is such that two types of consumers acquire the product with identical probabilities, $q^A = q^B$.*

In the case described above, the symmetric design is represented by $w = \frac{1}{2}$.

Definition 5. *The design is called specialized if the product characteristic $w \in \{0, 1\}$, which implies that only one type of consumers acquires the product.*

These two types of design represent different marketing strategies. A symmetric design intends to satisfy needs of both types of consumers, without giving preference to any of them, while the specialized one focuses on one type and neglects the other.

2.3 Monopoly Problem

The monopolist develops new product and introduces it to consumers who are engaged in WOM communication. In the model the monopolist chooses design of the product w and price P to maximize profits. To induce sales the monopolist advertises the product directly to an infinitesimal part of the population. The rest of the population is expected to find out about the product from neighbors who have acquired the product. The diffusion of information stops when there are no new acquisitions of the product.

The network literature usually distinguish two possible scenarios of information spreading. In the first an information propagates to some finite number of consumers and than stops, while in the second an information continues to propagate unboundedly. Let us give the precise definition of the latter case:

Definition 6. *We say that the global cascade of sales arises if ultimately some non-infinitesimal proportion of the population buys the product.*

Depending wether the global cascade of sales arises or not there are two techniques available to study information diffusion. The main results of the paper are developed for the case of finite sales, while in Section 8 we study the case of global cascade.

3 Cascade of Sales

In this section we derive the expression for the expected size of cascade of sales generated by one advertisement and study its properties. In the derivation of the expression we rely on the generating functions approach for multi-type nodes based on Newman (2003). The main focus of Newman's paper is heterogeneity of types in terms of degree distribution. Our paper adopts different perspective. While two types of consumers enjoy the same degree distribution, they differ in their willingness to purchase the product. This implies that a further propagation of information depends on the way in which different types are linked.

3.1 Generating Functions Approach

In the field of complex networks, generating functions were introduced by Newman et al. (2001) and since then have been widely used. A generating function encapsulates all the information about degree distribution, and thus completely characterizes a random network. The generating functions allow us to calculate various local and global properties, such as average degree, average size of component, etc.

In the case of nodes of two types we need to define generating functions associated with degree distribution and homophily level of each type of consumer. Recall that probability of having j links to consumers of the same type for a randomly selected consumer of type i with k links is given by $Pr(J = j|k, \rho^i)$, which is described in (1). The probability pseudo-generating function $F_0^i(x, y)$, where $i \in \{A, B\}$, is given by:

$$F_0^i(x,y) = \sum_{k=0}^{\infty} p(k) q^i \sum_{j=0}^{k} Pr(J = j|k, \rho^i) x^j y^{k-j} \tag{3}$$

This is a polynomial expression in x and y where the coefficient on $x^j y^{k-j}$ is the probability that a randomly chosen consumer of type i buys the product, given that she has j links to consumers of the same type and $k - j$ links to consumers of another type. These functions are known as pseudo-generating due to the fact that for $x = y = 1$ they do not sum to 1. This happens since not all consumers buy the product. Actually, $F_0^i(1, 1) = q^i$, which is the probability that a randomly chosen consumer of type i buys the product given that she is aware of it.

Using the binomial identity we can perform summation over j and the expression reduces to the following:

$$F_0^i(x,y) = \sum_{k=0}^{\infty} p(k) q^i [\rho^i x + (1-\rho^i) y]^k$$

Note, when $x = y$, we obtain the same generating function as in the case when there is only one type:

$$F_0^i(x,x) = \sum_{k=0}^{\infty} p(k) q^i x^k$$

This inheritance allows us to calculate a number of useful properties applying the same techniques as in the case of nodes of one type. Taking the k-th derivative and normalizing by the factor $k! F_0^i(1,1)$ one can recover degree distribution of consumers:

$$\frac{1}{k! F_0^i(1,1)} \left[\left(\frac{\partial}{\partial x} \right)^k F_0^i(x,x) \right]_{x=0} = p(k)$$

A moment of the degree distribution of order m can be calculated by deriving the generating function m times and multiplying each time by x:

$$\frac{1}{F_0^i(1,1)} \left[\left(x \frac{\partial}{\partial x} \right)^m F_0^i(x,x) \right]_{x=1} = \sum_k k^m p(k)$$

For example the average degree of consumer of type i is equal to $\frac{1}{q^i} \frac{\partial F_0^i(1,1)}{\partial x}$.

Apart of the characteristics intrinsic to all random graphs we also can calculate properties tailored to a type of consumer. For example, a degree distribution of links connecting the same type nodes for node of type i is given by:

$$\frac{1}{k! F_0^i(1,1)} \left[\left(\frac{\partial}{\partial x} \right)^k F_0^i(x,1) \right]_{x=0} = Pr(J = j | k, \rho^i) \times p(k)$$

A degree distribution of a neighbor of a randomly chosen consumer plays the important role in the analysis to come. Note it is not the same as the degree distribution of a randomly selected consumer, since the more links a consumer has the more often she is selected as a neighbor. A consumer with k links is found k-times more often through friends than a consumer with one link. Therefore, the probability to have a neighbor with k links is proportional to $kp(k)$. After normalization we obtain that the degree distribution of a neighboring consumer $\xi(k)$ is given by:

$$\xi(k) = \frac{kp(k)}{\sum_{j=1}^{\infty} jp(j)} = \frac{kp(k)}{z_1}$$

Using the degree distribution of neighboring consumer we can write a generating function characterizing degree distribution of a neighboring node of consumer of type i:

$$F_1^i(x,y) = \sum_{k=0}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^k = \frac{1}{z_1}[\rho^i x + (1-\rho^i) y] \left(\frac{\partial F_0^i(x,y)}{\partial x} + \frac{\partial F_0^i(x,y)}{\partial y} \right)$$

The important characteristic that affects the process of information diffusion or spreading of a disease in the network is the excess degree of a neighboring node. That is to say we want to find generating functions that characterize the probability that a neighboring consumer of type i has k links apart of the link which led us to this consumer. The excess degree distribution is given by:

$$\hat{\xi}(k) = \xi(k+1) = \frac{(k+1)p(k+1)}{z_1}$$

And the associated generating functions are:

$$\begin{aligned} \hat{F}_1^i(x,y) &= \sum_{k=0}^{\infty} \hat{\xi}(k) q^i [\rho^i x + (1-\rho^i) y]^k \\ &= \sum_{k=1}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^{k-1} \\ &= \frac{1}{z_1} \left(\frac{\partial F_0^i(x,y)}{\partial x} + \frac{\partial F_0^i(x,y)}{\partial y} \right) \end{aligned}$$

The generating function characterizing the degree distribution of second neighbors who buy the product is given by:

$$\sum_{k=0}^{\infty} p(k) q^i [\rho^i \hat{F}_1^i(x,y) + (1-\rho^i) \hat{F}_1^{\sim i}(x,y)]^k = F_0^i(\hat{F}_1^i(x,y), \hat{F}_1^{\sim i}(x,y)),$$

where $\sim i$ denotes the type of consumer different from type i.

Using this expression we can calculate z_2, the expected number of second neighbors of a randomly selected node. Since we are interested in the expected number of neighbors regardless of type we put $y = x$. To account for all neighbors in the following calculation we assume the probability to buy product for two types q^A and q^B equal to 1. This implies that $\hat{F}_1^i(x,y) = \hat{F}_1^{\sim i}(x,y)$, since now there is no difference between consumers of two types. Applying the method described above we arrive to the expression[8]:

$$\begin{aligned} z_2 &= \left[\tfrac{1}{F_0^i(1,1)} \times x \tfrac{\partial}{\partial x} F_0^i(\hat{F}_1^i(x,x)) \right]_{x=1, q^i=1} \\ &= \textstyle\sum_{k=0}^{\infty} k p(k) [\hat{F}_1^i(1,1)]^{k-1} \times \frac{\partial \hat{F}_1^i(1,1)}{\partial x} \\ &= \textstyle\sum_{k=0}^{\infty} k p(k) \sum_{n=1}^{\infty} \xi(n)(n-1) \\ &= \textstyle\sum_{n=1}^{\infty} n(n-1) p(n) \end{aligned}$$

[8]In the case when $q^i = 1$ for $i \in \{A, B\}$ all generating functions evaluated at 1 are equal to 1.

In the following analysis we assume that the underlying conditions are such that no giant component of consumers who buy the product arises. The case of giant cascade of sales is considered in Section 8. Let us denote by $H_1^i(x,y)$ generating functions characterizing probability distribution of size of finite components of buyers, induced by an information flow through a randomly chosen link to a consumer of type i. If the consumer of type i does not buy the product component is empty (since the information does not spread any further). This happens with the probability:

$$1-\sum_{k=1}^{\infty}\xi(k)q^i = 1-\hat{F}_1^i[1,1],\ i\in\{A,B\}$$

However, with complementary probability a consumer with k contacts buys the product and relays the information to neighbors. The further spread of information is subject to analogous considerations for $k-1$ additional links and is described by $\hat{F}_1^i[H_1^i(x,y), H_1^{\sim i}(x,y)]$. This leads us to the following system of self-consistency conditions for $H_1^i(x,y)$:

$$\begin{cases} H_1^A(x,y) &=& 1-\hat{F}_1^A[1,1]+x\hat{F}_1^A[H_1^A(x,y),H_1^B(x,y)] \\ H_1^B(x,y) &=& 1-\hat{F}_1^B[1,1]+y\hat{F}_1^B[H_1^A(x,y),H_1^B(x,y)] \end{cases}$$

where the leading factor x and y account for the fact that the first visited consumer buys the product.

On the basis of $H_1^i(x,y)$ we can define $H_0^i(x,y)$ generating functions of size of buyers components generated by advertisement to a randomly chosen consumer of type i. Since a randomly chosen consumer of type i does not buy the product with the probability $1-F_0^i(1,1)$ we have:

$$\begin{cases} H_0^A(x,y) &=& 1-F_0^A(1,1)+xF_0^A[H_1^A(x,y),H_1^B(x,y)] \\ H_0^B(x,y) &=& 1-F_0^B(1,1)+yF_0^B[H_1^A(x,y),H_1^B(x,y)] \end{cases}$$

As we have shown before the derivative of the generating function evaluated at the point $(1,1)$ gives us the first moment of a distribution. That is why the number of consumers who eventually buy the product if we advertise it to a randomly chosen consumer of type i is the sum $H_{0x}^i + H_{0y}^i$ evaluated at the point $(1,1)$. Recall that in the population there is proportion γ of consumers of type A and $1-\gamma$ of type B. Thus if we advertise the product to a randomly chosen consumer the expected number of purchases in vector form is given by the expression:

$$s(q^A,q^B,\rho,\gamma,z_1,z_2) = (\gamma\ 1-\gamma)\begin{pmatrix} H_{0x}^A + H_{0y}^A \\ H_{0x}^B + H_{0y}^B \end{pmatrix}$$

Omitting further derivations to the appendix, the resulting expression is given by the following lemma:

Lemma 1. *The expected number of consumers who buy the product if the monopolist advertises it to a randomly chosen consumer is given by an expression:*

$$s(q^A, q^B, \rho, \gamma, z_1, z_2) = (\gamma \;\; 1-\gamma) \left[\mathbf{I} + \mathbf{F_0'}(\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}')^{-1} \right] \binom{q^A}{q^B},$$

where z_1 *and* z_2 *are expected numbers of first and second neighbors correspondingly,* $\rho^A = \rho$, $\rho^B = 1 - \frac{\gamma}{1-\gamma}(1-\rho)$, $\hat{\mathbf{F}}_\mathbf{0}' = \frac{z_1^2}{z_2}\hat{\mathbf{F}}_\mathbf{1}'$ *and*

$$\hat{\mathbf{F}}_\mathbf{1}' = \frac{z_2}{z_1} \begin{pmatrix} q^A \rho^A & q^A(1-\rho^A) \\ q^B(1-\rho^B) & q^B \rho^B \end{pmatrix}$$

Proof See appendix □

The first term of the expression $(\gamma \;\; 1-\gamma)\mathbf{I}\binom{q^A}{q^B}$ is the probability that a randomly chosen consumer buys the product and transmits information further. The second term consists of two parts. The first part $(\gamma \;\; 1-\gamma)\mathbf{F_0'}$ is a vector with components showing the number of the first neighbors of type A and type B who buy the product. The second part is the vector $(\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}')^{-1}\binom{q^A}{q^B}$ with components that represent number of purchases generated by the flow of information through a randomly chosen link to a consumer of type A and B.

Note that as in the epidemic diffusion literature only first two moments of degree distribution z_1 and z_2 are relevant for the propagation of cascade of sales. This substantially reduces the amount of information about network structure that monopolist needs to know to make the optimal decision.

In a special case when consumers of both types have the same preferences towards the product and buy it with the same probability q the expression of cascade of sales reduces to well known expression of the average size of component of operational nodes[9]:

$$s(q^A, q^B, \rho, \gamma, z_1, z_2)|_{q^A = q^B = q} = q + \frac{q^2 z_1}{1 - q(z_2/z_1)}$$

In this case size of sales cascade is independent of such network characteristics as population composition γ and homophily level ρ. This points out that for homophily to have the impact on diffusion there should be a heterogeneity of types in terms of preferences towards both the product and friendship relationships.

4 Main Results

We begin our analysis by establishing a condition under which the global cascade of sales arises. With this condition in mind, we turn to the problem of the monopolist considering

[9]See Callaway et al. (2000)

the case when sales are finite. We derive the optimal price and product design solving the maximization problem in two steps. In the first step we fix the price and solve the problem for the optimal design of the product. In the second step we allow the monopolist to re-optimize with respect to the price. We complete our analysis by studying the implications of homophily level for price elasticity of demand and social welfare.

4.1 Arise of the Global Cascade of Sales

A WOM marketing campaign is regarded as successful if it induces multiple sales per advertisement. However, there are some exceptional cases when an information propagates to a significant part of the population. These were the case of movie advertisement "The Blair Witch Project" and diffusion of Hotmail accounts. In this section we identify the condition under which the monopolist acting optimally can sell the product to non-infinitesimal part of the population. We consider two cases, when the price is endogenous and forms part of the decision process of the monopolist and when price is exogenous.

From Lemma 1 we know that number of buyers of the product explodes when the denominator $\det(\mathbf{I}-\hat{\mathbf{F}}'_{\mathbf{1}})$ goes to 0. Thus the condition of appearance of the global cascade of sales is:

$$1 - q^A q^B \left(\frac{z_2}{z_1}\right)^2 (1-\rho^A-\rho^B) - \frac{z_2}{z_1}(q^A\rho^A + q^B\rho^B) \leq 0$$

In order not to favor any group of consumers in the following analysis we assume that consumers are partitioned into two groups of equal sizes, thus consumers of type A and B constitute half of the population. In this case the expression (2) implies that $\rho^A = \rho^B = \rho$. Substituting expressions for q^A and q^B and incorporating assumptions we obtain the following quadratic inequality:

$$w^2(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) - w(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \leq 0$$

In the expression, degree distribution is summarized by the ratio of expected number of second neighbors to expected number of first neighbors. This ratio tells us how efficient is a network in information diffusion. In particular, it shows how many second neighbors on average become aware of the product if a consumer shares the information with one of her first neighbor.

The following proposition summarizes the result:

Proposition 1. *For endogenous price P, if $z_2/z_1 \geq \min\left\{2, \rho^{-1}\right\}$ there exists non empty set $E(z_2/z_1, \rho)$ such that for any $(w, P) \in E(z_2/z_1, \rho)$ a global cascade of sales arises.* □

Proof See appendix □

In the framework of one-type nodes a paper by Molloy and Reed (1995) for the first time derives the condition for appearance of the giant component of connected nodes, which is $z_2/z_1 > 1$. In our case it is a necessary condition for a global cascade of sales to occur. One can easily check that $z_2/z_1 < 1$ does not satisfy condition in Proposition 1, since $\rho \in [0,1]$. Intuitively, for the information to spread unboundedly, there should exist a giant component of connected consumers upon which spreading may take place.

The condition in Proposition 1 is stronger than $z_2/z_1 > 1$ since not all consumers buy the product and consequently relay WOM further. One can separate the condition into two parts: $z_2/z_1 \geq 2$ and $z_2/z_1 \geq \rho^{-1}$. The first part of the condition tells us that independently of homophily level ρ, if z_2/z_1 is higher than 2 then a global cascade of sales occurs. This part of the condition comes from the case when maximal spread of WOM is attained for symmetric characteristic $\left(w = \frac{1}{2}\right)$, which mitigates differences between nodes and makes ρ irrelevant. Moreover, it resembles a condition from Callaway et al. (2000) for the appearance of a giant component of operational nodes, $z_2/z_1 \geq \frac{1}{p}$, where p is a probability that a node is operational. In our model in the case of the symmetric design $p = w = \frac{1}{2}$, since all consumers buy the product with the same probability.

The second part of the condition comes from the case when $\rho > \frac{1}{2}$ and the maximal spread of WOM is attained when the monopolist chooses a specialized design ($w \in \{0,1\}$). In this case information propagates only via consumers of one type. To fix ideas assume that the monopolist chooses $w = 1$ and thus only consumers of type A buy the product. The expected number of first neighbors of type A is ρz_1 and the expected number of second neighbors of type A, which can be attained through type A consumers is $\rho^2 z_2$. Substituting these numbers into the condition from Molloy and Reed (1995) we obtain $z_2/z_1 > \rho^{-1}$, which is exactly the same as the second part of the condition in Proposition 1.

In the analysis to come we also consider a case when the price is exogenously given and the monopolist only chooses characteristic of the product w. The following lemma establishes condition for the global cascade of sales to occur for exogenously given price:

Lemma 2. *For an exogenously given price P, if $z_2/z_1 \geq \frac{1}{1-P} \min\left\{2, \rho^{-1}\right\}$ there exists non-empty interval $[\underline{w}, \overline{w}]$, such that for any $w \in [\underline{w}, \overline{w}]$ a global cascade of sales arises.*

Proof See appendix □

Not surprisingly, in the case of exogenous price the condition of appearance of giant cascade of sales in the Lemma 2 is more strict, since not all consumers are willing to pay the price P for the product.

4.2 Optimal Design

In this section we consider the problem of the monopolist who takes the price as given and chooses the design of product to maximize profits. Without loss of generality we assume that a production cost of the product is zero. Thus profits are given by the product of the price and size of sales cascade. In the case of an exogenously given price Lemma 2 implies that there is no global cascade if $z_2/z_1 \leq \frac{1}{1-P} \min\left\{2, \rho^{-1}\right\}$. Thus the monopolist profits maximization problem subject to preferences frontier is the following:

$$\max_{w} P \times s(q^A, q^B, \rho, \frac{1}{2}, z_1, z_2)$$

$$\text{s.t.:} \quad \begin{aligned} q^A &= (1-P)w \\ q^B &= (1-P)(1-w) \end{aligned}$$

Before going to the results we develop some intuition. We already have seen the importance of homophily level in the Lemma 2 . In the following exercise let us assume that the society exhibits heterophily, which implies that nodes of type A are more often connected to nodes of type B. Assume further that a consumer of type A has bought the product. Since majority of her neighbors are of type B, a necessary condition for further spread of the information is attractiveness of the product to consumers of type B. However, once they buy the product, most of their neighbors are of type A and the process reiterates. Thus we can conclude that for a sufficiently low homophily level the optimal product design should be appealing to both groups of consumers. Assume now that homophily is sufficiently high and consumers of both types have majority of their links to the consumers of the same type. Would it be optimal to focus on consumers of one type, and forget about others? The question is non-trivial, since there are components of consumers of both types and if the monopolist focuses on one type all components of another type will be out of reach.

The following proposition summarizes the results:

Proposition 2. *For any exogenously given price P following holds:*

(a) if $\rho = \frac{1}{2}$ the function $s(\cdot)$ is horizontal line and all $w \in [0,1]$ are solutions to the maximization problem.

(b) if $\rho < \frac{1}{2}$ the function $s(\cdot)$ is quasi-concave and has its unique maximum at the point $w = \frac{1}{2}$.

(c) if $\rho > \frac{1}{2}$ the function $s(\cdot)$ is quasi-convex and its unique minimum is at the point $w = \frac{1}{2}$ and maxima are situated at points $w \in \{0,1\}$

Proof See appendix □

The first result states that for two groups of consumers of equal sizes if $\rho = \frac{1}{2}$ (which for $\gamma = \frac{1}{2}$ implies random mixing) the size of sales cascade is not affected by the product characteristic w. That is why heterogeneity of consumers preferences towards the product and towards linking both constitute key ingredients of the model.

The Proposition 2 confirms our intuition for the case of low homophily levels and most importantly states that the maximization problem has a threshold solution. More precisely, independently of a degree distribution and price, if ρ becomes higher than $\frac{1}{2}$, the optimal product design abruptly changes from symmetric $w^* = \frac{1}{2}$ to specialized $w^* \in \{0,1\}$. The explanation is the following: when ρ is higher than $\frac{1}{2}$ the majority of consumer's neighbors are of the same type as the consumer. That is why the design most attractive for a randomly selected consumer is the one that induces the highest sales of the product among her neighbors. The situation reiterates for every consumer that buys the product, reinforcing the optimality of the specialized design.

The obtained result does not depend on a degree distribution or price, which makes it easy to apply. The monopolist just needs to know whether homophily level of the society is higher or lower than $\frac{1}{2}$ to choose the optimal design of the product.

4.3 Demand

Incorporating the optimal design of the product into the expression for cascade of sales from Lemma 1 we obtain the following demand function:

$$Q(P,\rho,z_1,z_2) = \begin{cases} \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{2-z_2/z_1(1-P)}\right), & \rho \leq \frac{1}{2} \\ \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{\frac{1}{\rho}-z_2/z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

Note that for $\rho \leq \frac{1}{2}$ the demand is independent of homophily level ρ, since in this case the optimal design is given by the symmetric characteristic $w^* = \frac{1}{2}$. The symmetric design implies that both types of consumers buy the product with the same probability and mixing pattern does not matter. The following proposition summarizes main properties of the demand function:

Proposition 3. *The demand is given by the function $Q(P,\rho,z_1,z_2)$, which has following properties:*

(i) $Q(P,\rho,z_1,z_2)$ is continuous in ρ and for $\rho > \frac{1}{2}$ is increasing and convex in ρ.

(ii) $Q(P,\rho,z_1,z_2)$ is decreasing and convex in P.

(iii) A price elasticity of demand is increasing in ρ, for $\rho > \frac{1}{2}$.

Proof See appendix □

The first result states that for homophily level $\rho > \frac{1}{2}$ a classical demand (the demand with the incorporated optimal design) increases in ρ. In Proposition 2 we have seen that for $\rho > \frac{1}{2}$ the optimal design is specialized with characteristic w^* belonging to the set $\{0,1\}$. In this case a randomly chosen consumer has the majority of neighbors of the same type and a further increase of homophily increases this subset. To fix ideas assume that

$w^* = 1$ and $P = 0$. Thus if a consumer of type A gets the information about the product she and all her neighbors of type A acquire the product. Thus an increase in homophily level leads to a higher number of acquisitions in the neighborhood of a type A consumer.

The obtained result differs from intuitions of McPherson (2001), which argues that for higher homophily levels, information flows are localized and status quo of individuals tend to be maintained. We show that when further transmission of information depends on the adoption decision, an increase in homophily may actually produce higher spread of information. Higher levels of homophily induce a higher correlation of consumers' preferences making it easier for the monopolist to design a product information of which can penetrate further in the network.

The convexity part of the result *(i)* comes from the fact that an increase of homophily expands a subset of neighbors of the same type in a neighborhood of consumer. Moreover, each consequent increase of homophily operates upon the neighborhood of a higher number of consumers, which acquire the product. This gives rise to increasing returns in terms of number of buyers in homophily level.

The result *(ii)* has a similar nature as the result of convexity of demand in ρ. A price increase affects the decision of all consumers to acquire the product regardless of their position in a chain of buyers. Let us consider an example when a consumer who is situated earlier in a chain of buyers stops to acquire the product due to a price increase. In this case the whole branch of consumers that has received information about the product through this consumer stops to acquire the product. A further price increase has smaller impact on the demand, since chains of buyers become shorter.

Having two previous results at hand we are equipped to understand the third one. The result *(i)* implies that when $\rho > \frac{1}{2}$ an increase in ρ leads to higher sales and awareness of consumers about the product. Hence, a price increase affects decision of an increased number of consumers, which translates into an increased price elasticity of demand.

4.4 Optimal Price

We have seen the solution of maximization problem with respect to the optimal design of the product. In this subsection we relax the assumption of exogenous price and allow the monopolist to re-optimize with respect to a price. The optimal product design is independent of the price chosen by the monopolist, thus all results derived in the previous sections hold for the optimal price as well. The monopolist maximizes profits and solves the following problem with respect to price:

$$\max_{0 \leq P \leq 1} P \times \frac{1-P}{2} \left(1 + \frac{z_1(1-P)}{\frac{1}{\rho} - \frac{z_2}{z_1}(1-P)} \right)$$

In a price setting the monopolist faces usual trade-off: an increase in the price augments profits from each unit sold, but simultaneously decreases demand for the product.

However, in the presence of WOM communication there is an additional informational component of the trade-off. Since a consumer spreads information about the product only if she acquires it, a price increase lowers product awareness of consumers. The properties of the optimal pricing strategy are summarized in the following proposition:

Proposition 4. *The optimal price P^* for $\rho \geq \frac{1}{2}$ is decreasing in the homophily level, while for $\rho < \frac{1}{2}$, P^* is independent of the homophily level. The optimal price P^* is always lower than $\frac{1}{2}$.*

Proof See appendix □

The result is a direct consequence of the fact that the price elasticity of demand is increasing in homophily level. As we have seen in Proposition 3 for $\rho > \frac{1}{2}$ an increase in homophily implies that more consumers become aware about the product, and sales increase. The monopolist by reducing the price capture a higher fraction of informed consumers. Proposition 4 implies that the informational component outweighs an increase of profits per purchase generated by a higher price.

4.5 Social Welfare

Our model allows to address welfare implications of the homophily in explicit manner. A majority of the literature on homophily adopts a view that diversity of contacts is a socially desirable property per se (e.g. Moody, 2001). Although this assertion could be supported by evidence, no rigorous analysis has been made. A recent paper by Currarini et al. (2009) shows that welfare implications of homophily crucially depend on the structure of consumers preferences. In the following analysis we consider welfare implications of homophily in the framework of our model.

A consumer surplus is given by the following expression:

$$CS(P^*(\rho), \rho, z_1, z_2) = \int_{P^*(\rho)}^{1} Q(P, \rho, z_1, z_2) dP$$

We already have seen that for $\rho \geq \frac{1}{2}$, demand is increasing in homophily, and thus increase in ρ shifts the demand curve upwards. This happens, since more consumers become aware about the product. At the same time an increase in the homophily level by Proposition 4 leads to lower price. As a consequence more consumers buy the product for lower price and thus both effects lead to an increase in consumer surplus.

A producer surplus is the area below the price of the product and marginal cost curve (MC) of producer. In our case $MC = 0$, which leads to the following expression for the producer surplus:

$$PS(P^*(\rho), \rho, z_1, z_2) = P^*(\rho) \times Q(P^*(\rho), \rho, z_1, z_2)$$

Proposition 5. *Both consumer surplus and monopolistic profits are increasing in the level of homophily.* □

The Proposition 5 states that if information retransmission is subject to an adoption decision then society is better-off when homophily level is high. There are two driving forces of the result. First, the optimally constructed message propagates better in homogenous groups, which leads to an increase in awareness of consumers about the product. Second, the price reduction is more effective in facilitating diffusion of WOM in the case of higher homophily levels. These two effects are beneficial for both consumers and the monopolist.

4.6 Example of Classical Random Graph

In further analysis we will often refer to a special case of network structure known as a classical random graph. The notion of a random graph has been introduced by Paul Erdõs and Alfréd Rényi and since then it is the most studied model of graphs. Nodes connectivity in a random graph follows Poisson degree distribution and arises in infinite networks, where each node creates a link to any other node in the network with a uniform probability.

In our case probability that a randomly selected link connects two consumers of the same type is different from the probability that it connects consumers of different types. One can think about a network of N consumers of two types, where each consumer creates a link to any other consumer of the same type with probability $\frac{\rho z_1}{N}$ and to a consumer of another type with the probability $\frac{(1-\rho)z_1}{N}$. When N goes to infinity we obtain infinite network that is characterized by two Poisson degree distributions. One for links among consumers of the same type with the mean ρz_1 and another for links among consumers of different types with the mean $(1-\rho)z_1$. Recall, that the sum of two Poisson variables also follows Poisson distribution with the mean equal to the sum of means. That is why overall connectivity of a randomly chosen node follows Poisson distribution and the network is classical random graph with average connectivity given by z_1.

In the case of Poisson degree distribution the average connectivity z_1 is a sufficient characteristic of the network, and $z_2 = z_1^2$. This property allows us to study the effect of network density on the propagation of WOM and the optimal strategies of the monopolist.

As we already have seen the optimal design of the product does not depend on a degree distribution and is given by the expression in Proposition 2. Incorporating relation between the expected number of first and second neighbors into the demand function we obtain:

$$Q(P,\rho,z_1,z_2) = \begin{cases} \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{2-z_1(1-P)}\right), & \rho \leq \frac{1}{2} \\ \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{\frac{1}{\rho}-z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

Note that demand is continuous and does not depend on ρ when $\rho \leq \frac{1}{2}$. In the derivation we use the demand function for the case of $\rho > \frac{1}{2}$. In order to get results for the case of $\rho \leq \frac{1}{2}$ one needs to substitute $\rho = \frac{1}{2}$. Taking derivative with respect to z_1 of the demand function one can show that the denser is the network the higher is demand:

$$\frac{\partial}{\partial z_1} Q(P, \rho, z_1, z_2) = \frac{(1-P)^2 \rho}{2(1-(1-P)z_1\rho)^2} > 0$$

Solving maximization problem we obtain the expression for the optimal price,[10,11] which is given by:

$$P^* = \begin{cases} 1 - \frac{2-\sqrt{4-2z_1}}{z_1}, & \rho \leq \frac{1}{2} \\ 1 - \frac{1-\sqrt{1-z_1\rho}}{z_1\rho}, & \rho > \frac{1}{2} \end{cases}$$

The derivative of the optimal price with respect to homophily level ρ is negative, the thus optimal price is decreasing with homophily of the society:

$$\frac{\partial P^*}{\partial \rho} = -\frac{2 - z_1\rho - 2\sqrt{1-z_1\rho}}{2z_1\rho^2\sqrt{1-z_1\rho}} < 0$$

To study an effect of the network density on the optimal price we derive the expression (4.6) with respect to z_1:

$$\frac{\partial P^*}{\partial z_1} = -\frac{2 - z_1\rho - 2\sqrt{1-z_1\rho}}{2z_1^2\rho\sqrt{1-z_1\rho}} < 0$$

The derivative is negative, which implies that the optimal price P^* is decreasing in both average connectivity and homophily parameter ρ.

Turning to welfare implications of homophily and using the same line of arguments as we have outlined before one can show that an increase in z_1 leads to a higher consumer surplus. This happens since a denser network implies higher awareness of consumers about the product. An increase in the network density also augments benefits for the monopolist of price reduction. These both effects benefit consumers.

5 Targeted Advertisement

In the previous section we have considered the problem of the monopolist, which cannot distinguish consumers by type. The monopolist, restricted by the anonymity assumption, was advertising the product to a randomly chosen subset of the population. This formulation is relevant for an advertisement through the mass media, when the monopolist

[10] In the case of Poisson degree distribution the first order condition for the optimal price when $\rho > \frac{1}{2}$ reduces to $z_1\rho P^2 + 2P(1-z_1\rho) + z_1\rho - 1 = 0$.

[11] Condition of absence of a global cascade of sales in the case of the Poisson distribution is $z_1 < \min\{2, \rho^{-1}\}$.

cannot control who is watching or hearing an advertisement. However, in the case of a direct advertisement there is a possibility to target chosen group of consumers. For example, monopolist that is interested in students' community, can advertise a product in a university campus, etc.

In this section we are going to relax anonymity assumption and allow the monopolist to observe types of consumers. More precisely, we assume that the monopolist chooses a design of the product w and proportion α of consumers of type A in the subset that is selected for advertisement. Note, before proportion of consumers of type A which were receiving advertisement was fixed exogenously at the level γ, which is proportion of nodes of type A in the society. In the analytical part, for tractability of the problem, we assume that the price P is exogenously given. Thus the maximization problem of the monopolist becomes:

$$\max_{w,\alpha} P \times s(q^A, q^B, \rho, \alpha, z_1, z_2)$$

$$\text{s.t.:} \quad \begin{aligned} q^A &= (1-P)w \\ q^B &= (1-P)(1-w) \end{aligned}$$

The expression for sales cascade $s(q^A, q^B, \rho, \alpha, z_1, z_2)$ can be rewritten as a linear combination of number of purchases resulted from an advertisement to consumer of type A and of type B: $\alpha \times s^A(q^A, q^B, \rho, z_1, z_2) + (1-\alpha) \times s^B(q^A, q^B, \rho, z_1, z_2)$. Given the linear structure of the problem in terms of α it is easy to see that if $q^A \neq q^B$ the optimal targeting proportion has a corner solution. Namely, the solution depends whether a cascade of sales induced by an advertisement is higher if we advertise to a consumer of type A or of type B. In the case when $q^A = q^B$ both types of consumers buy the product with the same probability and thus all values of α on the interval $[0, 1]$ are optimal.

Proposition 6. *Targeting one type of consumers for advertisement is always the optimal strategy for the monopolist.* □

Since preference frontier is symmetric, without loss of generality we assume that the monopolist targets consumers of type A and hence $\alpha^* = 1$. Moreover, by symmetric nature of the problem if $\alpha^* = 1$ and some w^* is the solution then $\alpha^* = 0$ and $1 - w^*$ is a solution too. For the following analysis we assume that $\alpha^* = 1$.

The maximization problem with targeted advertisement for an arbitrary degree distribution quickly becomes intractable. In the following analysis we focus on the commonly employed structure of a network given by the classical random graph.

The intuition tells us that possibility of targeting of consumers for advertisement would unavoidably brings bias towards characteristics of the product favorable for consumers of targeted type. The bias itself can be of two forms. The first form is that a threshold level of homophily, which separates specialized design ($w \in \{0, 1\}$) and symmetric ($w = \frac{1}{2}$)

moves to a level lower than $\frac{1}{2}$. The second form is that for ρ lower than new threshold level the optimal design belongs to the interval $\left(\frac{1}{2}, 1\right)$ instead of being symmetric. The following proposition shows that the both types of bias are present in the case of targeted advertisement:

Proposition 7. *In the case of the Poisson degree distribution and exogenously given price, there is a threshold level* $\hat{\rho}_T(z_1, P) = \frac{3}{4} + \frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2 z_1^2}}{4(1-P)z_1}$*, such that for* $\rho \geq \hat{\rho}_T(z_1, P)$ *the monopolist will advertise and specialize only on one type of consumers* ($\alpha^* = 1$ *and* $w^* = 1$)*. For* $\rho < \hat{\rho}_T(z_1, P)$ *the optimal advertisement strategy is still* $\alpha^* = 1$*, but the optimal characteristic is given by the following expression:*

$$w^* = \rho - \frac{1 - 2\rho - \sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2+(1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)} > \frac{1}{2}$$

Proof See appendix □

The success of WOM campaign to a high extend depends on the effectiveness of the direct advertisement in inducing initial acquisitions of the product. In order to convert consumers of type A, who receive direct advertisement, into initial adopters the monopolist designs the product more attractive to them. The bias in the product design persists even when the society exhibits heterophily ($\rho < \frac{1}{2}$).

In the case of the Poisson distribution the rise in z_1 implies a higher diffusion of WOM in the network, since there are more channels for information to spread on. In the case when we approach the global cascade phase (z_1 goes to $2(1-P)^{-1}$) the threshold level $\hat{\rho}_T(z_1, P)$ goes to $\frac{1}{2}$ and the optimal characteristic for $\rho < \frac{1}{2}$ is $w^* = \frac{1}{2}$. Thus the optimal design becomes exactly the same as in the case of non-targeted advertisement. In this case the monopolist optimally sacrifices some initial adopters, and design the product in a way that WOM can penetrate further in the network.

The result implies that when we approach the global cascade phase the option to target advertisement to some group of consumers does not influence neither the optimal design of the product nor the demand. We regard this result as indication of robustness of the optimal design strategy that we obtained in our base-line model.

6 Non-Linear Shape of the Preference Frontier

The previous analysis develops results for the case of a linear preference frontier. However, one can think about many examples where a relationship between acquisition probabilities for two groups of consumer is not linear. These situations occur when attractiveness of the product for two types of consumers varies non-linearly in a characteristic. For example, longer guarantee on the product is attractive for both types, while red color of the product may be very welcome by one group of consumers and be unattractive for an another. In this situations the shape of the frontier can vary from concave to convex depending on a product in question.

In this section we want to address the robustness of obtained results to a change in curvature of the preference frontier that the monopolist faces. We consider CES functional form of the preference frontier, which allows to model variety of shapes. Thus probabilities to buy the product for two types of consumers are related in the following manner:

$$\left(q^A\right)^r + \left(q^B\right)^r = (1-P)^r$$

By varying the parameter r we can obtain shapes of the preferences frontier that include a bend inward circle ($r = 0.562$), a linear function ($r = 1$), a bend outward circle ($r = 2$) and everything in between.

6.1 Arise of the Global Cascade of Sales

Similarly to analysis in the case of a linear frontier it is important to establish conditions under which a global cascade of sales occurs. An essential parameter for appearance of the global cascade of sales is r, the degree of curvature. For example, if r goes to ∞, as we know from properties of CES function, the frontier becomes a step function and the best design of the product is $w = 1$ which implies $q^A = q^B = 1 - P$. In this case two types of consumers are fully satisfied with a product design and thus effectively there is only one type of consumers. The condition for appearance of global cascade in this case is well known from the networks literature, namely $\frac{z_2}{z_1} \geq \frac{1}{1-P}$.

For arbitrary values of r the condition of existence of a global cascade is given by the following inequality:

$$1 - (1-P)^2 w(1-w^r)^{\frac{1}{r}} \left(\frac{z_2}{z_1}\right)^2 (1-2\rho) - (1-P)\left(w + (1-w^r)^{\frac{1}{r}}\right)\frac{z_2}{z_1}\rho \leq 0$$

Due to an arbitrary power of the polynomial equation it is impossible to solve it analytically, however we can identify a broad set of parameters such that a global cascade of sales occurs. The results are summarized in the following proposition:

Proposition 8. *In the case of CES preference frontier with a curvature parameter r, global cascade of sales arises if $z_2/z_1 > 2^{\frac{1}{r}}$ or $\rho > \hat{\rho}_{NL}(z_2/z_1, r, P)$, where*

$$\hat{\rho}_{NL}(z_2/z_1, r, P) = \min_{0 \leq w \leq 1} \frac{1 - w(z_2/z_1)^2(1-P)^2(1-w^r)^{\frac{1}{r}}}{(1-P)(z_2/z_1)\left(w + (1-w^r)^{\frac{1}{r}}(1-2w(z_2/z_1)(1-P))\right)}.$$

Proof See appendix □

6.2 Problem of the Monopolist

The subsection consists of two parts: analytical and numerical. In the analytical part we focus on the case of an exogenous price and Poisson degree distribution and find conditions such that symmetric and specialized designs are solutions to the maximization problem. In the second part we endogenize a price and solve monopolist problem numerically for an arbitrary degree distribution.

Similarly to the analysis in Section 5 we assume that the degree distribution is Poisson and price is given exogenously. The monopolist faces a non-linear preferences frontier and chooses design of the product to maximize profits. The monopolist solves the following maximization problem:

$$\max_{w} P \times s(q^A, q^B, \rho, \frac{1}{2}, z_1, z_2)$$

$$\text{s.t.:} \quad q^A = (1-P)w$$
$$q^B = (1-P)(1-w^r)^{\frac{1}{r}}$$

We identify homophily level such that symmetric and specialized designs are solutions. The results are summarized by the following proposition:

Proposition 9. *For an exogenous price and Poisson degree distribution following holds:*

(a) For $r \leq 1$ there is $\hat{\rho}_{NL}(r, P, z_1) = \frac{1}{2} - \frac{2^{\frac{1}{r}}-2}{2z_1(1-P)}$ such that the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ and otherwise the optimal design is specialized $w^* \in \{0, 1\}$.

(b) For $r > 1$ the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ otherwise the optimal design belongs to the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.

Proof See appendix □

The Proposition 9 states that the optimal design has similar structure as in the case of linear preferences frontier. More precisely, for bend inwards frontier ($r \leq 1$) only symmetric and specialized designs are optimal. They are separated by new threshold value $\hat{\rho}_{NL}(r, P, z_1)$. For bend outwards frontier ($r > 1$) and sufficiently low levels of ρ, symmetric design is optimal. However for high levels of ρ the solution gradually changes from the symmetric to specialized.

In the case of bend outward shape of the frontier ($r > 1$) the optimal design is biased towards the symmetric design. The explanation is following: while the corner solutions are still as attractive as they were in the problem with $r = 1$ (endpoints are fixed), the symmetric solution becomes more appealing. This happens since in the case of $r > 1$ by moving to the center of symmetry from $q^A = 1$ we are gaining more of q^B, while

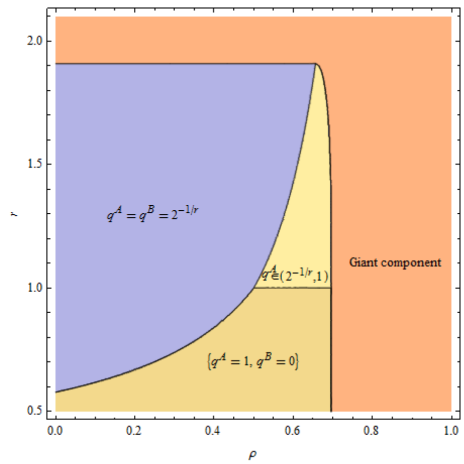


Figure 3: Diagram depicts a solution for the case of scale free distribution with pdf $Ck^{-3.34}$, where C is normalizing constant. In this case $z_1 = 1.23$ and $z_2 = 1.77$. Areas represent: symmetric solution (blue), asymmetric (light yellow), specialized (yellow) and area where a global cascade arises (red).

sacrificing the same amount of q^A as compared to the linear case. For the case when $r < 1$ the situation is reversed.

The threshold level of homophily $\hat{\rho}_{NL}(r, P, z_1)$ is increasing in z_1. Thus the denser is a network the more appealing becomes the symmetric design. The intuition is the same as for the case of targeted advertisement. Dense network implies higher diffusion of information and to penetrate further the product should be appealing for both types.

To check robustness of the obtained result we consider a numerical solution of the problem for the case of a scale free distribution. Figure 3 shows a diagram of the structure of the solution. One can see from the diagram that for $r < 1$ we have similar results as in a linear case. Namely, there is the threshold level of $\hat{\rho}_{NL}(r, z_1, z_2, P)$ such that for $\rho < \hat{\rho}_{NL}(r, z_1, z_2, P)$ the optimal solution is symmetric and for values of $\rho > \hat{\rho}_{NL}(r, z_1, z_2, P)$ the solution is specialized. In the case of $r > 1$ a structure stays the same but after $\rho = \hat{\rho}_{NL}(r, z_1, z_2, P)$ the solution gradually changes from symmetric to some intermediate value, which lies in the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.

7 Targeting One Type of Consumers

In this section we address the problem of monopolist who has an interest only in one type of consumers, for concreteness lets assume that this is type B. This situation could arise if the monopolist believes that consumers differ in their post purchasing behavior. For example, once a consumer of type B buys the product she becomes a customer loyal to the brand and continue to make purchases of the same brand, while consumers of type

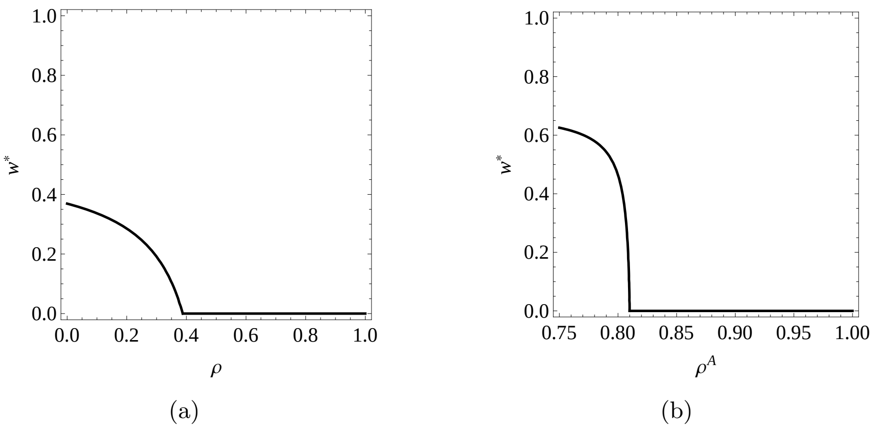


Figure 4: The optimal design when $z_1 = 1.7$, $z_2 = 2$. Figure (a) for the case of $\gamma = \frac{1}{2}$, figure (b) for the case of $\gamma = 0.8$

A may be accidental buyers. For the sake of simplicity we assume that the monopolist completely ignores consumers of type A. Assume further that the monopolist maximizes awareness of the brand and chooses price equal to 0. The main question is than: what is the optimal product design such that maximizes a number of purchases by consumers of type B?

The first guess could be that the monopolist should completely forget about consumers of type A and design a product as attractive as possible to consumers of type B. The first guess, however, turns out to be wrong for broad set of parameters. Assume for example that homophily level of the society is low, which implies that consumers of type B are mostly connected to consumers of type A. Hence, to spread, the information should be able to pass through consumers of type A. A Figure 4a illustrates the optimal product design for the case of groups of consumers of equal size ($\gamma = \frac{1}{2}$) and $z_1 = 1.7$ and $z_2 = 2$. Note that for $\rho \in [0, 0.39]$ the optimal design is such that there is a non zero probability for consumers of type A to buy the product. The result requires low levels of homophily and actually implies heterophily of the society. We already have seen similar answer in the case when the monopolist profits from both types of consumers.

Probably, the more surprising result is that although society exhibits homophily it may be optimal to make a product attractive for consumers of type A. The only requirement is that the proportion of consumers of type A in the society should be sufficiently high. A Figure 4b illustrates the optimal product design for the case when consumers of type A constitute 80% of the population ($\gamma = 0.8$) and the expected number of neighbors are as before $z_1 = 1.7$ and $z_2 = 2$. Note that $\rho \in [0.8, 1]$ implies that the society exhibits homophily and there is a range of $\rho \in [0.8, 0.81]$ such that the optimal product characteristic w is not zero. Another surprising feature of the result is that for a sufficiently small ρ it is optimal to construct the product more attractive to consumers of type A than B.

8 Global Cascade Phase

In previous sections we have seen what happens when WOM marketing campaign does not trigger a global cascade of sales. This is the case for a majority of WOM campaigns. However, WOM campaigns such as diffusion of Hotmail accounts and “The Blair Witch Project” (tiny budget movie) were so successful that a considerable fraction of the population became aware of the product. In this cases we can no longer apply techniques from the previous analysis.

So let us assume that conditions are such that a global cascade of sales arises, by Lemma 2 this happens when $\frac{z_2}{z_1} > \frac{1}{1-P}\min\{2, \rho^{-1}\}$. To determine the fraction of the population that buys the product we turn back to generating functions. However instead of looking on the distribution of sizes of cascades we would like to estimate a fractional size of a global cascade. Assume that by following randomly chosen link we arrive to a consumer of type i. Lets denote by u^i the probability that this consumer does not buy the product. This happens if a consumer does not like the product or price, which occurs with the probability $1 - q_i$ or if she likes it, but has not heard about it. We can write the system of self-consistency conditions for u^i as the following:

$$\begin{aligned} u^A &= 1 - q^A + q_A \sum_{k=1}^{\infty} \xi(k) \left[\rho^A u^A + (1-\rho^A) u^B\right]^{k-1} \\ u^B &= 1 - q^B + q^B \sum_{k=1}^{\infty} \xi(k) \left[\rho^B u^B + (1-\rho^B) u^A\right]^{k-1} \end{aligned}$$

In terms of generating functions system is:

$$\begin{aligned} u^A &= 1 - \hat{F}_1^A(1,1) + \hat{F}_1^A(u^A, u^B) \\ u^B &= 1 - \hat{F}_1^B(1,1) + \hat{F}_1^B(u^A, u^B) \end{aligned}$$

Having at hand u^A and u^B we can find probabilities that a randomly chosen consumer of type i does not form a part of a global cascade. These probabilities are different from u^A and u^B, due to the difference of degree distributions of a randomly selected and neighboring node. Note that the probability that a randomly chosen consumer does not have links to a giant component of buyers is equal to $F_0^i(u^A, u^B)$. Thus a randomly selected consumer does not buy the product with the probabilities:

$$\begin{aligned} v^A &= 1 - F_0^A(1,1) + F_0^A(u^A, u^B) \\ v^B &= 1 - F_0^B(1,1) + F_0^B(u^A, u^B) \end{aligned}$$

Calculating the linear combination with weights equal to proportions of consumer of type A and B in the society and subtracting from 1 we obtain the probability that a randomly selected consumer belongs to a giant component:

$$s = 1 - \gamma v^A - (1-\gamma) v^B = \gamma F_0^A(1,1) + (1-\gamma) F_0^B(1,1) - \gamma F_0^A(u^A, u^B) - (1-\gamma) F_0^B(u^A, u^B),$$

where

$$\begin{aligned}
u^A &= 1 - \hat{F}_1^A(1,1) + \hat{F}_1^A(u^A, u^B) \\
u^B &= 1 - \hat{F}_1^B(1,1) + \hat{F}_1^B(u^A, u^B) \\
\gamma(1-\rho^A) &= (1-\gamma)(1-\rho^B)
\end{aligned}$$

Note that the last expression insures that a number of links going from consumers of type A to consumers of type B equals to the number of links going from consumers of type B to consumers of type A. Since we choose γ, ρ^A and ρ^B exogenously we can assume that the condition holds.

As in the previous analysis we assume that there are equal proportions of consumers of type A and type B in the population ($\gamma = \frac{1}{2}$). This in turn implies that $\rho^A = \rho^B = \rho$ and the maximization problem of monopolist is summarized by the following lemma:

Lemma 3. *For two groups of equal sizes the maximization problem of monopolist becomes:*

$$\max_{q^A, q^B} \frac{1}{2}\left(q^A[1 - G_0(x)] + q^B[1 - G_0(y)]\right)$$

$$\begin{aligned}
\textit{s.t.:}\quad & x = 1 - \rho q^A[1 - \hat{G}_1(x)] - (1-\rho) q^B[1 - \hat{G}_1(y)] \\
& y = 1 - (1-\rho) q^A[1 - \hat{G}_1(x)] - \rho q^B[1 - \hat{G}_1(y)] \\
& 0 \le q^A, q^B, x, y \le 1
\end{aligned}$$

where $x = \rho^A u^A + (1-\rho^A)u^B$ is the probability that a randomly chosen link of a consumer of type A leads to a giant component of buyers and $y = \rho^B u^B + (1-\rho^B)u^A$ is the same for consumers of type B. In addition $G_0(x) = \sum_{k=0}^{\infty} p(k)x^k$ and $\hat{G}_1(x) = \sum_{k=0}^{\infty} \xi(k)x^{k-1}$. ***Proof*** *See appendix* □

Assuming a linear preferences frontier, $q^A = (1-P)w$ and $q^B = (1-P)(1-w)$ the maximization problem becomes:

$$\max_{w} \frac{1-P}{2}[1 - wG_0(x) - (1-w)G_0(y)]$$

$$\begin{aligned}
\text{s.t.:}\quad & x = 1 - (1-P)\Big(\rho w[1 - \hat{G}_1(x)] + (1-\rho)(1-w)[1 - \hat{G}_1(y)]\Big) \\
& y = 1 - (1-P)\Big((1-\rho) w[1 - \hat{G}_1(x)] + \rho(1-w)[1 - \hat{G}_1(y)]\Big) \\
& 0 \le w, x, y \le 1
\end{aligned}$$

A solution to the maximization problem is characterized by the following proposition:

Proposition 10. *In the case when population is divided into two equally sized groups, $\gamma = \frac{1}{2}$, for any degree distribution following hold:*

- *For $\rho < \frac{1}{2}$, $w = \frac{1}{2}$ is a local maximum, which gives higher profits than $w \in \{0, 1\}$.*

- *For $\rho > \frac{1}{2}$, $w = \{0, 1\}$ are local maxima, which give higher profits than $w = \frac{1}{2}$.*

- *For $\rho = \frac{1}{2}$, the interval $[0, 1]$ is the solution to the problem.*

The Proposition 10 indicates that the optimal design of the product has the same structural form as in the case where there is no global cascade of sales.

9 Conclusion

The importance of a word of mouth communication for a company's performance is well documented by a growing number of research. However, success of a WOM marketing campaign varies enormously between product categories and within. We show that a high variation in the performance of WOM campaigns can be explained by different homophily levels of a consumer network towards different products.

A key innovation of our paper is two-fold. First, we enrich a network structure by incorporating notion of homophily and study its impact on the optimal strategies of the monopolist. Second, the monopolist is allowed to construct a message to the network by choosing the design of product. We found a number of results: (i) for low levels of homophily the product, designed to attract both types of consumers is preferred to specialized products, even if there is no cost of producing more than one product; (ii) a price elasticity of demand is higher for products towards, which consumer network exhibits higher levels of homophily; (iii) a social welfare is increasing in homophily level; and (iv) a product designed to attract two types of consumers may be optimal even if the monopolist benefits only from one type.

Flexibility of the model allows us to outline several avenues for future research. The first one consists in introduction of influencers, consumers whose opinion affect opinion of many others. In the extension we want to match two observations: influencers on average enjoy higher average degree and their proportion in the population is small. The extension aimed to study the impact of homophily on information spreading in a network of "hub-and-spoke" type and the effect of presence of hubs on the optimal design of the product and price. In the case when society exhibits homophily influencers will be linked among themselves and will constitute core with access to a large share of consumers.

In the second extension we want to consider the optimal strategy for an entrant who faces presence of an incumbent firm on a market. We assume that a product is durable and consumer buys it only once. Hence, the optimal strategies for the monopolist that we have studied in the paper are optimal for incumbent firm as well. We want to study how homophily affects the optimal product design for the entrant and its effect on a variety of products that are produced. Finally, we plan to compare predictions of the model with observations from real markets.

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10 APPENDIX

Proof of Lemma 1

Let us find first what is the number of consumers of type A that buy the product if we advertise it to consumer of type A. The answer is:

$$\frac{\partial}{\partial x} H_0^A(x,y)\bigg|_{x=1,y=1} = H_{0x}^A(1,1)$$

With abuse of notation we assume that all function are being evaluated at point $(1,1)$:

$$H_{0x}^A = F_0^A + F_{0x}^A H_{1x}^A + F_{0y}^A H_{1x}^B$$

We can find H_{1x}^i by solving linear system of self-consistency conditions:

$$\begin{cases} H_{1x}^A = \hat{F}_1^A + \hat{F}_{1x}^A H_{1x}^A + \hat{F}_{1y}^A H_{1x}^B \\ H_{1x}^B = \hat{F}_{1x}^B H_{1x}^A + \hat{F}_{1y}^B H_{1x}^B \end{cases}$$

In vector form:

$$\begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix} \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

or in more compact way

$$\left(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}}\right) \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix},$$

where $\hat{F}_1^i = \sum_{k=1}^{\infty} \xi(k) q_k^i$ and

$$\hat{\mathbf{F}}'_{\mathbf{1}} = \begin{pmatrix} \hat{F}_{1x}^A & \hat{F}_{1y}^A \\ \hat{F}_{1x}^B & \hat{F}_{1y}^B \end{pmatrix} = \sum_{k=1}^{\infty} \xi(k)(k-1) \begin{pmatrix} q_k^A \rho^A & q_k^A (1-\rho^A) \\ q_k^B (1-\rho^B) & q_k^B \rho^B \end{pmatrix}$$

The number of consumers of type A who buy the product if consumer of type i finds out about the product from one of her friends H_{1x}^i goes to infinity when determinant of the matrix $\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}}$ goes to zero:

$$\Delta = \det \begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix}$$

The system has following solution:

$$\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1} \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

Thus we can find the number of consumers of type A who buy the product if consumer of type A receives direct advertisement:

$$H_{0x}^A = F_0^A + (F_{0x}^A \; F_{0y}^A)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

By doing analogous calculations we can get:

$$H_{0x}^B = (F_{0x}^B \; F_{0y}^B)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

$$H_{0y}^A = (F_{0x}^A \; F_{0y}^A)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

$$H_{0y}^B = F_0^B + (F_{0x}^B \; F_{0y}^B)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = F_0^B + (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

The total number of purchases resulted from direct advertisement to a consumer of type A is following:

$$H_{0x}^A + H_{0y}^A = F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix} + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix} =$$

$$= F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

If the monopolist advertises the product to consumer of type B:

$$H_{0x}^B + H_{0y}^B = F_0^B + (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Let us define:

$$F'_0 = \left(\begin{array}{cc} F_{0x}^A & F_{0y}^A \\ F_{0x}^B & F_{0y}^B \end{array} \right) = \sum_{k=1}^{\infty} kp(k) \left(\begin{array}{cc} q_k^A \rho^A & q_k^A (1-\rho^A) \\ q_k^B (1-\rho^B) & q_k^B \rho^B \end{array} \right)$$

The resulting number of purchases resulting from advertisement to consumers of type A and B in vector form is:

$$\mathbf{s} = \begin{pmatrix} H_{0x}^A + H_{0y}^A \\ H_{0x}^B + H_{0y}^B \end{pmatrix} = \begin{pmatrix} F_0^A \\ F_0^B \end{pmatrix} + \mathbf{F}'_{\mathbf{0}}(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Thus the number of purchases resulting from advertisement to a randomly drawn consumer is:

$$s = (\gamma\ 1-\gamma)\mathbf{s} = (\gamma\ 1-\gamma)\left[\begin{pmatrix} F_0^A \\ F_0^B \end{pmatrix} + \mathbf{F_0'}(\mathbf{I}-\hat{\mathbf{F}}_1')^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}\right]$$

Assuming that the probability to purchase the product does not depend on the number of neighbors that consumer has, namely $q_k^A = q^A$ and $q_k^B = q^B$ we obtain:

$$s = (\gamma\ 1-\gamma)\mathbf{s} = (\gamma\ 1-\gamma)\left[\mathbf{I} + \mathbf{F_0'}(\mathbf{I}-\hat{\mathbf{F}}_1')^{-1}\right]\begin{pmatrix} q^A \\ q^B \end{pmatrix}$$

Note that expression depends on the linear combination of probability to infect initial node $(\gamma\ 1-\gamma)\binom{q^A}{q^B}$ and $(\gamma\ 1-\gamma)\left(\mathbf{I}-\hat{\mathbf{F}}_1'\right)^{-1}\binom{q^A}{q^B}$ which is number of infected nodes if we follow randomly chosen link, with weight given buy $\frac{z_1^2}{z_2}$.

Proof of Proposition 1

The global cascade of sales arises when inequality holds:

$$w^2(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) - w(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \leq 0$$

We want to identify a condition such that there exists characteristic of the product w which satisfies the inequality and hence the global cascade of sales may arise. To this end we find the minimum of the expression and check when it is less than zero. A derivative of the expression with respect to w is $-(1-P)^2(z_2/z_1)^2(1-2\rho)(1-2w)$. Note that if $\rho < \frac{1}{2}$ coefficient of the term w^2 is positive and thus we have upward sloping parabola. In this case function has its minimum at the point $w = \frac{1}{2}$. Substituting to the expression and taking positive root we obtain a condition $z_2/z_1 > 2(1-P)^{-1}$. On the other hand, if $\rho > \frac{1}{2}$ we have downward parabola with maximum at $w = \frac{1}{2}$ and minima on the ends of the interval $[0,1]$, which implies that we have cascade if $\rho > \frac{z_1}{z_2(1-P)}$. Combining both parts we arrive to the following condition: $\frac{z_2}{z_1} > \frac{1}{(1-P)}\min\{2, \rho^{-1}\}$.

Note that the condition on network structure becomes less restrictive when price decreases. Thus if price is a part of decision process the monopolist can achieve highest diffusion when $P = 0$ and condition becomes $\frac{z_2}{z_1} > \min\{2, \rho^{-1}\}$.

Proof of Proposition 2

Substituting constraints to the objective function and deriving with respect to w we find:

$$(1-P)^2\frac{z_1^4(1-2w)(1-2\rho)(2z_1 + z_2(1-P)(1-2\rho))}{2(z_1^2 - z_1z_2\rho(1-P) - wz_2^2(1-P)^2(1-w)(1-2\rho))^2}$$

A denominator of the condition is always positive and thus sign depends on the numerator. Recall that we assume that we are in sub-critical phase with $\frac{z_2}{z_1} < 2(1-P)^{-1}$ and thus term $2z_1 + z_2(1-P)(1-2\rho)$ is always positive. The sign of the condition depends exclusively on values of ρ and w. Namely if $\rho < \frac{1}{2}$ derivative is positive for $w < \frac{1}{2}$ and negative afterwards. Thus, we can conclude that for $\rho < \frac{1}{2}$ objective function has unique maximum at the point $w = \frac{1}{2}$. In the case when $\rho > \frac{1}{2}$ results are reversed and the objective function has its minimum at a point $w = \frac{1}{2}$ and maxima lie on the boundaries, namely $w^* \in \{0, 1\}$. If $\rho = \frac{1}{2}$ all interval $[0, 1]$ satisfies first order condition.

Proof of Proposition 3

We analyze the second part of the functional form of demand. Results for the first part can be obtained by substituting $\rho = \frac{1}{2}$. The demand is decreasing and convex in P:

$$\frac{\partial}{\partial P}Q(P,\rho,z_1,z_2) = -\frac{1}{2}\left(1 + \frac{(1-P)z_1^2\rho(2z_1 - z_2(1-P)\rho)}{(z_1-(1-P)z_2\rho)^2}\right) < 0$$

The second derivative:

$$\frac{\partial^2}{\partial P^2}Q(P,\rho,z_1,z_2) = \frac{z_1^4\rho}{(z_1-(1-P)z_2\rho)^3} > 0$$

It is positive since by condition of no global cascade from Lemma 2 we know that $z_1 > (1-P)z_2\rho$. Moreover cross derivative of $Q(P,\rho,z_1,z_2)$ with respect to P and ρ is $-\frac{(1-P)z_1^4}{(z_1-(1-P)z_2\rho)^3}$, which is negative.

Lets turn to the elasticity of demand:

$$\begin{aligned} E_d &= \frac{\partial_P \log Q(P,\rho,z_1,z_2)}{\partial_P \log P} \\ &= -\frac{P}{1-P}\left(1 + z_1\left(\frac{1}{z_1-(1-P)z_2\rho} - \frac{1}{z_1+(1-P)(z1^2-z2)\rho}\right)\right) \end{aligned}$$

Taking derivative of $|E_d|$ with respect to ρ we obtain:

$$\frac{\partial}{\partial \rho}|E_d| = \frac{z_1^3 z_2 P(1-P)^2(z1^2-z2)\rho^2 + z_1^5 P}{(z1-(1-P)z_2\rho)^2(z_1-(1-P)z_2\rho + z_1^2\rho(1-P))^2} > 0$$

Which implies that elasticity of demand is increasing in ρ.

$$\frac{\partial}{\partial \rho}s^*(P,\rho,z_1,z_2) = \frac{(1-P)^2 z_1^3}{2(z_1-(1-P)z_2\rho)^2} > 0$$

Thus for $\rho > \frac{1}{2}$ function is increasing in ρ.

Proof of Proposition 4

Price is decreasing in the homophily level

The first order condition for P is:

$$\frac{(1-2P)z_1^2-(1-P)z_1[(1-P)z_2+(1-3P)(z_2-z_1^2)]\rho}{2(z_1-(1-P)z_2\rho)^2}-$$
$$-\frac{(1-P)^2(1-2P)(z_1^2-z_2)z_2\rho^2}{2(z_1-(1-P)z_2\rho)^2}=0$$

Let us fix expected number of friends z_1 and z_2 and call the expression on the left hand side $F(P,\rho)$. The second derivative of $F(P,\rho)$ with respect to P is:

$$F''_P(P,\rho)=\frac{3z_1^4\rho(z_1-z_2\rho)}{(z_1-(1-P)z_2\rho)^4}>0$$

It is positive since by conditions of no giant component we have $z_1>z_2\rho$. Thus function is convex. Evaluating function on the ends of the interval we have $F(0,\rho)=\frac{z_1+z_1^2\rho-z_2\rho}{2(z_1-z_2\rho)}>0$ and $F(1,\rho)=-\frac{1}{2}$. The first derivative with respect to P is negative at 0:

$$F'_P(0,\rho)=-1-\frac{z_1^2\rho(2z_1-z_2\rho)}{(z_1-z_2\rho)^2}<0$$

If $F(P,\rho)$ is convex in P, positive at 0 and negative at 1, we can conclude that function should intersect x-axis from above on the interval $[0,1]$. Hence, the derivative of the $F(P,\rho)$ evaluated for the optimal price $P=P^*$ is negative, $\frac{\partial}{\partial P}F(P^*,\rho)<0$.

Moreover $F(\frac{1}{2},\rho)=\frac{z_1^3\rho}{2(2z_1-z_2\rho)^2}>0$, which implies that the optimal price is always less than $\frac{1}{2}$. The derivative with respect to ρ is:

$$F'_\rho(P,\rho)=-\frac{(1-P)z_1^3\left[(1-P)^2z_2\rho-(1-3P)z_1\right]}{2(z_1-(1-P)z_2\rho)^3}$$

The sign of the derivative depends on the sign of the term in square brackets. Thus taking into account that $P>0$ the derivative is negative if $P>\bar{P}=1-\frac{3z_1}{2z_2\rho}+\frac{\sqrt{9z_1^2-8z_1z_2\rho}}{2z_2\rho}$. One can check that $F(\bar{P},\rho)>0$. The fact that $F(P,\rho)$ intersects x-axes from above, implies that $P^*>\bar{P}$ and thus the derivative is negative.

By implicit function theorem we know:

$$\frac{\partial P^*}{\partial\rho}=-\left.\frac{\frac{\partial}{\partial\rho}F(P,\rho)}{\frac{\partial}{\partial P}F(P,\rho)}\right|_{P=P^*}$$

Taking into account that $F'_P(P^*,\rho)<0$ and $F'_\rho(P^*,\rho)<0$ we can conclude that $\frac{\partial P^*}{\partial\rho}$ is negative and consequently the optimal price P^* is decreasing in ρ.

Profits are increasing in the level of homophily

Lets take two levels of homophily $\rho_2 > \rho_1$. By the Proposition 3 we know that for any fixed price P following holds $Q(P, \rho_2, z_1, z_2) > Q(P, \rho_1, z_1, z_2)$. Thus for any given price P the same is true for profits, namely $PQ(P, \rho_2, z_1, z_2) > PQ(P, \rho_1, z_1, z_2)$. Assume further that P_1^* is optimal price for ρ_1. The previous result states that $\pi(\rho_2, P_1^*) > \pi(\rho_1, P_1^*)$ and thus by optimality we know that $\pi(\rho_2, P_2^*) > \pi(\rho_2, P_1^*) > \pi(\rho_1, P_1^*)$, where P_2^* is optimal price for ρ_2.

Proof of Proposition 7

A derivative of sales function with respect to product characteristic is given by:

$$s'(w) = (1-P)\times$$
$$\times\frac{1 + z_1(1-P)(1-3\rho-(1-2\rho)[w^2(1-P)z_1 + 2w(1-\rho(1-P)z_1) + \rho(1-P)z_1])}{(1-wz_1^2(1-P)^2(1-2\rho)(1-w) - z_1\rho(1-P))^2}$$

Note that the denominator is always positive. It is easy to verify that for $\rho > \frac{1}{2}$ all terms in the numerator involving w are positive too. Thus if we prove that $s'(0) > 0$ then the derivative of sales function with respect to w is positive on the whole interval $[0, 1]$ and we can conclude that the optimal design is $w^* = 1$. Substituting $w = 0$ into the derivative and taking into account that $z_1 < \frac{1}{\rho(1-P)}$ we have:

$$s'(0) = \frac{1 + (1-P)z_1(1-2\rho)}{1-(1-P)z_1\rho} = 1 + \frac{(1-P)z_1(1-\rho)}{1-(1-P)z_1\rho} > 0$$

Thus for $\rho > \frac{1}{2}$ characteristic $w^* = 1$ is the solution to the problem.

When $\rho < \frac{1}{2}$ then all terms involving w are negative and numerator is decreasing function in w. Thus numerator has its minimum at $w = 1$ and condition for $s'(w) > 0$ on the interval $w \in [0, 1]$ is simply $s'(1) > 0$. This in turn implies that if $w^* = 1$ is maximum it is also the global maximum. The derivative at 1 is greater than zero if:

$$-2(1-P)^2z_1^2\rho^2 + z_1(1-P+3(1-P)^2z_1)\rho + 1 - (1-P)z_1 - (1-P)^2z_1^2 > 0$$

An expression on the left describes downward sloping parabola. The solution is:

$$\rho_1 = \frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2z1^2}}{4(1-P)z_1}$$

$$\rho_2 = \frac{3}{4} + \frac{1 + \sqrt{9 - 2z_1(1-P) + (1-P)^2z1^2}}{4(1-P)z_1}$$

It can be shown that $\rho_2 > 1$ and thus condition reduces to:

$$\frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1} < \rho < \frac{1}{2}$$

Combining the previous condition with the case of $\rho > \frac{1}{2}$, we know that $w^* = 1$ is solution if:

$$\rho > \hat{\rho}_T(z_1, P) = \frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1}$$

On the other hand for $\rho < \hat{\rho}_T(z_1, P)$ there is an interior solution which is given by:

$$w^* = \rho - \frac{1 - 2\rho - \sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2 + (1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)}$$

It is interesting to note that $w = \frac{1}{2}$ is never solution for $\rho < \frac{1}{2}$ since $s'(\frac{1}{2}) = \frac{4(1-P)}{(2-(1-P)z_1)(2+(1-P)z_1(1-2\rho))} > 0$, which implies that $w^* > \frac{1}{2}$.

Proof of Proposition 8

Lets denote by $\theta = \frac{z_2}{z_1}$ and by $\lambda(w)$ the following polynom:

$$\lambda(w) = 1 - (1-P)^2 w(1-w^r)^{\frac{1}{r}}\theta^2(1-2\rho) - (1-P)\left(w + (1-w^r)^{\frac{1}{r}}\right)\theta\rho$$

The global cascade of sales occurs if there is $\bar{w}$ such that $\lambda(\bar{w}) \leq 0$. One can readily obtain condition for ρ. We just need to take the derivative with respect to ρ and to show that it is negative. Thus there is the global cascade of sales if $\rho > \hat{\rho}(\theta, r)$, where

$$\hat{\rho}(\theta, r) = \min_{0 \leq w \leq 1} \frac{1 - w\theta^2(1-P)^2(1-w^r)^{\frac{1}{r}}}{(1-P)\theta\left(w + (1-w^r)^{\frac{1}{r}}(1 - 2w\theta(1-P))\right)}$$

From the previous analysis we know that candidates for maxima are extreme values and such that $q^A = q^B$, which in our case is $\left(\frac{1}{2}\right)^{\frac{1}{r}}$. Evaluating polynomial at 0 we have $\lambda(0) = 1 - (1-P)\theta\rho$ and at the point $\left(\frac{1}{2}\right)^{\frac{1}{r}}$ we have:

$$\lambda\left(2^{-\frac{1}{r}}\right) = 4^{-\frac{1}{r}}(2^{\frac{1}{r}} - (1-P)\theta)\left[2^{\frac{1}{r}} + (1-P)\theta(1-2\rho)\right]$$

From the first condition we can conclude that if $\rho > \frac{1}{\theta}$ then there is global cascade. From the second we see that if $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$ and $\rho < \hat{\rho} = \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)}$ then global cascade of sales arises. Lets consider a case when $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, but $\rho < \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)}$.

From the first condition we know that there is global cascade if $\rho > \frac{1}{\theta(1-P)}$ thus to insure existence of the global cascade we should prove that $\frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)} > \frac{1}{\theta(1-P)}$.

$$\frac{1}{2}(1-P) + 2^{\frac{1-r}{r}}\theta^{-1} > \theta^{-1}$$

$$\frac{1}{2}(1-P) > \theta^{-1}\left(1 - 2^{\frac{1-r}{r}}\right)$$

$$\theta > (1-P)^{-1}\left(2 - 2^{\frac{1}{r}}\right)$$

$$\theta > 2^{\frac{1}{r}}(1-P)^{-1}\left(2^{\frac{r-1}{r}} - 1\right)$$

There is the global cascade if the former condition holds. However, we have assumed that $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, which implies that former condition holds, since $\left(2^{\frac{r-1}{r}} - 1\right) < 1$. Thus we have shown that if $\theta > 2^{\frac{1}{r}}$ there is global cascade independently of the homophily level ρ.

Proof of Proposition 9

For the case of Poisson degree distribution size of sales cascade is given by:

$$s(w,P,r,z_1) = \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

The product characteristic $w^* = 0$ is global maximum if for any w, $s(0) \geq s(w)$:

$$\frac{1-P}{2(1-z_1\rho(1-P))} \geq \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

since $1-P \geq 0$ by definition, we have

$$\frac{1}{2(1-z_1\rho(1-P))} - \frac{(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))} \geq 0$$

Note further that denominators of two fractions are positive due to condition of no global cascade, thus the sign of the expression depends on the numerator of combined terms, which is:

$$
\begin{aligned}
-4(1-P)^2 w(1-w^r)^{\frac{1}{r}} z_1^2 \rho^2 + 4(1-P)w(1-w^r)^{\frac{1}{r}} z_1(1+(1-P)z_1)\rho + \\
+1 - w - (1-w^r)^{\frac{1}{r}}(1+(1-P)wz_1(2+(1-P)z_1)) \quad \geq 0
\end{aligned}
$$

Note that expression describes downward sloping parabola and thus our condition will be in the form $\rho_1 \leq \rho \leq \rho_2$, where ρ_1 and ρ_2 are solutions to the quadratic equation:

$$
\rho_1 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{z_1(1-P)} - \frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)
$$

$$
\rho_2 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{z_1(1-P)} + \frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)
$$

The condition should hold for all w and thus we should find the maximum of ρ_1 and the minimum of ρ_2. In order to do this we should identify maximum and minimum of the term with w. Taking derivative of this term with respect to w we have:

$$
-\frac{(1-w^r)^{-\frac{1+r}{r}}\left[(1-2w^r) + \left(w^{r+1} - (1-w^r)^{\frac{1+r}{r}}\right)\right]}{w^2}
$$

Independently of r one can see that first derivative is zero at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. It can be proved that for $r < 1$, $-(1-2w^r) - \left(w^{r+1} - (1-w^r)^{\frac{1+r}{r}}\right)$ is negative for $w < \left(\frac{1}{2}\right)^{\frac{1}{r}}$ and positive afterwards, which implies that minimum is at $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ and maxima lay at borders.

Substituting back values of w into condition for ρ we obtain:

$$
\frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)} < \rho < \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2z_1(1-P)}
$$

Since $z_1 \leq 2^{\frac{1}{r}}$ the minimum of last term is 1. Taking it into account we can rewrite the condition as:

$$
\frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)} < \rho < 1
$$

On the other hand for $r > 1$, one can show that $w^* = 0$ is never a solution. Lets evaluate derivative of sales at $w = 0$ for the case when $r > 1$ we have:

$$
\left.\frac{\partial s}{\partial w}\right|_{w=0} = \frac{(1-P)(1+(1-P)z_1(1-2\rho))^2}{2(1-(1-P)z_1\rho])^2} > 0
$$

It is always positive which implies that $w^* = 0$ is never solution for $r > 1$.

The symmetric design $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ is global maximum if for any w, $s\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}\right) \geq s(w)$:

$$\frac{1-P}{2^{\frac{1}{r}} - (1-P)z_1} \geq \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

The denominators are positive due to no global cascade condition and thus sign of the expression depends on the numerator of combined fraction, which is:

$$\begin{aligned} -2(1-P)(w + (1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}})z_1\rho + 2 - (2-2w^r)^{\frac{1}{r}} + \\ +(1-P)(1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}}z_1 - w(2^{\frac{1}{r}} - (1-P)z_1) \quad \geq 0 \end{aligned}$$

Note that the line is downwards sloping if for any w, $(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}) > 0$. Thus to prove that it has downward slope we should prove that the minimum of the term $w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}$ is greater or equal to 0. The first derivative is:

$$1 - 2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} - \left(\frac{(1-w^r)^{\frac{1}{r}}}{w}\right)^{1-r} \tag{4}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r < 1$ the expression is negative for $w < \left(\frac{1}{2}\right)^{\frac{1}{r}}$, since term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} > 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} > 1$ (by properties of frontier). The expression is positive for $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, because term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} < 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} < 1$. This implies that minimum lies at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ where the expression equals to zero. Thus the line has negative slope. And condition becomes:

$$\rho < \hat{\rho}_1 = \min_w \left\{ \frac{1}{2}\left(1 - \frac{2^{\frac{1}{r}}(w + (1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}})}\right)\right\}$$

we can show that for $r < 1$ the expression with w has its maxima on the borders and thus, evaluating at $w = 0$ we have:

$$\rho < \frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)}$$

The case when $r > 1$

Lets rewrite the expression (4):

$$1 - (2^{\frac{1+r}{r}}(1-2w^r) + w^{r-1})(1-w^r)^{\frac{1-r}{r}}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r > 1$ and $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\min\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\min\{(2^{\frac{1+r}{r}}(1-2w^r)+w^{r-1})\} = 2^{\frac{1-r}{r}}$. Thus minimum of the product of two terms is equal to 1 and this implies that for $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$ expression is negative. For $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\max\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\max\{(2^{\frac{1+r}{r}}(1-2w^r)+w^{r-1})\} = 2^{\frac{1-r}{r}}$, which implies that expression is positive. Thus minimum of the expression is at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ line has negative slope.

This leads us again to the condition:

$$\rho < \hat{\rho}_2 = \min_w \left\{ \frac{1}{2} \left(1 - \frac{2^{\frac{1}{r}}(w + (1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}})} \right) \right\}$$

Thus we can establish that there is $\hat{\rho}_2$ such that if $\rho < \hat{\rho}_2$ than the optimal characteristic is $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. Note that condition $\hat{\rho}_1 \neq \hat{\rho}_2$ since we optimize for different values of r. If $\rho > \hat{\rho}_2$ then solution belongs to $(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1)$, since as we have seen $w = 1$ is always not optimal for the case of $r > 1$.

Proof of Lemma 3

Let us denote by $x = \rho u^A + (1-\rho)u^B$ and by $y = \rho u^B + (1-\rho)u^A$ than we can rewrite system of equations as following:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$u^A = 1 - q^A + q^A \hat{G}_1[x]$$
$$u^B = 1 - q^B + q^B \hat{G}_1[y]$$

Or equivalently:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$x = \rho[(1-q^A) + q^A \hat{G}_1(x)] + (1-\rho)[(1-q^B) + q^B \hat{G}_1(y)]$$
$$y = \rho[(1-q^B) + q^B \hat{G}_1(y)] + (1-\rho)[(1-q^A) + q^A \hat{G}_1(x)]$$

Substituting $\gamma = \frac{1}{2}$ we obtain following maximization problem of the monopolist:

$$\max_{q^A, q^B} \frac{1}{2} \left[q^A + q^B - q^A G_0(x) - q^B G_0(y) \right]$$

s.t.

$$
\begin{aligned}
x &= 1 - \rho q^A - (1-\rho) q^B + \rho q^A \hat{G}_1(x) + (1-\rho) q^B \hat{G}_1(y) \\
y &= 1 - (1-\rho) q^A - \rho q^B + (1-\rho) q^A \hat{G}_1(x) + \rho q^B \hat{G}_1(y)
\end{aligned}
$$

Proof of Proposition 10

The FOC of the monopolist problem is:

$$-G_0(x) - wG_0'(x)\frac{\partial x}{\partial w} + G_0(y) - (1-w)G_0'(y)\frac{\partial y}{\partial w} = 0$$

The derivatives of constraints with respect to w are following:

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w \hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho) w \hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

The case when firm's action has no effect.

Interesting case arises when $\rho = \frac{1}{2}$. It seems that w has no effect on the size of global cascade of sales . Substituting $\rho = \frac{1}{2}$ we can rewrite the problem as following:

$$\max_w 1 - wG_0(x) - (1-w)G_0(y)$$

s.t.

$$
\begin{aligned}
&x = \tfrac{1}{2} + \tfrac{1}{2} w \hat{G}_1(x) + \tfrac{1}{2}(1-w)\hat{G}_1(y) \\
&y = \tfrac{1}{2} + \tfrac{1}{2} w \hat{G}_1(x) + \tfrac{1}{2}(1-w)\hat{G}_1(y) \\
&0 \le w \le 1, 0 \le x \le 1, 0 \le y \le 1
\end{aligned}
$$

Note that in this case $x = y$ for any w. This implies that maximization problem of the monopolist in the case of $\rho = \frac{1}{2}$ does not depend on w:

$$\max_w 1 - G_0(x)$$

s.t.

$$
\begin{aligned}
&x = \tfrac{1}{2} + \tfrac{1}{2}\hat{G}_1(x) \\
&0 \le x \le 1
\end{aligned}
$$

Thus eventual outbreak is the same for all values of w and moreover it's size is equal to the giant component of connected consumers.

The case when specialized design is optimal.

We want to check when it is optimal to focus on the first group or equivalently when $w = 1$ is the solution. Note that $w = 1$ is corner solution that is why it is enough to show that derivative of $\frac{\partial s}{\partial w}\big|_{w=1}$ is non-negative:

$$-G_0(x) - G_0'(x)\frac{\partial x}{\partial w} + G_0(y) > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$
$$y = \rho + (1-\rho)\hat{G}_1(x)$$

The derivative of first constraint with respect to w is:

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y)$$

Thus we have

$$\frac{\partial x}{\partial w} = \frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)}$$

Substituting it to the maximization problem we obtain:

$$G_0(y) - G_0(x) - G_0'(x)\frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)} > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$
$$y = \rho + (1-\rho)\hat{G}_1(x)$$

Let us rewrite the first equation as:

$$[G_0(y) - G_0(x)] + G_0'(x)\frac{\rho(1 - \hat{G}_1(x)) - (1-\rho)[1 - \hat{G}_1(y)]}{1 - \rho\hat{G}_1'(x)} \geq 0$$

The first term is non-negative when $y \geq x$ and the condition is following:

$$\rho + (1-\rho)\hat{G}_1(x) \geq 1 - \rho + \rho\hat{G}_1(x)$$

$$(2\rho - 1)[1 - \hat{G}_1(x)] \geq 0$$

since $\hat{G}_1(x) \leq 1$ for all $x \in [0, 1]$ the condition is $\rho \geq \frac{1}{2}$.

The same happens with the second term when $\rho > \frac{1}{2}$. Note that $\rho > \frac{1}{2}$ implies that $\hat{G}_1[x] < \hat{G}_1[y]$ consequently $1 - \hat{G}_1[x] > 1 - \hat{G}_1[y]$. Multiplying both sides by ρ and taking into account that $\rho \geq 1 - \rho$ for $\rho \geq \frac{1}{2}$ we have:

$$\rho[1 - \hat{G}_1(x)] \geq \rho[1 - \hat{G}_1(y)] \geq (1 - \rho)[1 - \hat{G}_1(y)]$$

Thus we have proved that $w = 1$ is locally optimal if $\rho > \frac{1}{2}$ independently of degree distribution.

The case when symmetric design is optimal

The symmetric design $w = \frac{1}{2}$ is optimal if following holds:

$$-G_0(x) - \frac{1}{2}G_0'(x)\frac{\partial x}{\partial w} + G_0(y) - \frac{1}{2}G_0'(y)\frac{\partial y}{\partial w} = 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\rho\hat{G}_1(x) + \frac{1}{2}(1 - \rho)\hat{G}_1(y)$$

$$y = \frac{1}{2} + \frac{1}{2}(1 - \rho)\hat{G}_1(x) + \frac{1}{2}\rho\hat{G}_1(y)$$

and

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1 - \rho)\hat{G}_1(y) + (1 - \rho)(1 - w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1 - \rho)\hat{G}_1(x) + (1 - \rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1 - w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

It is easy to check that $x = y$ satisfies our conditions on x and y, thus we have:

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Moreover it is possible to show (it should be) that system of equations for x and y has only two solution. The first one is $x = y = 1$. Thus we have:

$$G_0'(x)\left[\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}\right] = 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

The first solution to the FOC equation is $G_0'(x) = 0$ which implies $x = 0$. This obviously does not satisfy second equation, thus the only possibility left is $\frac{\partial x}{\partial w} = 0$

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

Solving previous system and substituting $y = x$ we have:

$$\frac{\partial x}{\partial w} = -\frac{1 - 2\rho - \hat{G}_1(x) + 2\rho\hat{G}_1(x) - \rho\hat{G}_1(x)\hat{G}_1'(x) + \rho\hat{G}_1'(x) + \frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x) - \frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2 + \frac{1}{4}[\hat{G}_1'(x)]^2 + \rho\hat{G}_1'(x) - 1}$$

$$\frac{\partial y}{\partial w} = \frac{1 - 2\rho - \hat{G}_1(x) + 2\rho\hat{G}_1(x) - \rho\hat{G}_1(x)\hat{G}_1'(x) + \rho\hat{G}_1'(x) + \frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x) - \frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2 + \frac{1}{4}[\hat{G}_1'(x)]^2 + \rho\hat{G}_1'(x) - 1}$$

Note that $\frac{\partial x}{\partial w} = -\frac{\partial y}{\partial w}$ and thus we have that:

$$G_0'(x)\left[\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}\right] = 0$$

This implies that $w = \frac{1}{2}$ is always the critical point. What is left to proof is that it is maximum when $\rho < \frac{1}{2}$.

SOC of the problem

$$-2G_0'(x)\frac{\partial x}{\partial w} - wG_0''(x)\left(\frac{\partial x}{\partial w}\right)^2 - wG_0'(x)\frac{\partial^2 x}{\partial w^2} + 2G_0'(y)\frac{\partial y}{\partial w} - (1-w)G_0''(y)\left(\frac{\partial y}{\partial w}\right)^2 - (1-w)G_0'(y)\frac{\partial^2 y}{\partial w^2}$$

SOC when $w = \frac{1}{2}$

$$-\frac{4z(1-2\rho)(1-\hat{G}_1(x))}{\left(2-\hat{G}_1'(x)\right)\left(2+(1-2\rho)\hat{G}_1'(x)-2\right)^2}\times$$

$$\times\left((1-2\rho)\left(\hat{G}_1'(x)^2 + 2\hat{G}_1'(x) + \hat{G}_1''(x)(1-\hat{G}_1(x))\right)\hat{G}_1(x) + 8\hat{G}_1(x) + (1-2\rho)\left(2-\hat{G}_1'(x)\right)\hat{G}_1'(x)\right)$$

Thus we can conclude that $w = \frac{1}{2}$ is local maximum for $\rho < \frac{1}{2}$ if:

$$2 - \hat{G}_1'(x) > 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Let us denote by $F(x) = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$ then solution to equation x^* is such that $F(x)$ crosses 45 degree line in x^* from above since $F(0) = \frac{1}{2}$ Thus we can conclude that $F'(x^*) < 1$. Thus $\frac{1}{2}\hat{G}_1'(x) < 1$ and consequently $\hat{G}_1'(x) < 2$, which in turn implies that our condition always holds.

When $w = \frac{1}{2}$ ***should be preferred to*** $w = 1$***?***

Recall that in the case when $w = \frac{1}{2}$ the size of the giant component is given by:

$$S(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}G_0(x_m^*)$$

$$x_m^* = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x_m^*)$$

On the other hand if $w = 1$ we have:

$$S(1) = \frac{1}{2} - \frac{1}{2}G_0(x_b^*)$$

$$x_b^* = 1 - \rho + \rho\hat{G}_1(x_b^*)$$

Due to the monotonicity of the $G_0(x)$ we know that $S(\frac{1}{2}) > S(1)$ whenever $x_m^* < x_b^*$. So basically we should see how ρ affects solution to fixed point problem, since first equation is just particular case of the second. Using IFT we have:

$$\frac{\partial x}{\partial \rho} = -1 + \hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial \rho}$$

$$\frac{\partial x}{\partial \rho} = -\frac{1 - \hat{G}_1(x)}{1 - \rho\hat{G}_1'(x)}$$

Note that x is solution to fixed point of $1 - \rho + \rho\hat{G}_1(x)$ at x^* it should cross the 45 degree line and this in turn implies that $\rho\hat{G}_1'(x) < 1$ thus we have shown that $\frac{\partial x}{\partial \rho} < 0$. Thus in turn implies that if $\rho < \frac{1}{2}$ $x_b^* > x_m^*$ and thus $S(\frac{1}{2}) > S(1)$ on the other hand if $\rho > \frac{1}{2}$ thus $S(\frac{1}{2}) < S(1)$