Chaos and Bifurcation in 2007-09+ Financial Crisis

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Abstract

The impact of increasing leverage in the economy produces hyperreaction of market participants to variations of their revenues. If the income of banks decreases, they mass-reduce their lendings; if corporations sales drop, and due to existing debt they cannot adjust their liquidities by further borrowings, then they must immediately reduce their expenses, lay off staff, and cancel investments. This hyperreaction produces a bifurcation mechanism, and eventually a strong dynamical instability in capital markets, commonly called systemic risk. In this article, we show that this instability can be monitored by measuring the highest eigenvalue of a matrix of elasticities. These elasticities measure the reaction of each sector of the economy to a drop in its revenues from another sector. This highest eigenvalue - the spectral radius - of the elasticity matrix, can be used as an early indicator of market instability and potential crisis. Grandmont (1985) and subsequent research showed the possibility that the “invisible hand” of markets become chaotic, opening the door to uncontrolled swings. Our contribution is to provide an actual way of measuring how close to chaos the market is. Estimating elasticities and actually generating the indicators of instability will be the topic of forthcoming research.

Keywords systemic risk, systemic crisis, econophysics, macroeconomics, bifurcation, system stability, chaos

1 Introduction

The global aspect of the subprime-originated financial crisis in 2007-08 is the contagion of risks which is better described by the butterfly effect, which started regularly being mentioned after the series of financial “unthinkables” that took place in September of 2008, starting with the nationalization of government sponsored enterprises Freddie Mac and Fannie Mae, the demise of the investment bank Lehman Brothers a week later, the fire-sale of another one Merrill Lynch to Bank of America on the same day, and the government bail-out of the insurance giant AIG just two days later.

“Butterfly effect” is a term used to describe a phenomenon such that small changes at the initial stage result in a huge difference in long-term behavior. The current financial crisis started in the U.S. real estate market and spread to all over the world, and people are still debating when and how this crisis will be over. Such a phenomenon is formally defined in dynamical systems as sensitive dependence on initial condition. When a dynamical system possesses a sensitive dependence on initial condition together with cyclical behavior, the system often exhibits chaos, [22].

In dynamical systems theory, a bifurcation refers to a structural modification of the system behavior upon a continuous change in the parameters of its equations. A catastrophe occurs when, following a bifurcation, a small change in parameters discontinuously alters the equilibrium state of the economy. During
the 2007-09+ subprime crisis, we did observe such a catastrophic event, where a mild evolution of economic parameters ended into a drastic shift in financial interactions. On the brink of the crisis, the economy was in what physicists call a “meta-stable equilibrium”, that is, an equilibrium state that is destroyed by a very small perturbation - like a dry forest totally burning upon the scratch of a match - leading to a series of catastrophic events, until another basin of attraction is reached, i.e. another stationary evolution mode, another cycle or, even, a strange attractor as chaos theory may predict.

Although the economy as a whole is a very complex body (system) which is hard to describe as a finite dimensional dynamical systems, its behavior during the crisis leads us to state the hypothesis that its major drivers can be described by such a system.

In this paper we suggest that the current financial crisis was mainly caused by a breakdown of the dynamic stability of the financial system, according to some catastrophic mechanism. More precisely, we start from a mathematical model in $\mathbb{R}^n$ (the dimension $n$ will be specified in the next section) of the financial system that exhibits a stationary state equilibrium. The financial activities are considered as continuous perturbations of this equilibrium: when the perturbation is small enough, the equilibrium persists and the economy remains stable. When the perturbation is too big, the equilibrium collapses and a financial crisis emerges. Furthermore, we show that the critical size of perturbations that destroy the equilibrium shrinks when financial actors react more rapidly and intensely to other actors they are in business with, leading to a meta-stable equilibrium and a catastrophe. The critical perturbation size is directly related to the debt and borrowing capacity, the leverage, and the market liquidity. In other words our mathematical model shows the causal relation between leverage and market instability.

Based on these observations, we propose the principles of methodology to build an early indicator of the global system instability. The details of such indicator still need to be worked out and tested, as all economic indices involved in this methodology are not readily available. Estimating elasticities and actually generating the indicators of instability will be the topic of forthcoming research.

In Section 2 we provide an intuitive view of the chaos in the current financial crisis and relevant mathematics background. Section 3 will be fully devoted to the structural stability and perturbation analysis of the financial system. Section 4 is devoted to the aftermath of the crisis and the chaotic behavior in the default mechanism.
2 Glimpse of Chaos

Financial crisis is generally defined as a situation in which some financial institutions or assets suddenly lose a large part of their value. The current (2007 - 2009+) crisis started in a small sector of global economy called “the U.S. real estate market”. In the United States the housing bubble started to form in 2001 following the burst of the tech bubble, and the September 11 events. After the Federal Reserve’s rate cut and the development of securitization (CDO, MBS) and credit derivatives (CDS), a lot of money was made available for lending (housing, consumers, corporates, etc.). These drove interest rates down, both T-bonds and corporate spreads, which resulted in a bond rally. That induced capital inflow. Therefore even more money was made available, and this feedback loop created the credit bubbles. In parallel, private equity investors and venture capitalists who lost a lot in the crack of the Nasdaq in 2000 had to deliver returns to their investors and started becoming short-termists. A large portion of investment in small and medium companies was done through LBO.

The increasing debt of companies in general made their balance sheets more and more sensitive to the level of credit spreads. As long as credit spreads were going down, corporations could refinance themselves, but the least loss of confidence from lenders would (and did) put them in financial squeeze. Many people started investing in real estate. The Case-Shiller Home Price Index had its peak in the second quarter of 2006 [27], and the U.S. house prices have drastically decreased since. Many banks and financial entities, both regional and global, went bankrupt [4] [15]. Companies, both small and large, went under as well [1]. In both cases the main cause was the lack of solvency and liquidity. During the development of the crisis, the damage seemed to be only getting more severe, massive, and unpredictable with questions about a “double dip”. Fourth year in the crisis, the current situation seems to be stabilizing, but the future still looks unpredictable. In the meantime, the blame has been aimed at financial engineers (also known as “quants”) for having supported the creation of esoteric financial derivatives and used faulty mathematical models to evaluate them.

Avoiding the question of pricing model validity, which appears to us as a side question, we try to understand the financial, then economic crisis in its dynamical aspects.

Here we get a glimpse of chaos: what started locally has spread globally with unpredictable severity. This demonstrates sensitive dependence on initial condition. The mathematical models which used to work well in the past, all of a sudden do not work any longer, as this happens when a dynamical system experiences a bifurcation.

Typically chaos is found in dynamical systems that possess nontrivial recurrence (i.e. which cannot be isolated), and indeed there is “recurrence of risks”

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1 collateralized debt obligation, mortgage backed security, and credit default swap
2 leveraged buyout
behind the current financial crisis. In dynamical systems theory recurrence is produced by the feedback loop. The typical road to chaos is as follows:

- A bifurcation breaks the stability of some equilibrium.
- Through a catastrophe the asymptotic behavior suddenly shifts from the formal equilibrium to a completely different behavior.
- When recurrence occurs, symbolic dynamics takes place and the asymptotic behavior is chaotic.

For instance, securitization was one of the causes that created a bifurcation and eventually chaos. Although the original purpose of the securitization was to diversify default risk, this “originate to distribute” practice spawned too many risky loans which were destined to default. As a result the risk was disseminated globally as opposed to diversified, then boomeranged back to the issuer of the loans as well as to the borrowers. This is because all the financial transactions were made in a closed system. If the Masters of the Universe on our planet had managed to pass all those risks to other denizens of the universe, we would not be having any of the problems we are having now, for the Earth aggregately acts as a source of risks. This was for instance the case of local systemic crisis such as the Asian-Russian one in 1997-98, which eventually was rapidly absorbed.

To visualize the situation, let us consider the following feedback model. If we consider the so-called fixed and variable cash flows among financial segments during the real estate boom, cash flows that one usually calls “fixed” are the scheduled ones, such as salaries, contributions to pension and other investment plans, coupons, installments, etc. Other cash flows are said “variable” because they are at will: investments, loans, dividends, savings, etc. In fact, both types of cash flows are impacted by economic conditions, the less variable ones not necessarily being those called “fixed”. For example, the cash flow from households to industry in exchange of goods and services remains almost constant regardless of the economic condition. Noninterest income for banks such as ATM fees and credit card fees are not affected by economic deterioration, for banks can always raise those fees. Such cash flows are rather steady, while payments from subprime loans, which are subject to default, are in practice more variable cash flows.

At a macroeconomic level, variations of aggregate cash flows could be explained by shifts in the classical Hick’s IS-LM curves. However in order to assess the actual market stability, we do not deduce variations of cash flows from a model, but from empirical observations. Market instability, which is our core target of study, may possibly result from a behavior that is predicted by a model, e.g. [18], but it may also well be the consequence of the economy departing from classical models.
To model the 2007 - 2009+ financial crisis which started in the U.S. real estate market, let us now broadly divide the economy into several segments in the spirit of [7]. Typically home buyers (HB), which we do not distinguish from general consumers, get financing from local mortgage lenders (ML) which include the mortgage divisions of large banks. To facilitate financing with limited fund, these mortgages are sold to large banks (LB - not only banks but other financial institutions that function like banks, for example brokers and insurance companies) and the government sponsored enterprises (GSE - Fannie Mae and Freddie Mac) for securitization - they are sliced, diced, and repackaged as mortgage backed securities (MBSs) which is a special kind of asset backed securities (ABSs) or collateralized debt obligations (CDOs). Part of these MBSs are kept within the banks (super-senior tranches) or are securitized again (ABS-squared) and sold in secondary mortgage market. They are also sold to investors. The investors (I) consist of many funds from all over the world, such as pension funds, mutual funds, academic endowments, state employees' retirement funds, sovereign funds etc. The wealth of many people, directly and indirectly, depends on such funds, so the impact of these funds on the real sector of economy is immense.

Speculative hedge funds are here excluded, since in this framework they are just intermediaries, hence their presence merely affects the whole cycle. They are indeed an important segment of the financial system to understand its short term reactions. They act as liquidity providers and sometimes destabilize the whole system. However they rarely are solely responsible for financial bubbles. In our framework, we can merge them with the general investors' class “I”, as this does not significantly affect the overall behavior of the model.

Another key player in the real sector is corporations (C). Home buyers pay their mortgage installments to mortgage lenders and large banks, and they do so largely thanks to the wages paid mostly by corporations which in return, get financed mostly by large banks and investors.

We also have here excluded the government as a financial actor, as we wish to understand the dynamics of the free market. Government actions, when they occur, are directed mainly to banks and government sponsored enterprises, and indirectly to corporations and consumers by the fiscal policy. The framework we describe here provides a clear basis to understand the potential impact of such or such government actions.

The cash flows among these six segments, HB, ML, LB, GSE, C, and I, can be further classified into two groups, variable cash flows and fixed cash flows as explained previously.

Variable cash flows include equity investments, debt investments (commonly called loans), and dividends which include payments that act like dividends. (Figure 1)
1. Equity investment
   (a) HB to HB: home buyers invest in houses, and sell those to one another.
   (b) C to C: companies invest into each other.
   (c) I to LB, GSE, C: investors buy stocks of LB, GSE, and C.

2. Loan (debt investment)
   (a) ML to HB: mortgage loans
   (b) LB to HB: credit cards and other financing
       LB to ML: purchasing mortgages for securitization
       LB to LB and GSE: secondary MBS market
       LB to C: bank loans to companies
   (c) GSE to ML, LB: guarantees mortgages by purchasing them and creating MBS.
   (d) I to LB, GSE, C: investors buy bonds issued by LB, GSE, and C.

3. Dividends
   (a) LB, GSE, C to I: investors earn dividends from the LB, GSE, and C stocks they invested.
   (b) I to HB: investors pay pensions to their client, which work like dividends.

4. Consumption: HB to C

   Fixed cash flows are coupons, which include not only bond coupons but payments from fixed rate mortgages and other conventional loans, minimum payments for adjustable rate mortgages and credit card loans, salaries, and contribution to retirement fund and other money market funds. Also included are premiums for credit default swaps (CDSs) to issuing financial institutions from counterparties. (Figure 2)

5. Coupons
   (a) HB to ML, LB: mortgage and other financing payments, credit card debt payments
   (b) ML to LB, GSE: although LB and GSE purchase mortgages from ML for securitization and guarantee them by holding the resulting MBSs, the payments from HB are still directly made to the original lender ML. So we interpret that there is a cash flow from ML to GSE and LB.
   (c) LB to LB, GSE, I: coupons for MBS and ABS of MBS markets. Also included are CDS premiums.
Figure 1: Variable cash flows among six agents

(d) GSE to LB, I: GSE pays coupons to MBS investors. When GSE started buying and guaranteeing more MBSs the financial crisis had already developed [11]. We consider cash flows during normal economy, so do not include this special case which will create a loop from GSE to itself.

(e) C to LB, I: companies pay coupons to the bond holders.

6. Salary: C to HB

7. Contributions (e.g. pension fund): HB to I

When the real estate bubble burst in 2007, home buyers started getting behind their payments. So the financial segments at the receiving end of cash flows involving MBS experienced significant default and write downs (Figure 3), which lead to the bail out of Fannie Mae and Freddie Mac in September of 2008. The blow was more severe for large banks, for they not only invested in those CDOs but also insured against them by selling CDSs. Just a week later Lehman Brothers declared bankruptcy and Merrill Lynch was sold to Bank of America on the same day. Several days later, the insurance giant AIG was bailed out.

The September 2008 saga froze market liquidity, and corporations started not being able to borrow money. This resulted in mass bankruptcy of companies of all size, and consecutively massive unemployment. The market value of companies and banks plunged, and so did the return to the investors and the home buyers who contributed to them. This resulted in “victims” of the financial crisis. In the U.S. real estate market, not all home buyers contributed to the crisis. There are people who used to be prime lenders and never delinquent in their payments before, but ended up falling behind due to the side effects...
Figure 2: Fixed cash flows among six agents

Figure 3: Cash flows affected by a decreased demand of MBS
of the financial crisis, such as plunging house price, soaring interest rates on
loans, and job loss [8] [26]. Commercial mortgage holders are like prime lenders
in that sense. Likewise there are banks which used to serve prime lenders and
people with sound financial basis but eventually fell victim to the crisis as well.
Although it is tempting to distinguish the “sinners” and “victims” of the fi-
nancial crisis, in the end they are not distinguishable at all. Dynamically, once
the crisis is on, whether a default comes from an initially insolvent borrower
or from a borrower who became insolvent doesn’t matter in the evolution of
the crisis. The purpose of this paper is to think of not the responsibilities but
the evolution of the financial crisis and the transition from equilibrium to chaos.

3 Stability of Financial Equilibrium

3.1 Propagation of Wealth

Keeping the mathematical intuition from the previous section in mind, we start
a formal stability analysis of a financial equilibrium. We model the economy in
\( \mathbb{R}^n \). The dimension \( n \) is the number of main segments of the financial system
under consideration. For instance, the model in Section 2 is in dimension \( n = 6 \).
for generalization purpose.

Let the wealth vector \( w(t) \) represent the wealth (asset) of all segments at
time \( t \), \( w(t) = (w_1(t), w_2(t), \ldots, w_n(t)) \) where \( w_i(t) \) is the wealth of the market
segment \( i \). We define the asset \( w_i(t) \) of each segment \( i \) at time \( t \) as the sum of
the equity and debt,

\[
w_i(t) = E_i(t) + D_i(t)
\]

(1)

In terms of liquidity, the asset can be again divided into liquid asset and illiquid
asset,

\[
w_i(t) = L_i(t) + K_i(t)
\]

(2)

The illiquid asset \( K_i(t) \) consists of fixed asset (property, equipment, human
capital etc.) and assets that are not readily convertible to cash. Liquid asset
consists of cash or assets which can be easily converted to cash.

To analyze the wealth level of each segment, we make the following assump-
tions:

(a) For all segments only the illiquid asset \( K_i(t) \) has an internal growth factor
\( \gamma_i(t) \). “Internal” means the value of \( K_i(t) \) changes without incoming or
outgoing cash flow. This “growth” factor can be negative, which represents
a drop in value of \( K_i(t) \) (e.g. real estate price decline).

(b) There is one interest rate \( r \) banks use to lend and borrow money.

For each agent \( i \) and the time interval \( [t, t+1] \), we have the following wealth
dynamics for the debt \( D_i(t) \) which we account at the face value discounted at
the interest rate \( r \), illiquid asset \( K_i(t) \), and liquidity \( L_i(t) \):

10
\[ D_i(t + 1) = (1 + r)D_i(t) + \Delta D_i(t) \quad (3) \]

\[ K_i(t + 1) = (1 + \gamma_i(t))K_i(t) + \Delta K_i(t) \quad (4) \]

\[ L_i(t + 1) = L_i(t) + \sum_{j \neq i}^n F_{ij}(t) - \sum_{k \neq i}^n F_{ki}(t) - \Delta K_i(t) \quad (5) \]

\[
\left( \sum_{j \neq i}^n F_{ij} \text{ denotes } \sum_{j=1}^n \left( F_{ij} - F_{ii} \right) \right)
\]

where

- \( \Delta D_i(t) \) is new loans less reimbursement of the existing balance.
- \( \Delta K_i(t) \) is new investment less realization, that is, the money spent to increase \( K_i(t) \), such as purchasing equipment and hiring workers, less money saved by reducing \( K_i(t) \), such as selling equipment and firing workers.
- \( F_{ij}(t) \) is the fund transferred from the segment \( j \) to \( i \) at time \( t \). This can be seen as an “investment” with expected returns \( F_{ji}(t+h) \) after \( h \) units of time later. If \( F_{ij}(t) \) is a debt reimbursement, the expected return is the saving from the reduced interest payment.

Therefore by Equation (2)

\[ w_i(t + 1) = w_i(t) + \sum_{j \neq i}^n F_{ij}(t) - \sum_{k \neq i}^n F_{ki}(t) + \gamma_i(t)K_i(t) \quad (6) \]

The internal growth \( \gamma_i(t)K_i(t) \) of \( K_i(t) \) can be interpreted as a result of “self-investment”. For example, active investment on houses by the members of HB at time \( t \) will increase or decrease the value of \( K_i(t) \) by a positive or negative growth rate \( \gamma_i(t) \). For homogeneity reason we can replace \( \gamma_i(t)K_i(t) \) by \( F_{ii}(t) \), and Equation (6) becomes

\[ w_i(t + 1) = w_i(t) + \sum_{j=1}^n F_{ij}(t) - \sum_{k \neq i}^n F_{ki}(t) \quad (7) \]

More traditionally, the wealth variation \( \Delta w_i = w_i(t + 1) - w_i(t) \) is decomposed into equity and debt as in Equation (1), \( \Delta w_i(t) = \Delta E_i(t) + \Delta D_i(t) \). In this equation, the change in equity value results from the change in the expectation of discounted future cash dividends whereas the debt change is new loan less the payment of existing debt. The term \( \gamma_i(t)K_i(t) \) accounts for the equity variation.

The global wealth \( S(w) \) is the sum of all wealth, thus at a given time \( t \),

\[ S(w(t)) = \sum_{i=1}^n w_i(t) \quad (8) \]
In an economic rally, the global wealth grows steadily and people often use high leverage to maximize the return on their investment. In the next section we will show that when the economy is too much leveraged, the chance for the global wealth to experience strong downfalls is high due to a bifurcation in the dynamical system of wealth.

3.2 Optimal Investment: Equilibrium State

Recall the six financial agents introduced in Section 2, home buyers (HB), local mortgage lenders (ML), large banks (LB), government sponsored enterprises (GSE), corporations (C), and investors (I). On the brink of the 2007 Credit Crisis, all six agents were holding a heavy portion of mortgages or MBSs in their investment portfolios. To take advantage of the historically low interest rates, cheap credit which was made available from securitization, and increasing house price, people borrowed heavily and invested in real estate or related financial product. As a result these highly leveraged investments were susceptible to even a small drop in the cash inflows (what we call “default”) or a sudden increase in cash outflows such as payment jumps in adjustable rate mortgages (ARMs). We model this situation by a nonlinear programming problem (NLP) with multiple objective functions and constraints, and by solving the NLP we find a dynamical system \( f \) of wealth and an equilibrium situation, that is, a stationary state \( \tilde{w} \) such that

\[
    f(\tilde{w}) = \tilde{w}
\]

Recall the wealth composition \( w_i(t) = E_i(t) + D_i(t) = L_i(t) + K_i(t) \) introduced in Section 3.1 where

\[
    D_i(t + 1) = (1 + r)D_i(t) + \Delta D_i(t) \tag{9}
\]

\[
    K_i(t + 1) = (1 + \gamma)K_i(t) + \Delta K_i(t) \tag{10}
\]

\[
    L_i(t + 1) = L_i(t) + \sum_{j \neq i}^n F_{ji}(t) - \sum_{k \neq i}^n F_{ki}(t) - \Delta K_i(t) \tag{11}
\]

We already mentioned that \( F_{ji}(t) \) is the flow of funds (“investment”) from \( i \) to \( j \) at time \( t \). We further assume the following for \( F_{ji}(t) \).

(a) This “investment” \( F_{ji}(t) \) is meant to receive a stream of returns - with uncertainly - either from the very same counterpart, or from the system in general, but we shall ignore this second case and assume that we only expect returns from the same counterpart.

(b) The rate of return is not necessarily linear as a function of \( F_{ji} \). For instance if \( j \) is LB and \( F_{ji} \) is a debt reimbursement, it can be seen as a debt reduction, which implies the rate of return is the interest rate of the debt, but missing a payment implies penalties which increase this interest rate. We may therefore observe threshold effects.
Let $F_{ij}(s)$ be the return at time $s$ for the investment $F_{ji}(t)$ at $t$. The total return can possibly be spread over several times $s$. Let $U(F_{ij}(s))$ be the utility of the agent $i$’s receiving $F_{ij}$ from $j$ at $s$. The present value of this utility is $e^{-\alpha s}U(F_{ij}(s))$. Therefore the total utility of the agent $i$ for the investment $F_{ji}(t)$ is

$$J(x) = \sum_{s>t} e^{-\alpha s} E[U(F_{ij}(s)|F_{ji}(t) = x] - x$$

(12)

The utility function $U(x)$ is assumed increasing and concave, such that $U(0) = 0$ and $U'(0) = 1$, for the following reasons:

- increasing in $x$ (the more money one receives, the better it is)
- concave (utility increases at a decreasing rate - diminishing marginal utility)

(d) We assume that each agent $i$ tries to maximize $\sum_{j=1}^n J(F_{ji}(t))$ within bounds driven by liquidity, realization of illiquid assets, and borrowing capacity in the following manner:

- Positive liquidity $L_i(t) \geq 0$ means that negative liquidity is immediately converted to a debt increase.
- Maximum realization rate $-\Delta K_i(t) \leq \kappa_i(t)$ means that for each $t$, there is a limit $\kappa_i(t)$ to converting illiquid asset into liquidity.
- Borrowing capacity constraint $\Delta D_i(t) \leq D_{i\text{max}}(t+1) - (1+r)D_i(t)$ where $D_{i\text{max}}(t+1)$ depends on $E_i(t)$ and market conditions like credit rating. For instance $D_{i\text{max}}(t)$ can be set as the maximum debt level an agent can afford for a bankruptcy risk less than a given level. It may happen that these constraints are incompatible. In this case one of the agents is in default: the debt constraint is released up to the minimum level that resets the other ones.

The above utility function and investment constraints naturally lead us to find an optimal investment level $F_{ij}$s (decision variables) for all agents $1 \leq i \leq n$ via a nonlinear programming problem (NLP) [14], [31] with $n$ objective functions and $3n$ constraints:

NLP: \[
\begin{align*}
\text{max} & \quad z_i = \sum_{j=1}^n J(F_{ji}(t)) \\
\text{subject to} & \quad L_i(t) \geq 0 \quad (14) \\
& \quad -\Delta K_i(t) \leq \kappa_i(t) \quad (15) \\
& \quad \Delta D_i(t) \leq D_{i\text{max}}(t+1) - (1+r)D_i(t) \quad (16) \\
& \quad 1 \leq i \leq n, \ t \geq 0
\end{align*}
\]

where $t = 0$ is the beginning of an economic period (e.g. a business cycle, a past financial crisis, a future forecast etc.) This NLP is a Pareto optimality
problem in economics. Let $F^*_{ij}(t)$, $1 \leq i, j \leq n$ be the optimal solution of the NLP (13) for each $t$. Solving NLP (13) gives a dynamical system

$$w^*_i(t + 1) = f(w^*_i(t))$$

(17)

where $w^*_i(t + 1)$ is given by Equation (7) with optimal $F_{ij}(t)$s under the constraints (14)-(16). The dynamical system $f$ is in fact random as the returns on the illiquid part $\gamma_i K_i$ are uncertain. It is linear only during $[t, t + 1]$ and nonlinear due to the constraints and depends only on the utility functions $J(F_{ij})s$.

We also define the “saturated system” $\hat{f}$ in which, at each step, agents choose the maximum possible debt and hit the borrowing capacity constraint. In such a system, through time, the borrowing capacity may become occasionally negative. In this case no new debt is contracted and all effort is made to come back to acceptable bounds as fast as possible. In other words, liquidity and capacity to sell illiquid assets are used to reduce the debt, and bring it within the bounds of nonnegative borrowing capacity. One of the features observed during the crisis is the fact that the “optimal” system hits its constraints and becomes equal to the saturated one.

Define now:

$$\bar{f}(w^*_i(t)) = E[w^*_i(t + 1)|w^*_i(t)]$$

(18)

This $\bar{f}$ is a deterministic dynamical system. In the following section we analyze the stability breakage of $\bar{f}$. Indeed, if $\bar{f}$ becomes unstable, then the actual random system $f$ will also display the same instability.

We shall assume that, being in a closed economy the asset level and the borrowing capacity of each market agent are bounded, and due to the diminishing marginal utility, the wealth level in constant dollars (i.e. discounted by the interest rate $r$) reaches an equilibrium $\tilde{w}_i$. In other words, every agent has become as rich as it can be. Once the wealth level stabilizes, so do the constraints (14) - (16), and a “small” changes in them does not affect the optimal solution of NLP (13), therefore the equilibrium of wealth $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)$ is a fixed point of $\bar{f}$,

$$\bar{f}(\tilde{w}) = \tilde{w}$$

(19)

However large changes in the constraints can break the equilibrium and change the optimal solution $F_{ij}s$. When the new optimal $F_{ij}s$ are lower than those of the equilibrium, some economic agents, if not all, experience drops in the cash inflows. In the next section we will analyze when perturbations can be absorbed by the system keeping the equilibrium stable, and when perturbations break the equilibrium and propagate in the system.

### 3.3 Perturbation Analysis

This section is devoted to the perturbation analysis of the deterministic system $\bar{f}$ near the equilibrium $\tilde{w}$. In order to ease notations, we shall remove the
“overline” and denote it by \( f \). We assume as well that \( F_{ij} \) represent expected cash flows and not the actual ones. We shall also drop the * symbol of the optimal solution of the NLP and simply write \( w(t + 1) = f(w(t)) \).

Applying the optimal solution \( F^*_ij \), \( 1 \leq i, j \leq 6 \) to the wealth dynamics Equation (7), we get

\[
w^*_i(t + 1) = w^*_i(t) + \sum_{j=1}^{n} F^*_ij(t) - \sum_{k \neq i}^{n} F^*_ki(t)
\]

For each \( t \), let \( f_{ij} = f_{ij}(t) \) be the proportion of wealth transferred from the agent \( j \) to \( i \), that is, \( F_{ij}(t) = f_{ij}w_j(t) \). With this notation

\[
w_i(t + 1) = w_i(t) + \sum_{j=1}^{n} f_{ij}w_j(t) - \sum_{k \neq i}^{n} f_{ki}w_i(t) \tag{20}
\]

Putting Equation (20) for all \( i \) from 1 to 6 in a column vector, we get

\[
w(t + 1)^\top = (I + F - F^\#)w(t)^\top \equiv Tw(t)^\top
\]

where \( \top \) denotes the transpose,

\[
F = \begin{pmatrix}
f_{11} & f_{12} & \cdots & f_{1n} \\
f_{21} & f_{22} & \cdots & f_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
f_{n1} & \cdots & f_{nn}
\end{pmatrix}
\]

and \( F^\# \) the diagonal matrix whose entries are \( \sum_{k \neq i}^{n} f_{ki} \) for \( 1 \leq i \leq n \),

\[
F^\# = \text{diag} \left( \sum_{k \neq 1}^{n} f_{k1}, \sum_{k \neq 2}^{n} f_{k2}, \cdots, \sum_{k \neq n}^{n} f_{kn} \right)
\]

Therefore we can rewrite Equation (17) as

\[
f(w(t)) = T(w(t)) \cdot w(t)^3 \tag{21}
\]

and Equation (19) means that

\[
f(\tilde{w}) = \tilde{w} \tag{22}
\]

\footnote{This is the dot product of tensors. For a rank-2 tensor \( A \) (matrix) and a rank-1 tensor (vector) \( v \), their dot product is a rank-1 tensor defined as follows: \( A \cdot v = \hat{e}_iA_{ij}v_j = \sum_i \hat{e}_i(\sum_j A_{ij}v_j) \) where \( \hat{e}_i \) is the \( i \)th unit vector, \( A_{ij} \) is the \( ij \)th entry of \( A \), and \( v_j \) is the \( j \)th entry of \( v \).}
implies that one of the eigenvalues of $T = I + F - F^\#$ is 1; this should go

In real life, a stationary equilibrium is always perturbed, so unexpected changes of wealth occur frequently. Here we mean by unexpected change a diversion from the equilibrium, which is caused by changes of net cash inflows and equity levels. A drop in cash inflow can occur in so-called variable payments upon decision of financial actors, but it can as well occur in so-called fixed payments, for instance due to the default, job loss etc.

Let $B$ be the Jacobian matrix of the system $f$ and $\hat{B} = B(\hat{w})$, that is, the Jacobian matrix at the equilibrium $\hat{w}$. The stability of $\hat{w}$ means that all eigenvalues of $B$ have modulus less than 1. This matrix is not directly observable, but “flows of funds” $F_{ji}$ are available [17] and their co-variations can be measured.

Let us assume that one of the segments $j$ experiences a drop in wealth $x_j(t)$ at time $t$ and let $a_{ij} = a_{ij}(t)$ be the “elasticity coefficient” of its payments to segment $i$, so that upon a wealth decrease $x_j(t)$, the cash flow from segment $j$ to segment $i$ is reduced by $a_{ij}x_j$. Elasticities depend among others on borrowing capacities, which again depend on two things, leverage (i.e. debt vs. equity) and credit rating. They can be measured by a lagged regression coefficient of flows of funds with respect to one another. The resulting drop of wealth of segment $i$ at time $t + 1$ is:

$$x_i(t + 1) = \sum_{j \neq i} a_{ij}x_j(t) - \sum_{k \neq i} a_{ki}x_i(t)$$  \hspace{1cm} (23)

This equation, however, reflects only the drop in cash flow due to changes in counterparty payments and not the internal drop of equity level. For example, an overnight drop of bank stock prices due to fear for bank runs reduces the aggregate wealth of banks, but this is not due to reduced cash inflows of banks. To accommodate this internal drop of wealth, we introduce an equity drop factor $\delta_i(t)$. The amount $\delta_i x_i(t)$ measures the reduction in wealth of the segment $i$ at time $t$, which will consequently lead $i$ to reduce its payments at time $t + 1$. Then the change of wealth $x_i(t + 1)$ of $i$ at time $t + 1$ can be expressed as the sum of internal equity drop and external counterparty default,

$$x_i(t + 1) = \delta_i(t)x_i(t) + \sum_{j \neq i} a_{ij}x_j(t) - \sum_{k \neq i} a_{ki}x_i(t)$$  \hspace{1cm} (24)

Just like we considered the internal growth of the illiquid asset as “self-return on investment”, we consider the internal equity drop as “self-loss”, and set

$$\delta_i(t)x_i(t) = a_{ii}x_i(t)$$
With this notation
\[ x_i(t + 1) = x_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t) - \sum_{k \neq i}^{n} a_{ki}x_i(t) \] (25)
and the wealth after drop\(^4\) of \(w(t)\) is
\[ w_i(t + 1) - x_i(t + 1) = w_i(t) + \sum_{j=1}^{n} f_{ij}w_j(t) - \sum_{k \neq i}^{n} f_{ki}w_i(t) - x_i(t + 1) \] (26)

Let \(A\) be the \(n \times n\) matrix of elasticities, with entries \(a_{ij}\), \(A = (a_{ij})_{1 \leq i,j \leq n}\), and let \(A^\#\) is the diagonal matrix with entries \(\sum_{k \neq i}^{n} a_{ki}\) for \(1 \leq i \leq n\),
\[ A^\# = \text{diag} \left( \sum_{k=2}^{n} a_{k1}, \ldots, \sum_{k \neq i}^{n} a_{ki}, \ldots, \sum_{k=1}^{n-1} a_{kn} \right) \]

Matrices \(A\) and \(B\) are related by the equation:
\[ B = I + A - A^\# \] (27)
Which means:
\[ b_{ii} = 1 + a_{ii} - \sum_{k \neq i}^{n} a_{ki} \quad \text{and} \]
\[ b_{ij} = a_{ij} \quad \text{for } i \neq j \] (28)(29)

Differentiating Equation (21) using \(x = dw\) yields:
\[ B(w(t)) \cdot x = T(w(t)) \cdot x + DT(w(t)) \cdot (x, w(t)) \] (30)
where \(DT(w(t)) \cdot (x, w(t))\) is a rank-1 tensor
\[ DT(w(t)) \cdot (x, w(t)) = \hat{e}_i \frac{\partial T_{ij}}{\partial w_k} x_k w_j \] (31)

Swapping the indices \(j\) and \(k\) converts Equation (31) to
\[ DT(w(t)) \cdot (x, w(t)) = \hat{e}_i \frac{\partial T_{ik}}{\partial w_j} x_j w_k = (DT(w(t)) \cdot w(t)) \cdot x \] (32)

Therefore replacing Equation (32) for part of Equation (30) and cancelling the dot product with \(x\) in each term yields
\[ B(w(t)) = T(w(t)) + DT(w(t)) \cdot w(t) \] (33)

\(^4\)Note that the drop in equity is itself the result of the dynamics among asset managers and traders when confidence disappears in the stocks of a given sector. In this paper, we will not model this dynamics and simply consider such an event as shock in the market, for we are more interested in the result of such shock.
Which can be also seen as:

\[ b_{ij} = \frac{\partial f(w(t))_i}{\partial w_j(t)} = T_{ij} + \sum_{k=1}^{n} \frac{\partial T_{ik}}{\partial w_j} w_k \]

Rewriting this equation in terms of elasticities and flows of funds, we get:

\[ A(w(t)) = F(w(t)) + DF(w(t)) \cdot w(t) \]  \hspace{1cm} (34)

That is:

\[ a_{ij} = f_{ij}(w(t)) + \sum_{k=1}^{n} \frac{\partial f_{ik}}{\partial w_j}(w(t))w_k(t) \]

This equation clearly shows the two components of elasticities: on the one hand the direct proportionality coefficients of flows of funds with respect to the global wealth of the segments, on the other hand, the sensitivity of these coefficients to the wealth. As we mentioned above, this sensitivity results from the impact of a wealth drop on the leverage, therefore on the financial latitude of the segment as a whole.

The matrix \( B \) may take a different shape if it represents a reaction of the various market segments to a sudden shock in inflows. The equilibrium \( \tilde{w} \) is a fixed point of the function \( f \), thus an eigenvector of \( T \) associated with eigenvalue 1. Initially, we assume this equilibrium to be stable, which implies that the eigenvalues of \( B(\tilde{w}) \) have modulus smaller than 1. When leverage increases and/or the global wealth of the sectors decreases, the borrowing capacity drops immediately, and the elasticities tend to increase sharply. This concavity is an effect of the overreaction of market participants under liquidity shortage.

When the market is highly leveraged, the elasticities reach such a level that one or more of the eigenvalues of \( B(w(t)) \) have modulus above 1.

In this case, perturbations propagate. For instance, if variations in the money flow are due to some default in payments, then default becomes structurally installed. This is the situation we now have: government’s bailouts of large banks and corporations or attempts to restructure home mortgages are evidences of installed defaults.
3.4 Stability Indicator

The above perturbation analysis naturally leads us to define a “market stability indicator” $\rho$ as the spectral radius, i.e. the modulus of the largest eigenvalue of the Jacobian matrix $B(w(t))$.

$$I(t) = \rho(B(w(t)))$$ (35)

The higher the indicator, the more unstable the market. In stable market conditions, the equilibrium point $\tilde{w}$ is an attractor. The eigenvalues of $\tilde{B} = B(\tilde{w})$ have modulus less than 1 and, when the market is close enough to the equilibrium and, as a consequence, in its basin of attraction, the stability indicator $I(t)$ is also below the critical value 1.

The stability indicator is a time dependent, local indicator. As such, it is not a Lyapunov exponent of the system, as would be for instance the spectral radius $\rho(\tilde{B})$, which is not observable. This is the price to pay for $I(t)$ to be measurable and monitored. Nevertheless, when $I(t) < 1$ then perturbations of the system tend to be absorbed and disappear. On the contrary, when $I(t) > 1$ then most of the perturbations contain a component that will increasingly propagate within the system, either as a propagation of default, or simply as an increase of leverage making liquidity constraints tighter and tighter and reactions to variations of income stronger and stronger.

Typically, after a period of instability followed by an actual drop in wealth, the market temporarily stabilizes in a recession state and the stability indicator shrinks. Then government actions to exit the recession, such as quantitative easing, put again the market in an unstable state.

Rather than waiting for the market to blow up, monitoring this indicator would provide useful signals to the governments and the central banks on when and where to intervene in order to ease the system at a much lower cost than when the bubble bursts.

4 Financial Crisis: Breakage of Stability

4.1 What caused crisis?

In this section we investigate how the result of Section 3.3 can be applied to the current financial crisis, being referenced as “2007 - 2008/09/10 Financial Crisis”, which many consider having started in 2007.

Its origin goes further back, however. The U.S. housing boom which started in the early 2000’s eventually put residential real estate market in a saturated status. The U.S. home ownership rate reached above 69% average nationwide by the second quarter of 2004 and stayed that level until the first quarter of
The U.S. house price reached its peak during the 2nd quarter of 2006 and steadily declined for three years. In the mean time, the Federal Funds Rate rose from 1% (June 25, 2003) to 5.25% (June 29, 2006). These significantly lowered the borrowing capacity of HB, and resulted in a mass default of residential mortgages, which started surfacing in early 2007. This means the debt constraint of NLP changed a lot enough to break the stability of the optimal solution, and default set in HB and spread to all other segments. We will use well-known results in dynamical systems to explain this phenomenon. More details on the theories in dynamical systems which we use throughout the paper can be found in standard text books in dynamical systems, such as [24], [20], and [8] to name a few.

Thus we have the following feedback loop (Figure 5). The arrows represent the cash flows which experience significant drop (this would be our definition of “default”). This is the least bad scenario, that is, their counter cash flows stay normal. The worst scenario would be that all cash flows be in default. Such a case would have occurred with high probability without government intervention.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Combined cash flows and related default risk}
\end{figure}

In this section, we study the possibility and speed of “default” contagion throughout the system, where the word “default” means the any drop in a flow of funds from one of the segments.

Recall Equation (26)

\[ w_i(t + 1) - x_i(t + 1) = w_i(t) + \sum_{j=1}^{n} f_{ij} w_j(t) - \sum_{k \neq i}^{n} f_{ki} w_i(t) - x_i(t + 1) \]

which can be rewritten as
\[ w(t + 1) - x(t + 1) = Tw(t) - Bx(t) \]  

(36)

The vector \( x(t) \) is a drop in wealth \( w(t) \), and if \( x(t) \) is infinitesimal, we can write

\[ x(t) = -dw(t) \]

The linear approximation of \( f(w(t)) = Tw(t) \) for this increment is

\[ f(w(t) - dw(t)) \approx f(w(t)) - Df(w(t))dw(t) \]  

(37)

The left side \( f(w(t) - dw(t)) \) is the wealth at \( t + 1 \) when some agents experience default at time \( t \). Combining Equations (36) and (37), we have

\[ w(t + 1) - x(t + 1) = f(w(t) - dw(t)) \approx f(w(t)) - Df(w(t))dw(t) = Tw(t) - Bx(t) \]

Therefore the matrix \( \tilde{B} = I - \tilde{A} - \tilde{A}^\# \), where \( \tilde{A} = A(\tilde{w}) \) is the elasticity matrix at the equilibrium point \( \tilde{w} \), is the Jacobian matrix \( Df(\tilde{w}) \) of the function \( f \) at the equilibrium. The equilibrium \( \tilde{w} \) is a fixed vector of the matrix \( T \), thus there is an eigenvector of \( T \) associated with eigenvalue 1. Initially, we assume this equilibrium to be stable, which implies that the eigenvalues of \( \tilde{B} \) have modulus smaller than 1. When leverage increases and/or the global wealth of the sectors decreases, the borrowing capacity drops immediately, and the elasticities tend to increase sharply, and there is a higher chance of at least one eigenvalue of \( B \) becoming greater than 1 in magnitude. To explain what happened during the second half of 2006 and early 2007, we perturb the wealth transition function \( f \) by building a one-parameter family of maps \( \{f_\mu\} \), and analyze the behavior of the new equilibrium for the perturbed map under reduced borrowing capacity of home buyers (HB).

Recall the NLP (13),

\[
\begin{align*}
\text{NLP:} & \quad \max \quad z_i = \sum_{j=1}^{n} J(F_{ij}(t)) \\
\text{subject to} & \quad L_i(t) \geq 0 \\
& \quad -\Delta K_i(t) \leq \kappa_i(t) \\
& \quad \Delta D_i(t) \leq D_{i \max}(t + 1) - (1 + r)D_i(t) \\
& \quad 1 \leq i \leq n, \quad t \geq 0
\end{align*}
\]

We found an optimal solution \( w(t) \) for all \( t \) and built the wealth transition function \( f \) by \( f(w(t)) = w(t + 1) \). We build a one-parameter family of maps
by repeating this process for a changed constraint. More precisely, consider the following NLP for a $\mu > 0$.

\[
\text{NLP: } \max z_i = \sum_{j=1}^{n} J(F_{ij}(t))
\]

subject to

\[
L_i(t) \geq 0
\]

\[-\Delta K_i(t) \leq \kappa_i(t) \quad 1 \leq i \leq n,\]

\[
\Delta D_1(t) \leq D_{1\text{max}}(t + 1) - \mu - (1 + r)D_1(t) \quad (39)
\]

\[
\Delta D_i(t) \leq D_{i\text{max}}(t + 1) - (1 + r)D_i(t) \quad 2 \leq i \leq n, \quad t \geq 0
\]

This NLP is the same as NLP (13) except that the maximum borrowing capacity of HB is uniformly lowered by $\mu$ for all $t$ as described in Constraint (39). This simulates the fact that the series of defaults on home mortgages triggered the 2007 subprime crisis. Let $w_\mu(t) = f(w_\mu(t))$ be the optimal solution for NLP (38) and define $f_\mu$ by $f(w_\mu(t)) = w_\mu(t + 1)$. For the same reason as in NLP (13) there is an equilibrium $\tilde{w}_\mu$ such that $f(\tilde{w}_\mu) = \tilde{w}_\mu$. Repeat this process for a reasonable range of $\mu$, that is, the new maximum borrowing capacity $D_{i\text{max}}(t + 1) - \mu$ should be realistic. Then we have a one-parameter family of maps $\{f_\mu\}$ which is a continuous\(^5\) perturbation of $f$ from the original NLP. When some agents experience default in cash flows by $x_\mu(t)$ at an equilibrium $\tilde{w}_\mu$, the wealth at the next stage is

\[
w_\mu(t + 1) - x_\mu(t + 1) = T_\mu \tilde{w}_\mu - B_\mu x_\mu(t) = \tilde{w}_\mu - B_\mu x_\mu(t)
\]

where the matrices $T_\mu$ and $B_\mu$ are determined as in Section 3.3. For each $f_\mu$ and its equilibrium $\tilde{w}_\mu$, there are corresponding wealth transition matrix $T_\mu$ and the Jacobian of $f_\mu$, $B_\mu$.

When the market leverage is low, the elasticities in the matrix $B_\mu$ are low, so the eigenvalues of $B_\mu$ are small in magnitude. In this case the “perturbed” equilibrium $\tilde{w}_\mu$ preserves the stability type of the original equilibrium $\tilde{w}$ and remains as attractor. The default is absorbed by the market and the wealth level remains at the equilibrium at $\tilde{w}_\mu$. This is case (i) of Section 3.3. This is what would have happened if HB had not been as highly leveraged as it was: the house value and eventually the wealth of all other agents would have dropped yet eventually stabilized, without out further damage to the economy, thus no financial crisis we are having now. When the market is highly leveraged due to reduced borrowing capacity, the elasticities reach such a level that one or more of the eigenvalues of $B_\mu$ have modulus above 1. In this case, perturbations propagate. This is case (ii) of Section 3.3. Default which started in HB spread to other sectors and has become structurally installed. This is the situation we

\(^5\)Need to prove that $\{f_\mu\}$ depends continuously on $\mu$ when flows of funds do.
now have: government’s bailouts of large banks and corporations or attempts to restructure home mortgages are evidences of installed defaults. (Figure 4.1)

Figure 6: An illustration of bifurcation for $n = 1$. The original function $f$ is perturbed by a one-parameter family of functions $\{f_\mu\}$ and a bifurcation is shown by the slope of the tangent line at the equilibrium for each graph. The equilibrium of each $f_\mu$ preserves the stability type of that of $f$ since the tangent line has slope less than 1 in magnitude, $|f'_\mu(\tilde{w}_\mu)| < 1$. The stability type of the equilibrium changes for $f_{\mu_0}$ since $|f'_{\mu_0}(\tilde{w}_{\mu_0})| > 1$, so the fixed point $\tilde{w}_{\mu_0}$ of $f_{\mu_0}$ is repelling.

Alternative figure

Once we enter an unstable stage, what happens next? When there is a bifurcation that breaks stability, different possibilities may occur:

1. The equilibrium becomes a saddle with at least one of the eigenvalues becoming a real number greater than 1. In this case, the market moves away from the old equilibrium and shifts towards another attracting equilibrium or a more complicated attractor. (Figure 7)

2. The largest eigenvalues are a pair of complex conjugate numbers with modulus greater than 1. This is called an “Andronov-Hopf bifurcation”, in which case the equilibrium evolves from a sink to a cycle. (Figure 8)

In general, after a crisis, even if the market would a priori follow the catastrophic path towards a new, deterred, equilibrium, such as deflation for instance, it is very probable that the government policy will consist in doing everything in its power to avoid such a shift to lock-in, therefore putting the economy into a cyclic behavior. As a consequence, in either case, one can expect that, posterior to a crisis, the economy enters a period of intense oscillations. Normally, the
Figure 7: An attracting equilibrium becomes a saddle. Nearby points move away from the saddle in the horizontal direction and drift toward another sink or an attractor.
frequency of oscillations should be related to the imaginary part of the eigenvalues. However, because of year end tax reporting and usually yearly investment planning, the market tends to be subject to a forced, rather than free, oscillator, with yearly frequency.

More precisely, in the case of Andronov-Hopf bifurcation, there is an invariant closed curve (cycle) near the original equilibrium. Although the wealth transition matrix $T(t)$ and the perturbed one $(T_BB)(t)$ are linear during the time period $[t, t + 1]$, the wealth propagation over all time is nonlinear. Due to this nonlinearity, the system will cycle for a while, resulting in lots of bouncing back. As the financial crisis develops, the wealth of some of the segments is low and the linear approximation does not work. Economic recovery will go through a cycle, and each of the market segment will behave in an oscillating manner. We will see a sequence of growth and recession of each segment until the system stabilizes and reach a new equilibrium of wealth $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)$.

In the other case of an equilibrium shift, the government action is the major source of nonlinearity, and the above scenario occurs in a similar manner.

### 4.2 Contagion

Now that we know how the financial crisis started, we investigate why it spread across the globe, that is, why a series of local defaults became a global scale financial crisis. We suggest that there is chaos in the damages from the 2007-08 financial crisis, and to prove this, we use symbolic dynamics, a very effective tool in dynamical systems to analyze a chaotic behavior of a system.

The basics of symbolic dynamics which we will use to analyze the contagion of systemic risk in the 2007-2009 financial crisis can be found in books such
Recall the combined cash flow chart (Figure 5) in Section 4.1. As defined in Section 3.2, “default” means an unexpected drop in cash flow, therefore each arrow between any two agents in the combined cash flow chart indicates that there is default risk between them. Assign numbers 1 to 6 to HB, ML, LB, GSE, C, and I in that order, and define a matrix $C = (c_{ij})_{1 \leq i,j \leq 6}$ such that

$$c_{ij} = \begin{cases} 
1 & \text{if there is default risk from } i \text{ to } j \\
0 & \text{if there is no default risk from } i \text{ to } j 
\end{cases}$$

The $ij$th entry of this matrix $C$ is 1 if there is a cash flow, thus default risk as well, from the agent $i$ to $j$, and 0 if not. In this sense $C$ tracks the “transition” of default among economic agents.

$$C = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix} \quad (41)$$

Using $C$ as a transition matrix, we build a space of six symbols

$$\Sigma_C^+ \subset \{ s \in \Sigma_6^+ \mid c_{s_k s_{k+1}} = 1 \text{ for } k = 0, 1, 2, \ldots \} \quad (42)$$

Each element of $\Sigma_C^+$ represents a sequence of possible defaults from the combined cash flow chart (Figure 5). For example, the sequence (124361...) represent a chain of default initiated by home owners which was typical in 2007: home buyers defaulted on their mortgage payment to mortgage lenders (word 12), which again defaulted on the reimbursement to Fannie Mae and Freddie Mac (24), which affected the MBS and CDO market, causing write-downs and even bankruptcy of large banks (word 34). Investors experienced huge loss from those large banks in their investment portfolio (word 36), which subsequently delivered less-than-expected to home buyers (i.e. general consumers). Another example has the same beginning but different middle path. The sequence (124351...) follows the same path up to the point 4th digit, that is, large banks are affected by defaults which were initiated by home buyers. This time, the lending by large banks to corporations get reduced due to insufficient fund at banks, that is, liquidity is frozen. As a result many corporations lay off their employees and even go bankrupt, causing reduced income to the households. This is default from C to BH (word 51). During the crisis both scenarios took place, and with the unemployment rate near 10% (put a cite) and high mortgage delinquency rate (put a cite), the sequences from the two examples continue after returning to HB (the second appearance of 1). We call $C$ the default transition matrix, $\Sigma_C^+$ the default space, and a member of $\Sigma_C^+$ a default
Although some entries of the default matrix \( C \) is 0, all the entries of the matrix \( C \) become positive after two iterations, \( C^2 > 0 \).

\[
C^2 = \begin{pmatrix}
5 & 2 & 5 & 3 & 4 & 3 \\
2 & 3 & 3 & 1 & 2 & 3 \\
5 & 3 & 6 & 3 & 4 & 4 \\
3 & 1 & 3 & 3 & 2 & 1 \\
4 & 2 & 4 & 2 & 4 & 3 \\
3 & 3 & 4 & 1 & 3 & 4 \\
\end{pmatrix}
\]

This means that one agent will eventually reach all the agents including itself. This property implies two things in financial crisis. First, the damage from bad loans will return to the issuers of those loans, as well as the borrowers, and due to the nonlinearity of the system, the degree of damage will be far more severe than that of the initial benefit (fees and commissions). Second, default which took place in any one sector will eventually affect all other sectors through domino effect. This explains why the damage in the financial sector spread to the real sector and how defaults in the U.S. real estate market resulted in a global financial crisis. Dynamically speaking, \( C^2 > 0 \) implies that the subshift of finite type \( \sigma_C \) is topologically mixing on \( \Sigma_C^+ \). Therefore by definition of topological mixing, for any pair of open sets \( U \) and \( V \) in \( \Sigma_C^+ \) there is a positive integer \( n_0 \) such that \( \sigma^n_C(U) \cap V \neq \emptyset \) for any \( n \geq n_0 \). A financial crisis version of this is “any bad default will eventually ruin all the sectors in economy”. (A weaker version, topologically transitivity can be interpreted as “if timed well, part of the economy can be salvaged from the contagion of bad defaults”.)

We already know that \( \sigma_C \) is expansive on \( \Sigma_C^+ \). This means that any two default chains, even very close to each other at the beginning, will eventually get farther apart than a uniform distance. Two default chains \( s \) and \( t \) are close if they share the same word up to certain stage, that is, \( s = (s_1 s_2 \ldots s_n s_{n+1} \ldots) \) and \( t = (t_1 t_2 \ldots t_n t_{n+1} \ldots) \) where \( s_{n+1} \neq t_{n+1} \). In this case

\[
d(s, t) = \sum_{k=n+1}^{\infty} \frac{d(k, k)}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}}
\]

however

\[
d(\sigma^n_C(s), \sigma^n_C(t)) = d((s_{n+1} s_{n+2} \ldots), (t_{n+1} t_{n+2} \ldots)) \geq \frac{1}{2}
\]

In financial crisis, this means that even a single bad mortgage loan, after changing hands several times, can end up in different hands in different places hurting different agents. This is of course due to securitization AND the feedback effect from the combined cash flow chart (Figure 5). In dynamical systems such a feedback is called recurrence, and when there is a nontrivial (cannot be isolated
Indeed this financial crisis looks chaotic: there is still lots of debates whether the crisis is over or not (put a cite); the Federal Reserve downgraded its assessment of the U.S. economic prospects [16]; people are talking about double-dip (put a cite) to list a few symptoms. The default model via symbolic dynamics is an extreme case scenario. It assumes all economic agents keep defaulting, yet it gives us an idea on the mechanism of defaults and why certain government’s policies have not worked as well as people have hoped. Since the middle of 2008, the government has bailed out Fannie Mae, Freddie Mae, several too-big-to-fail banks, and implemented quantitative easing in both private and real sectors [12]. Our model shows that bailing out a certain economic agent does not necessarily prevent all default originating from it, and instead, it reduces the number of eventually fixed default chains. A default chain is eventually fixed if it ends with a repeating symbol such as \((s_1s_2\ldots s_{n-1}sns_n\ldots) = (s_1s_2\ldots s_{n-1}s_n)\). The repetition of one symbol implies self-default, that is, continuing decrease of the wealth of the economic agent the symbol represents.

The continuing decline of the U.S. house prices is a good example, which corresponds to the default chain \((111\ldots) = (\bar{T})\). If home buyers used their tax refund from the Economic Stimulus Act of 2008 (need a cite?) to buy things or pay mortgage or other loans, it reduced words 15 (HB to C), 12 (HB to ML), and 13 (HB to LB), respectively from default chains. However if they saved the money, it reduced words 11 (HB to HB). In case of GSE bail-outs, if government’s backing of Fannie Mae and Freddie Mac is successful, then these GSEs will keep buying mortgages from mortgage lenders to facilitate further lending, so no more word 42 (GSE to ML) in any default chain. This will also reduce 21 (ML to HB). However we can still have default chains that contains words 43 (GSE to LB) and 46 (GSE to I), which happens when MBS and CDOs are not demanded as much as expected, or GSE stocks and bonds lost value to incur losses to investors. As for bail-outs of large banks, if the bail-out fund was used to unfreeze liquidity, then words 35 (LB to C) and 31 (LB to HB) were reduced from default chains, which can further reduce words 51 (C to HB: corporations can keep operating without laying off or firing their employees) and 15 (HB to C: home buyers can keep using their credit card, which will boost consumption of goods and services from corporations). However if the banks hoarded cash to raise their cash reserve or used the bail-out fund for retention bonus to keep their talented employees, then the bail-out decreased only the word 33 (LB to LB) and did not affect other default from large banks.

A care should be taken however, when we apply this symbolic dynamics model to real life situation. In our model any self-default is represented by a repeating symbol, and it does not show the weight of that agent in the economy. For example, a steady decrease of house prices is represented by \((111\ldots) = (\bar{T})\) yet it does not specify the scale of the loss: whether the house prices decline 5% a year or 10% a year, is just \((\bar{T})\) in the symbol space \(\Sigma_C^+\). Nevertheless our model
suggests how future quantitative easing and bail-outs could be improved. For example, instead of just giving a lump sum to an economic agent (or a subset of an agent), a specific fund allocation could be accompanied, and the allocation is determined such that continuing default chains could be tamed and eventually reduced to a desired level. In the language of the familiar analogy, it is to make the butterfly flap less so as to get a tropical storm instead of a hurricane.

Compare the 2007-2008 financial crisis with the Asian-Russian crisis from 1997-98, the last global scale financial crisis before the current one, and see whether we can apply the same chaos model to it. The Asian crisis started in Thailand with the financial collapse of Thai baht when, after severe speculative attacks in May 1997, the Thai government gave up protecting its currency and decided to float the exchange rate. This resulted in a rapid depreciation of Thai baht and an exodus of foreign capital. This created turmoil in the currency market, and the effort to keep the currency from declining further led to domestic interest rate hike. Consequently, the Thai stock market dropped 75% and many companies bankrupted. Other Asian countries such as Indonesia, South Korea, and the Philippines followed its path [29] [19]. The subsequent drop in commodity prices eventually induced Russia into default.

Although the series of national defaults in Asia, then in Russia caused panic and hit stock markets worldwide, the dynamic structure of the events is quite simple. Those Asian countries collectively act as an isolated point which interacts with investors (I) via equity investment, with large banks (LB) via loans, and with Russia via commodity trading. The collapse of stock, bond and currency markets in the Asian countries and Russia was due to a sudden and severe erosion of international investors’ confidence, which resulted in a massive exodus of foreign capital. Although there was a brief worldwide stock market crash [30] there was not the kind of chaos observed in the 2007-08 crisis because there was no financial instrument backed by assets of those countries and resulting feedback loop. (Figure 9)

Assign numbers 1 to 4 to international investors (I), large banks (LB), Asian Countries, and Russia in that order to build a transition matrix of defaults $D$, which shows there was no contagion of default indeed.

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ (44)

At variance, the March 2000 Nasdaq collapse and the burst of the tech bubble could be considered as seed of the current financial crisis. We could summarize the mechanism as follows.

(1) Venture capitalists (VCs) and private equity investors, followed by general investors incurred huge losses due to coarse mispricing of internet securities.
Figure 9: The left picture shows the cash flows among stricken countries and foreign investors. The right picture is the flow of default, which forms only a self-recurring loop for each stricken sector.

(2) In order to catch back, they had to show rapid profits and, for this purpose, started investing in a leveraged manner. Typically, VCs invested in LBO (leveraged buy out) structures rather than straight equity, pushing SMEs (small medium enterprises) to massively borrow and leverage on their existing balance sheet.

(3) At the same time, the banking industry developed the CDS and CDO markets, hence easing access to credit by corporations in general, making all these LBO structures possible and profitable in the short term.

Obviously, this put the world at extreme sensitivity to the slightest increase of credit spreads, as a vast portion of the economy suddenly became non profitable because of the debt burden.

4.3 Work in Progress

Let $D_i(t)$ be the event that the agent $i$ is in default at time $t$. Let $p_{ij}(t+1)$ be the probability of the default of agent $i$ at time $t$ causing the default of agent $j$ at time $t+1$,

$$p_{ij}(t+1) = P(D_j(t+1)|D_i(t))$$

Let $P(t) = (p_{ij}(t))_{1 \leq i,j \leq n}$, the probability transition matrix.

Let $p(t) = (p_1(t), p_2(t), ... , p_n(t))$ where $p_i(t) = \frac{x_i(t)}{\sum_{j=1}^{n} x_j(t)}$ and $x_i(t)$ is the default amount of agent $i$ at time $t$. Then $p(t)$ is a probability vector since $\sum_{i=1}^{n} p_i(t) = 1$. 
We have the following questions and try to answer them:

1. Is there an equilibrium of default, that is, a vector

   \[ (p_1, p_2, \ldots, p_n) = \lim_{t \to \infty} (p_1(t), p_2(t), \ldots, p_n(t)) \]

   In other words, can we have Perron-Frobenius type result, i.e., there is an equilibrium \((p_1, p_2, \ldots, p_n)\) such that

   \[ (p_1, p_2, \ldots, p_n) = \lim_{t \to \infty} (p_1(t), p_2(t), \ldots, p_n(t)) \]

   In our case the probability matrix \(P(t)\) depends on \(t\), so \(p(t) = p(0) \prod_{i=0}^{t} P(i)\) while Perron-Frobenius theorem states \(p(t) = p(0)P^t\) for a time-independent probability \(P\). Here \(p(0) = (1, 0, 0, \ldots, 0)\) since we are modeling the systemic risk which started from HB.

2. Is there a \(\sigma\)-invariant Markov measure whose support is the subshift of finite type \(\Sigma_C\)?

3. Is there a topological equivalent of above \((p_1, p_2, \ldots, p_n)\), i.e. an invariant set \(\Lambda = \cap_{i=0}^{\infty} f^i(U)\) where \(f(U) \subset \text{int}(U)\)?

4. Can we determine how quantitative easing should be done? In other words, can we determine the amount of cash injection which each agent should receive and how it should be allocated to reduce “bad” default sequence? Government action should be taken so that as many components as possible of the equilibrium \((p_1, \ldots, p_n)\) is zero, i.e. prevent contagion by isolating default and work on the particular agent in default.

5. Can we use our method (predict a financial crisis via market stability indicator, avoid contagion by manipulating the probability matrix \(P(t)\) via quantitative easing) to “tame” the bubble, thus avoid the crisis. That is, instead of having “financial distress/crisis \(\rightarrow\) artificial bubble to control the damage \(\rightarrow\) market frenzy (over-investment) \(\rightarrow\) bubble bursts \(\rightarrow\) another distress/crisis” cycle, the government “deflates” current bubbles gradually and form new bubbles, then deflates them before they get too big and create a new batch etc.

5 Conclusion

In normal market conditions, the risk is usually monitored using techniques such as VaR (value at risk) or the volatility measures, which in some sense measure “the size of the waves” in order to guarantee that a given financial institution can face it. When a crisis occurs, it appears more important to estimate the “distance to the chute” than the “size of the waves”, indeed, the dynamic part dominates the random part of the evolution laws. In this article, we addressed
this question by trying to identify when the market equilibrium becomes unsta-
bile. For this purpose, following classical chaos theory, we look at the so-called
Jacobian matrix of the dynamical system near the equilibrium and ask the ques-
tion of its highest eigenvalue.

The entries of this matrix corresponds to how a given segment of the econ-
omy (banks, corporations, investors, consumers etc.) reacts in its spendings
and investments to variations of its income. The sensitivity coefficients of the
outflows with respect to the inflows, which we call here “elasticities”, strongly
depend on the borrowing capacities of the financial actors, and their general
leverage. When debt-to-wealth ratio is high, these elasticities tend to increase
sharply. As a consequence, the highest eigenvalue of the Jacobian matrix passes
the instability threshold, putting the market in high risk of turmoil.

We define a “market stability indicator” as the spectral radius of the Ja-
ocbian matrix of the system. The higher the indicator, the more unstable the
market. We recommend that governments monitor this indicator in order to an-
ticipate crises and avoid costly actions, such as quantitative easing, themselves
resulting in unstable markets again.

If we strictly follow the conclusions of this study, first, one should expect
several periods of significant market oscillations: rallies followed by more or less
rapid falls. Second, incentive actions such as quantitative easing should be used
carefully in view of their long-term effects in order not to be lured by a seeming
recovery which is just the upward side of the oscillation.

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