ВЫСШАЯ ШКОЛА ЭКОНОМИКИ

Lyudmila Egorova

RECOGNITION OF STOCK EXCHANGE PROCESSES AS A POISSON PROCESS OF EVENTS OF TWO TYPES: MODELS WITH STIMULATION AND LEARNING

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The behavior on a stock exchange can be represented as a reaction to the flow of events of two types entering the financial market – "regular" events and "crises". Broker's wealth depends on how successfully she identifies which event – crisis or not – occurs at the moment. It was shown in [1] that successful identification of the 'regular' events a little more than in half of the cases allows the player to have a nonnegative average gain.

We expand the model in [1] by introducing the possibility of a player to learn on her behavior and to receive the award for the "correct" behavior. It turns out that the ability to analyze actions and learn from them sometimes allows the player successfully identify regular events even less than in half of the cases to obtain a nonnegative expected payoff.

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Поведение биржи можно представить как реакцию на поток событий двух типов, поступающих на финансовый рынок, – «регулярных» событий (случающихся очень часто) и «кризисных» (случающихся намного реже). От того, насколько успешно брокер опознает, какое событие – кризис или не кризис – пришло в данный момент, зависит его благосостояние. В работе [1] было показано, что успешная идентификация частых «регулярных» событий чуть более, чем в половине случаев, позволяет игроку иметь положительный средний выигрыш.

Цель данной работы – расширение использованной в [1] модели введением возможности игрока обучаться на своих действиях и получать награду за «правильное» поведение. Оказывается, что возможность анализировать свои действия и обучаться на них позволяет игроку для получения неотрицательного ожидаемого выигрыша успешно распознавать регулярные события иногда даже меньше, чем в половине случаев.

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1. Introduction

In [1] stock exchange work was presented as a flow of coming events of two types – “regular” events (happen very often) and “crisis” (happen rare). What event came at a particular moment is not known beforehand and the player has to identify an unknown event. His wealth depends on how successfully he can recognize events. It was shown that successful identification of frequent “regular” events in a little more than half of the cases allows the player to have a positive average gain, and hence errors in the recognition of “crisis” events are not fatal. The basic model contains some important assumptions about the kind of processes. It is known that the actual flows of stock exchange events are not simplest. It can be better described as a piecewise stationary stochastic processes with unknown switch points. Stationarity of real data for S&P500 has been tested in [1] and periods of crisis and periods of absence of shocks show stationary time series indeed. However, we can not speak of stationary series in the long time. Aware of these shortcomings of our model, we consider it as a first step to the study of real stock exchange and the basis for constructing more sophisticated models.

In this work we extend mathematical model from [1] allowing player to learn on her own behavior and to receive an award for the “correct” behavior. Each of the proposed new models allows the player to have more freedom in her decisions and make mistakes in regular events more often.

The structure of the paper is as follows. In Section 2 we remind the problem and its solution from [1], Section 3 studies a model with stimulation for correct recognition, and Section 4 studies a model with learning. Section 5 contains data analysis for our model, Section 6 concludes. All proofs and derivations of the formulas are given in the Appendix.

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2. Basic model

We remind the problem statement in [1] and its solution. The flow of events of two types – type Q (*quiet*) and type R (*rare*) – enters the device. Each of them is the simplest, i.e. stationary, ordinary and has no aftereffects [3]. The intensity of the flow of Q-events is equal to λ, the intensity of the flow of R-events is equal to μ, where $\lambda \gg \mu$ (Q-events are far more frequent).

The device should recognize coming event X. If an event Q occurs and device identified it correctly, then it gets a small reward a, if the error occurred, and the event Q is recognized as the event R, then the device is "fined" by an amount b. The probabilities of such outcomes are known and equal p_1 and q_1, respectively. Similarly for R-events – correct identification of event R will give the value of c, where $c \gg a$, and incorrect one will give loss $-d$, $d \gg b$. After each coming event received values of "win" / "loss" are added to the previous amount (Fig. 1).

How large on average will be the amount received for the time t?

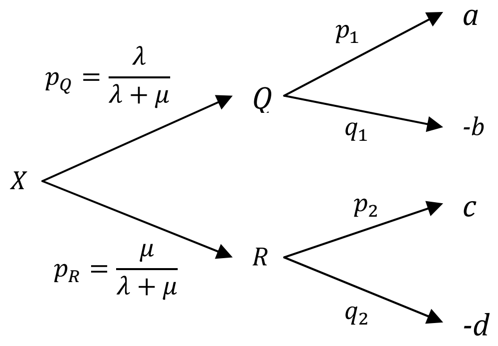


Fig. 1. The general scheme of identification of a random event ***X***

One possible implementation of a random process $Z(t)$, which is equal to the sum of all values of a random variable X received at the time t, is given on Fig. 2.

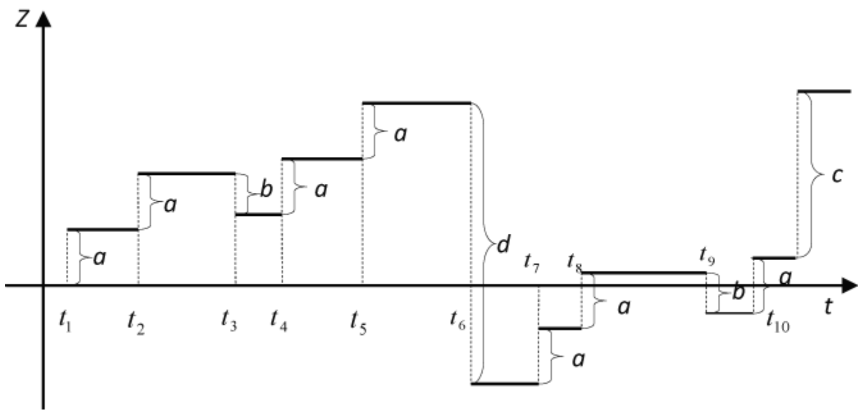


Fig. 2. One possible implementation of a random process $\boldsymbol{Z(t)}$

Theorem 1. The expected value of Z is equal to

$$E[Z] = (\lambda(p_1 a - q_1 b) + \mu(p_2 c - q_2 d))t.$$

The proofs of this and next statements are given in Appendix.

We used this model to estimate the stock exchange behavior [1]. The event X can be interpreted as a signal received by a broker about the changes of the economy that helps him to decide whether the economy is in “a normal mode” or in a crisis. The values a, b, c, d also have some meaning in such interpretation. If the event Q occurs (which means that the economy is stable), and broker correctly recognizes it, then he can get a small income (value a). If the event Q will be taken instead of R, he will loose the amount of $-b$. If the R-event occurred (crisis) and it was not recognized correctly, the broker will lose more (value $-d$). If he could forecast a crisis, he can earn a good deal of money on this – correct identification of the event of type R gives the broker the value c.

Such outcomes correspond to the opening of the long and short positions in a period of growth and recession in the work of the trader. A long position means that the broker buys assets to sale some time later at a higher price. A short position means that the broker sells assets with the hope of further buying at a lower price.

A long position will bring a small income a and a significant loss of $-d$ to the trader, when the market is growing (“regular” event) and falls (“crisis”), respectively. It will be the opposite with the short positions: trader will lose some value $-b$ in case of economic growth, but he can earn a considerable amount of c in the case of strong fall in the crisis.

In [1] we evaluated parameters of this model using stock index S&P500 [5] and showed that flows’ intensities are $\lambda = 246, \mu = 4$, and values of parameters are $a = 0.6\%, b = 0.6\%, c = 2.8\%, d = 2.9\%$ (values are measured in a percentage of the value of the index).

3. The model with stimulation

Let us extend the basic model from [1] adding new conditions.

Since the intensity of regular events Q is much higher than the intensity of rare events R, regular events often happen one by one and form a sequence of these “peaceful” events. So, we can suggest that the device can “learn” on such sequences and turn them to its advantage raising the winnings from regular events.

It means that if the event Q has been correctly detected by the device consecutively k times, then it gets a higher award $a + \varepsilon$ (not a as in the basic model) for the recognition of the Q-events.

Definition. The experience function S_i on step i is a random variable equal to the number of consequently correctly recognized Q-events (we designate an event of correct recognition of Q-type event as A) that occurred by this step

$$S_0 = 0,$$

$$S_i = \begin{cases} S_{i-1} + 1, \text{if A occured,} \\ 0, \text{if } \overline{\text{A}} \text{ occured(Aohas not occured).} \end{cases}$$

The model is graphically depicted on Fig. 3.

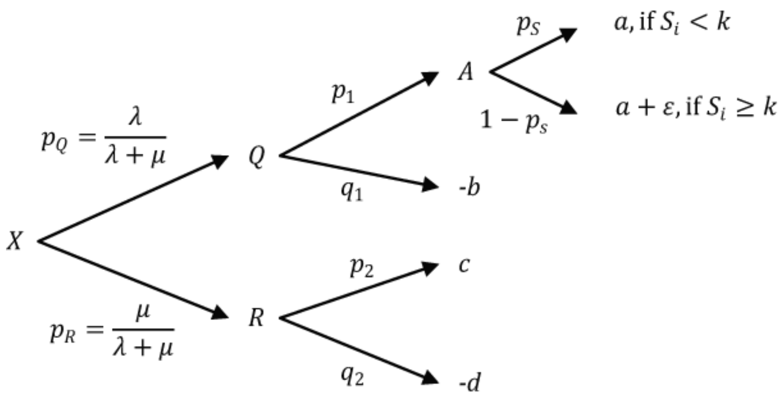
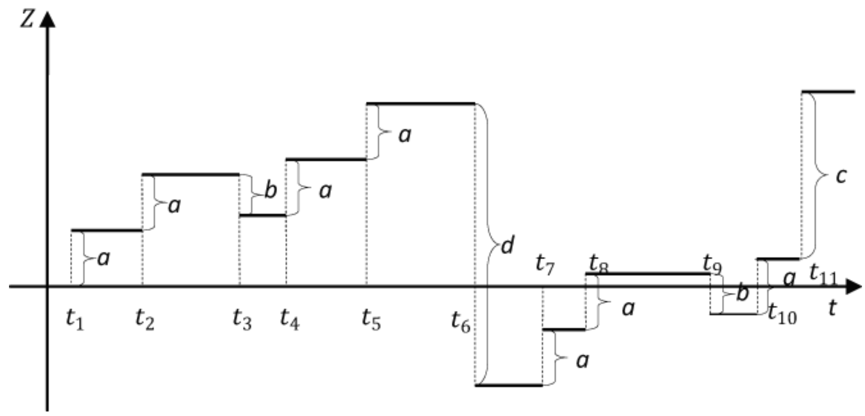


Fig. 3. Scheme of model with stimulation

The experience function $S_i(t)$ for implementation of random process $Z(t)$ from Fig. 2 is shown on Fig. 4.



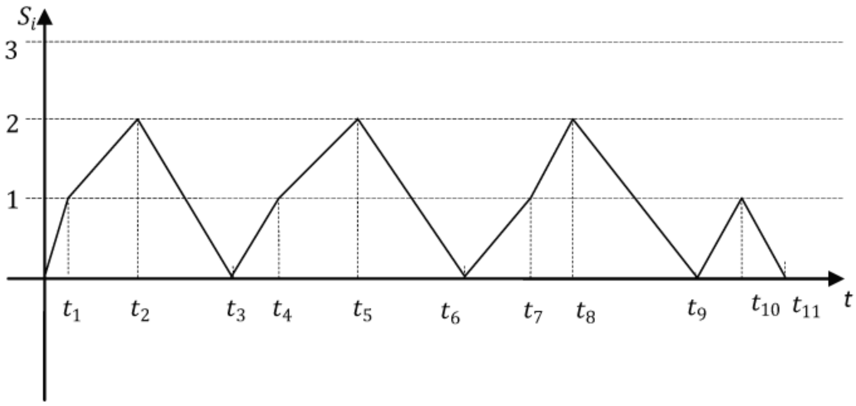


Fig. 4. One possible implementation of $\boldsymbol{Z(t)}$ and its experience function $\boldsymbol{S_i(t)}$

Theorem 2. The expectation of total gain in the model with stimulation is

$$E(Z) = E\left(X^{(1)}\right) \cdot \left[\lambda_{N_Z} \cdot \frac{\Gamma\left(k-1, \lambda_{N_Z}\right)}{\Gamma(k-1)} + (k-1) \cdot \left(1 - \frac{\Gamma\left(k, \lambda_{N_Z}\right)}{\Gamma(k)}\right)\right] +$$
$$+E\left(X^{(2)}\right) \cdot \left[\lambda_{N_Z} \cdot \left(1 - \frac{\Gamma(k-1, \lambda_{N_Z})}{\Gamma(k-1)}\right) + (1-k) \cdot \left(1 - \frac{\Gamma(k, \lambda_{N_Z})}{\Gamma(k)}\right)\right], \quad (3.1)$$

where $\lambda_{N_Z} = (\lambda + \mu)t$ is the intensity of flow of unknown events (both Q and R), $X_i^{(1)}$ and $X_i^{(2)}$ are the random variables for the total gain in case $i < k$ and $i \geq k$, respectively. Their distributions can be obtained by selection of relevant cases from the distribution of general variable X_i

$$\Pr\{X_i = x\} = \begin{cases} p_R q_2, \text{ if } x = -d, \\ p_Q q_1, \text{ if } x = -b, \\ p_Q p_1, \text{ if } x = a \text{ and } i < k, \\ p_Q p_1 \left(1 - \left(p_Q p_1\right)^k\right), \text{ if } x = a \text{ and } i \geq k, \\ 0, \text{ if } x = a + \varepsilon \text{ and } i < k, \\ \left(p_Q p_1\right)^{k+1}, \text{ if } x = a + \varepsilon \text{ and } i \geq k, \\ p_R p_2, \text{ if } x = c. \end{cases}$$

So, the distribution law of $X_i^{(1)}$ $(i < k)$ is

$$\Pr\left\{X_i^{(1)} = x\right\} = \begin{cases} p_R q_2, \text{ if } x = -d, \\ p_Q q_1, \text{ if } x = -b, \\ p_Q p_1, \text{ if } x = a, \\ p_R p_2, \text{ if } x = c. \end{cases}$$

and for $X_i^{(2)}$ $(i \geq k)$ the distribution law is

$$\Pr\left\{X_i^{(2)} = x\right\} = \begin{cases} p_R q_2, \text{ if } x = -d, \\ p_Q q_1, \text{ if } x = -b, \\ p_Q p_1 \left(1 - \left(p_Q p_1\right)^k\right), \text{ if } x = a, \\ \left(p_Q p_1\right)^{k+1}, \text{ if } x = a + \varepsilon, \\ p_R p_2, \text{ if } x = c. \end{cases}$$

4. The model with learning

Now we consider more complicated model where our device will also learn on its actions: if Q-event was successfully recognized k times consequently (it means that k times the device received an award a), then it will further recognize an event of Q correctly with greater probability $p_1^* = p_1 + \delta > p_1$.

Denote S_i as an experience function on step i, it is a random variable of the number of consequently correctly recognized Q-events. This experience function is defined almost like an experience function in the previous Section, but this function is changed if an event $X_i = a$ occurs, i.e. our device successfully detected coming event Q

$$S_0 = 0,$$
$$S_i = \begin{cases} S_{i-1} + 1, \text{if } a \text{ occurred}, \\ 0, \text{if } \bar{a} \text{ occurred } (a \text{ has not occurred}). \end{cases}$$

The graph for the model is given on Fig. 5.

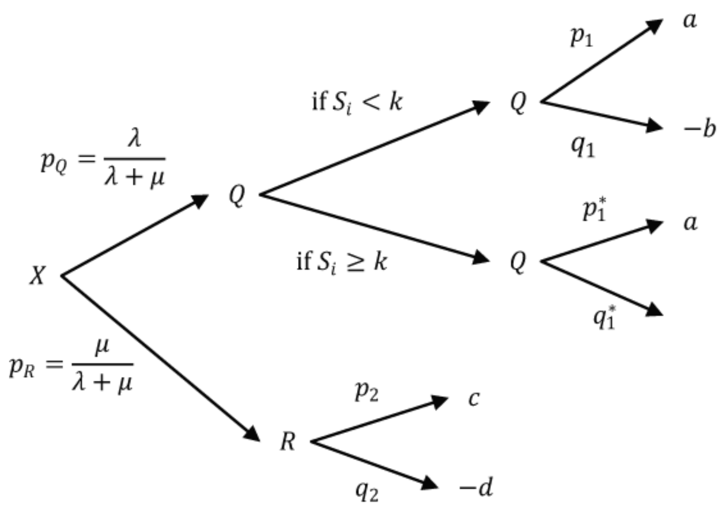


Fig. 5. The graph for model with learning

The question is still about the expected value of the total gain, but now we have to know the probabilities $P\{S_i < k\}$ and $P\{S_i \geq k\}$, because the random variable of single winning X_i takes values $-d, -b, a, c$ with probabilities

$$\Pr\{X_i = x\} = \begin{cases} p_R q_2, \text{if } x = -d, \\ p_Q[q_1 \Pr\{S_i < k\} + q_1^* \Pr\{S_i \geq k\}], \text{if } x = -b, \\ p_Q[p_1 \Pr\{S_i < k\} + p_1^* \Pr\{S_i \geq k\}], \text{if } x = a, \\ p_R p_2, \text{if } x = c. \end{cases}$$

Theorem 3. The probability $P\{S_i < k\}$ is equal to

$$P\{S_i < k\} = p_Q(q_1 - q_1^*)\sum_{j=0}^{k-1} P\{S_{i-1-j} < k\}(p_Q p_1)^j$$
$$+\frac{p_Q q_1^* + p_R}{p_Q q_1 + p_R}\left(1 - (p_Q p_1)^k\right).$$

Theorem 4. The sequence $P\{S_i < k\}$ has the limit

$$\lim_{i\to\infty} P\{S_i < k\} = \frac{(p_Q q_1^* + p_R)(1 - (p_Q p_1)^k)}{(p_Q q_1 + p_R) - p_Q(q_1 - q_1^*)(1 - (p_Q p_1)^k)} \qquad (4.1).$$

Fig. 6 illustrates the sequence of $P\{S_i < k\}$ and its limit with the example when $k = 8, q_1 = 0.3, q_1 = 1, \delta = 0.2$.

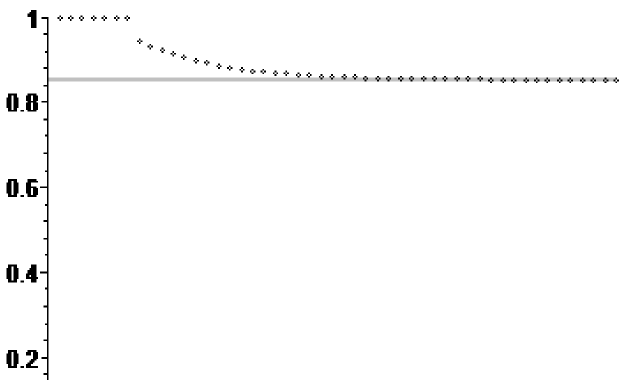


Fig. 6. Number of events $\boldsymbol{i}$ occurred is given on the axis OX and probabilities $\boldsymbol{P\{S_i < k\}}$ are presented on the axis OY, the line is defined by $\boldsymbol{P\{S_i < k\}}$ from formula (4.1)

We can use the formula (3.1) to compute the expected gain in this model.

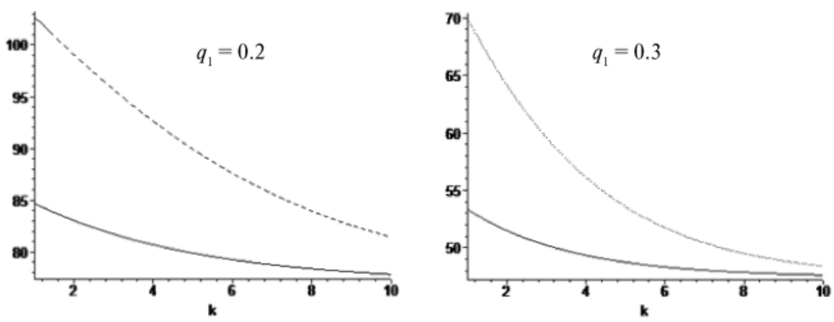
5. Examples

5.1. The expected gain as a function of k

We will draw the expected gain as a function of k for each model taking the one year horizon in each case. The parameters are taken from S&P500 analysis in [1] are $a = 0.6, -b = -0.6, c = 2.8, -d = -2.9, \lambda = 246, \mu = 4$. Let the probability of errors in R-events be $q_2 = 1$ (it means that the player can not predict crisis at all), and the probability of errors in regular events q_1 will be 0.2, 0.3, 0.4 and 0.46. For the new models the parameters are $\varepsilon = 0.05$ (which defines the increase of a when the player successfully detected Q-events k times consequently) and $\delta = 0.1$ (which defines the increase of p_1 when the player successfully detected Q-events k times consequently).

The expected gain $E(Z)$ in the basic model for all cases is 77, 47, 18 and 0.2, respectively.

You can see the dependence $E(Z)$ on k on the Fig.7.



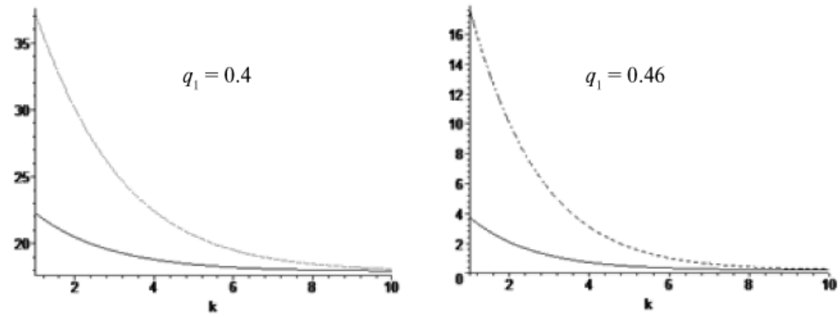


Fig. 7. The solid line corresponds to the expected gain in the model with stimulation and dotted line corresponds to the model with learning

Some values of expected gain can be seen in Tables 1 and 2.

Table 1. The expected gain for $\boldsymbol{q_1 = 0.2}$

$E(Z)$ when $q_1 = 0.2$	$k = 1$	$k = 10$	$k = 30$
The model with stimulation	85	80	78
The model with training	103	90	81
The basic model [1]	77	77	77

Table 2. The expected gain for $\boldsymbol{q_1 = 0.46}$

$E(Z)$ when $q_1 = 0.46$	$k = 1$	$k = 5$	$k = 10$
The model with stimulation	3.7	0.5	0.2
The model with training	17.6	1.7	0.3
The basic model [1]	0.2	0.2	0.2

Thus, when the error probabilities come closer to critical values (for which the expectation is zero), the expected gains in both models tend to the expected gain in the basic model much faster for the same k (moment of transition to a more advantage condition).

5.2. The expected gain as a function of q_1

Now the expected gain $E(Z)$ will be considered as a function of probability error q_1. Obviously $E(Z)$ increases in these advanced models comparing with the basic model, but will the critical value of q_1 also increase (we denote critical value as a value of q_1, that gives zero expected gain $E(Z)$)?

Parameters are still the same: $a = 0.6, -b = -0.6, c = 2.8, -d = -2.9, \lambda = 246, \mu = 4$, $\varepsilon = 0.05, \delta = 0.1$. Let $k = 5$.

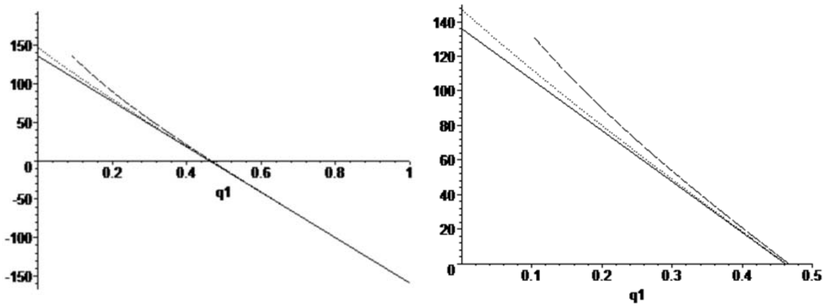


Fig. 8. The solid line corresponds to the expected gain in the basic model, dashed line – to the model with stimulation and dotted line corresponds to the model with learning

Because of the condition $q_1^* = q_1 - \delta \in [0,1]$ the values of q_1 in the third model must be in the interval $[\delta, 1]$. The comparison of the results is given in Table 3.

Table 3. The expected gain as a function of q_1 when $k = 5$

$E(Z)$	**Critical** q_1	$E(Z)$ **when** $q_1 = 0$	$E(Z)$ **when** $q_1 = \delta$
The basic model [1]	0.461	136	106
The model with stimulation	0.462	147	112
The model with training	0.466	-	132

If $k = 10$ then all critical values of q_1 (when $E(Z) = 0$) will be equal to 0.461. This means that if k increases then q_1 decreases to the critical value of the basic model.

On Fig. 9 the graphs for $\delta = 0.2$ are given.

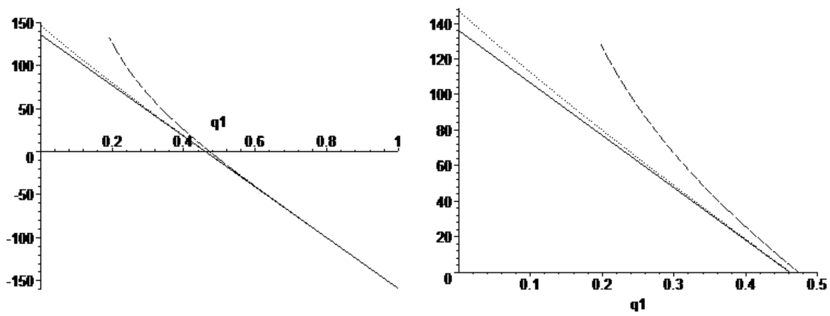


Fig. 9. The solid line corresponds to the expected gain in the basic model, dashed line – to the model with stimulation and dotted line corresponds to the model with learning

The probability q_1 will increase if parameter δ increases or parameter k decreases (Table 4). In the basic model q_1 is equal to 0.461.

Table 4. Critical value q_1 for different k

	$\boldsymbol{k=3}$ **Critical** $\boldsymbol{q_1}$	$\boldsymbol{k=5}$ **Critical** $\boldsymbol{q_1}$	$\boldsymbol{k=10}$ **Critical** $\boldsymbol{q_1}$
The model with stimulation $\varepsilon = 0.05$	0.464	0.462	0.461
The model with training $\delta = 0.1$	0.477	0.466	0.461
The model with stimulation $\varepsilon = 0.05$	0.464	0.462	0.461
The model with training $\delta = 0.2$	0.497	0.473	0.461
The model with stimulation $\varepsilon = 0.05$	0.464	0.462	0.461
The model with training $\delta = 0.3$	0.522	0.485	0.462
The model with stimulation $\varepsilon = 0.05$	0.464	0.462	0.461
The model with training $\delta = 0.4$	0.554	0.504	0.465

If the player does not try to recognize coming events but "toss up a coin" to decide (this corresponds to the model with $q_1 = \frac{1}{2}, q_2 = \frac{1}{2}$), then she will have negative gain $E(Z) = -0.2\%$ in a year.

6. Conclusion

We considered new models based on [1] adding award for 'successful behavior' as increase in gain and as increase in the probability of correct recognition, which means that the player can train on his past actions and accumulate experience. Both of these models allows the player to enlarge the total gain and to make more mistakes, because she can get more in the sequence of correctly detected events.

Appendix. Proofs of the theorems

Proof of Theorem 1.

The device does not know what event came at a time i, so received gain from recognition that event is a random variable X_i with discrete law of distribution. Since the flows of Q-events and R-events are simplest (and hence stationary), and probabilities p_1 and p_2 do not depend on time, all X_i is distributed equally as the random variable X with the law of distribution

$$\Pr\{X = x\} = \begin{cases} p_d, \text{if } x = -d, \\ p_b, \text{if } x = -b, \\ p_a, \text{if } x = a, \\ p_c, \text{if } x = c. \end{cases}$$

As flows of Q- and R-events are simplest, i.e. stationary, ordinary and has no aftereffects, then superposition of these flows will also be a simplest flow with intensity $\lambda + \mu$ [3]. Hence, the probability that coming unknown event is Q is equal to $p_Q = \frac{\lambda}{\lambda+\mu}$ and the probability that coming unknown event is R is equal to $p_R = \frac{\mu}{\lambda+\mu}$. Then p_d (the probability that the random variable X takes the value $-d$) is equal to the probability that the event occurred is the R-type and the device has not recognized it, i.e. $p_d = p_R q_2$. We can find other probabilities similarly.

Let $F_X(x)$ be the distribution function of a payoff X. The total value of received payoffs for time t equals to

$$Z = \sum_{i=1}^{N_Z} X_i,$$

where all X_i are random variables of gain of one event and they have the same distribution by the law of distribution of X, and N_Z is the number of events occurred during the time t, it is distributed according to the Poisson distribution with parameter $(\lambda+\mu)t$ (for the flow of events is the simplest flow with the intensity $\lambda+\mu$).

This sum of a Poisson number N_Z terms, where N_Z and X_i are independent, is called a compound Poisson random variable. Its distribution is given by a pair of $P((\lambda+\mu)t;\ F_Z(x))$, and the explicit form of the distribution function can be obtained by applying the formula of total probability with hypotheses $\{N_Z=m\}$

$$F_Z(x)=\sum_{m=0}^{\infty}P\{X_1+\cdots+X_m\le x\}P\{N_Z=m\}=$$
$$=\sum_{m=0}^{\infty}F_X^{(m)}(x)\frac{((\lambda+\mu)t)^m}{m!}e^{-\lambda t},$$

where $P\{N_Z=m\}=P_m(t)=\frac{((\lambda+\mu)t)^m}{m!}e^{-\lambda t}$, $F_X^{(m)}(x)$ – m-fold convolution of $F_X(x)$, $F_X^{(m)}=F_X^{(m-1)}*F, F*F-$ the distribution law of variable X_1+X_2 with probabilities $P(X_1+X_2)=\sum_{i=1}^{4}P(X_1=x_i)P(X_2=s_j-x_i)$, where x_i denotes a possible value of X_1 (it can be $-d,-b,a,c$) and s_j is a possible value of X_1+X_2.

Then the expected value of Z is equal to

$$E[Z]=\sum_{j=0}^{\infty}E[Z|N_Z=j]P\{N_Z=j\}=E[X]\sum_{j=0}^{\infty}jP\{N_Z=j\}=E[X]\,E[N_Z]=$$
$$=\left(-q_2\frac{d\mu}{\lambda+\mu}-q_1\frac{b\lambda}{\lambda+\mu}+p_1\frac{a\lambda}{\lambda+\mu}+p_2\frac{c\mu}{\lambda+\mu}\right)(\lambda+\mu)t=$$
$$=(\lambda(p_1a-q_1b)+\mu(p_2c$$
$$-q_2d))t.$$

Proof of Theorem 2.

Suppose A is an event of correct recognition of Q-type events and the probability of A is $p_a = p_Q p_1 = \frac{\lambda}{\lambda+\mu} p_1$. If the Q-event comes, we should choose the value of winning a or $a+\varepsilon$ according to the experience function S_i. We defined the experience function S_i on i step in the following way

$$S_0 = 0,$$

$$S_i = \begin{cases} S_{i-1} + 1, \text{if A occurs,} \\ 0, \text{if } \bar{\text{A}} \text{ occurs (does not occur A),} \end{cases}$$

The experience function on step i can take values from 0 to i with some probabilities. For example, the probability of $S_i = i$ is equal $P\{S_i = i\} = p_a^i$, for another values $k = 0,1,2,\dots,i-1$ the probabilities are $P\{S_i = k\} = p_a^k \cdot (1-p_a)$.

Obviously $p_s = \Pr\{S_i < k\} = 1$ for $i < k$. For $i \geq k$

$$p_s = 1 - \Pr\left\{\bar{A}\underbrace{A..A}_{k}\right\} = 1 - p_a^k.$$

So, the probability p_s is equal to

$$p_s = \begin{cases} 1, \text{if } i < k, \\ 1 - p_a^k, \text{if } i \geq k. \end{cases}$$

Let X_i be a random variable of gain in the model with stimulation. X_i depends on number i: for $i < k$ the probability to get value $a+\varepsilon$ is zero and for $i \geq k$ this probability is positive. Then X_i takes values $-d, -b, a, a+\varepsilon, c$ with probabilities

$$\Pr\{X_i = x\} = \begin{cases} p_R q_2, \text{if } x = -d, \\ p_Q q_1, \text{if } x = -b, \\ p_Q p_1, \text{if } x = a \text{ and } i < k, \\ p_Q p_1 \left(1 - \left(p_Q p_1\right)^k\right), \text{if } x = a \text{ and } i \geq k, \\ 0, \text{if } x = a + \varepsilon \text{ and } i < k, \\ \left(p_Q p_1\right)^{k+1}, \text{if } x = a \text{ and } i \geq k, \\ p_R p_2, \text{if } x = c. \end{cases}$$

It will be convenient to divide random variable X_i into two variables $X_i^{(1)}$ for $i < k$ and $X_i^{(2)}$ for $i \geq k$. Their distributions can be obtained from the law of the random variable X_i by selection the relevant cases.

Let $\lambda_{N_Z} = (\lambda + \mu)t$ be intensity of flow of unknown events. Then for compound Poisson random variable of total winnings $Z = \sum_{i=1}^{N_Z} X_i$ we have

$$E[Z] = \sum_{j=0}^{\infty} E[Z|N_Z = j] P\{N_Z = j\} =$$

$$\sum_{j=0}^{k-1} E\left(\sum_{i=1}^{N_Z} X_i^{(1)} \middle| N_Z = j\right) P\{N_Z = j\} + \sum_{j=k}^{\infty} E\left(\sum_{i=1}^{k-1} X_i^{(1)} \middle| N_Z = j\right) P\{N_Z = j\} +$$

$$+ \sum_{j=k}^{\infty} E\left(\sum_{i=k}^{N_Z} X_i^{(2)} \middle| N_Z = j\right) P\{N_Z = j\}.$$

The sum is divided into two parts in the last formula because before kth term all X_i are $X_i^{(1)}$ and after kth term all of them are equal to $X_i^{(2)}$. We take $k > 1$, for $k = 1$ is the case of basic model.

The first sum is

$$\sum_{j=0}^{k-1} E\left(\sum_{i=1}^{N_Z} X_i^{(1)} \middle| N_Z = j\right) P\{N_Z = j\} = E\left(X^{(1)}\right) e^{-\lambda_{N_Z}} \sum_{j=0}^{k-1} j \frac{\left(\lambda_{N_Z}\right)^j}{j!}$$

We will use formula 5.24.3 from из [2]

$$\sum_{n=0}^{k} \frac{1}{n!} x^n = \frac{1}{k!} e^x \Gamma(k+1, x), \qquad (A1)$$

where $\Gamma(a, z)$ is incomplete gamma function defined as

$$\Gamma(a, z) = \int_{z}^{\infty} e^{-t} t^{a-1} dt.$$

In our case it will be

$$\sum_{j=0}^{k-1} j \frac{(\lambda_{N_Z})^j}{j!} = \sum_{j=1}^{k-1} \frac{(\lambda_{N_Z})^j}{(j-1)!} = \sum_{m=0}^{k-2} \frac{(\lambda_{N_Z})^{m+1}}{m!}$$

$$= \lambda_{N_Z} \frac{1}{(k-2)!} e^{\lambda_{N_Z}} \Gamma(k-1, \lambda_{N_Z}).$$

Then first sum in the expression for expectation of total gain is

$$\sum_{j=0}^{k-1} E\left(\sum_{i=1}^{N_Z} X_i^{(1)} \middle| N_Z = j \right) P\{N_Z = j\} = E[X^{(1)}] \lambda_{N_Z} \frac{\Gamma(k-1, \lambda_{N_Z})}{\Gamma(k-1)}.$$

Let us find the second sum

$$\sum_{j=k}^{\infty} E\left(\sum_{i=1}^{k-1} X_i^{(1)} \middle| N_Z = j \right) P\{N_Z = j\} = \sum_{j=k}^{\infty} E\left((k-1) X^{(1)} \right) P\{N_Z = j\} =$$

$$= (k-1) E(X^{(1)}) \sum_{j=k}^{\infty} P\{N_Z = j\} = (k-1) E[X^{(1)}] e^{-\lambda_{N_Z}} \sum_{j=k}^{\infty} \frac{(\lambda_{N_Z})^j}{j!}.$$

We can find this sum using formula (A1)

$$e^{-\lambda_{N_Z}} \sum_{j=k}^{\infty} \frac{(\lambda_{N_Z})^j}{j!} = e^{-\lambda_{N_Z}} \left[\sum_{j=0}^{\infty} \frac{(\lambda_{N_Z})^j}{j!} - \sum_{j=0}^{k-1} \frac{(\lambda_{N_Z})^j}{j!} \right] =$$

$$= e^{-\lambda_{N_Z}} \left[e^{\lambda_{N_Z}} - \frac{1}{(k-1)!} e^{\lambda_{N_Z}} \Gamma(k, \lambda_{N_Z}) \right] = 1 - \frac{\Gamma(k, \lambda_{N_Z})}{\Gamma(k)}.$$

So the second sum is equal to

$$\sum_{j=k}^{\infty} E\left(\sum_{i=1}^{k-1} X_i^{(1)} \middle| N_Z = j\right) P\{N_Z = j\} = (k-1)E\big(X^{(1)}\big)\left(1 - \frac{\Gamma(k, \lambda_{N_Z})}{\Gamma(k)}\right).$$

The last sum in our formula can be represented as

$$\sum_{j=k}^{\infty} E\left(\sum_{i=k}^{N_Z} X_i^{(2)} \middle| N_Z = j\right) \cdot P\{N_Z = j\} = \sum_{j=k}^{\infty} E\left(\sum_{i=k}^{j} X_i^{(2)}\right) \cdot P\{N_Z = j\} =$$

$$= E\big[X^{(2)}\big] \cdot \sum_{j=k}^{\infty} (j - k + 1) \cdot P\{N_Z = j\} =$$

$$= E\big[X^{(2)}\big] \cdot \left[\sum_{j=k}^{\infty} j \cdot P\{N_Z = j\} + (1-k) \sum_{j=k}^{\infty} P\{N_Z = j\}\right] =$$

$$= E\big(X^{(2)}\big) \cdot \left[\lambda_{N_Z} \cdot \left(1 - \frac{\Gamma(k-1, \lambda_{N_Z})}{\Gamma(k-1)}\right) + (1-k) \cdot \left(1 - \frac{\Gamma(k, \lambda_{N_Z})}{\Gamma(k)}\right)\right].$$

Hence the expectation of total gain is equal to

$$E(Z) = E\left(\sum_{i=1}^{N_Z} X_i\right) =$$

$$= E\big(X^{(1)}\big) \cdot \left[\lambda_{N_Z} \cdot \frac{\Gamma\big(k-1, \lambda_{N_Z}\big)}{\Gamma(k-1)} + (k-1) \cdot \left(1 - \frac{\Gamma\big(k, \lambda_{N_Z}\big)}{\Gamma(k)}\right)\right] +$$

$$+ E\big(X^{(2)}\big) \cdot \left[\lambda_{N_Z} \cdot \left(1 - \frac{\Gamma(k-1, \lambda_{N_Z})}{\Gamma(k-1)}\right) + (1-k) \cdot \left(1 - \frac{\Gamma(k, \lambda_{N_Z})}{\Gamma(k)}\right)\right].$$

Proof of Theorem 3.

S_i – an experience function on step i – is a random variable equal to the number of consequently correctly recognized Q-events,

$$S_0 = 0,$$

$$S_i = \begin{cases} S_{i-1} + 1, \text{if } a \text{ occurred}, \\ 0, \text{if } \bar{a} \text{ occurred } (a \text{ did not occur}), \end{cases}$$

The experience function on step i can take values from 0 to i with some probabilities.

We have to know the probabilities $P\{S_i < k\}$ and $P\{S_i \geq k\}$, because now the random variable of single winning X_i takes values $-d, -b, a, c$ with probabilities

$$\Pr\{X_i = x\} = \begin{cases} p_R q_2, \text{if } x = -d, \\ p_Q[q_1 \Pr\{S_i < k\} + q_1^* \Pr\{S_i \geq k\}], \text{if } x = -b, \\ p_Q[p_1 \Pr\{S_i < k\} + p_1^* \Pr\{S_i \geq k\}], \text{if } x = a, \\ p_R p_2, \text{if } x = c. \end{cases}$$

Because the probability $P\{S_i < k\}$ is equal to

$$P\{S_i < k\} = 1 - P\{S_i \geq k\} =$$

$$= P\{S_i = 0 \text{ or } S_i = 1 \text{ or } S_i = 2 \text{ or } \dots \text{ or } S_i = k-1\} =$$

$$= P\{S_i = 0\} + P\{S_i = 1\} + P\{S_i = 2\} + \cdots + P\{S_i = k-1\},$$

we will describe all terms.

$P\{S_i = 0\}$ is a probability of experience function to get value 0 on step i, i.e. the device incorrectly recognized coming event Q or it was event R. Hence,

$$P\{S_i = 0\} = P\{X_i = -b \text{ or } X_i = c \text{ or } X_i = -d\} =$$

$$= p_Q[q_1 P\{S_{i-1} < k\} + q_1^* P\{S_{i-1} \geq k\}] + p_R p_2 + p_R q_2 =$$

$$= p_Q[q_1 P\{S_{i-1} < k\} + q_1^* P\{S_{i-1} \geq k\}] + p_R.$$

$P\{S_i = 1\}$ is a probability of experience function to get value 1 on step i, i.e. the device correctly recognized coming event Q on step i and did a mistake on step $i - 1$. It means that $S_{i-1} = 0$ and then happens $X_i = a$ (it can be with probability $p_Q p_1$)

$$P\{S_i = 1\} = \left[p_Q[q_1 P\{S_{i-2} < k\} + q_1^* P\{S_{i-2} \geq k\}] + p_R\right](p_Q p_1).$$

In the same way we can find another probabilities

$$P\{S_i = 0\}, P\{S_i = 1\}, \dots, P\{S_i = k-1\}.$$

For example

$$P\{S_i = k-1\} =$$

$$= \left[p_Q\left[q_1 P\{S_{i-1-(k-1)} < k\} + q_1^* P\{S_{i-k} \geq k\}\right] + p_R\right](p_Q p_1)^{k-1}.$$

Now we can evaluate a sum $P\{S_i < k\} = P\{S_i = 0\} + P\{S_i = 1\} +$ $+P\{S_i = 2\} + \cdots + P\{S_i = k-1\}$ taking $P\{S_j \geq k\} = 1 - P\{S_j < k\}$:

$$P\{S_i < k\} = P\{S_i = 0\} + P\{S_i = 1\} + \cdots + P\{S_i = k-1\} =$$

$$= p_Q \sum_{j=0}^{k-1} \left[q_1 P\{S_{i-1-j} < k\} + q_1^*\left(1 - P\{S_{i-1-j} < k\}\right)\right](p_Q p_1)^j +$$

$$+p_R \sum_{j=0}^{k-1} (p_Q p_1)^j =$$

$$= p_Q(q_1 - q_1^*) \sum_{j=0}^{k-1} P\{S_{i-1-j} < k\}(p_Q p_1)^j + (p_Q q_1^* + p_R) \sum_{j=0}^{k-1} (p_Q p_1)^j.$$

The last sum is geometric progression and the answer is

$$P\{S_i < k\} =$$

$$= p_Q(q_1 - q_1^*) \sum_{j=0}^{k-1} P\{S_{i-1-j} < k\}(p_Q p_1)^j + (p_Q q_1^* + p_R) \frac{(p_Q p_1)^k - 1}{p_Q p_1 - 1}.$$

Using the known dependences between the probabilities, we can express

$$p_Q p_1 - 1 = p_Q - p_Q q_1 - 1 = -p_R - p_Q q_1.$$

Finally the probability $P\{S_i < k\}$ is

$$P\{S_i < k\} = p_Q(q_1 - q_1^*) \sum_{j=0}^{k-1} P\{S_{i-1-j} < k\} \left(p_Q p_1\right)^j$$

$$+ \frac{p_Q q_1^* + p_R}{p_Q q_1 + p_R} \left(1 - \left(p_Q p_1\right)^k\right). \qquad (A2)$$

Proof of Theorem 4.

Let us denote $P\{S_i < k\} = z_i$, $p_Q(q_1 - q_1^*) = \alpha, 0 < \alpha < 1$, $p_Q p_1 = \beta, 0 \leq \beta < 1$, $\frac{p_Q q_1^* + p_R}{p_Q q_1 + p_R} \cdot \left(1 - \left(p_Q p_1\right)^k\right) = \gamma, 0 < \gamma < 1$, and $\gamma = \frac{(1-\beta^k)(1-\alpha-\beta)}{(1-\beta)}$, then the equation (A2) is written as

$$z_i = \alpha \cdot \sum_{s=i-k}^{i-1} \beta^{i-1-s} \cdot z_s + \gamma$$

or

$$z_i = \alpha \cdot (z_{i-1} + \beta z_{i-2} + \beta^2 z_{i-2} + \cdots + \beta^{k-1} z_{i-k}) + \gamma.$$

As all $z_i = P\{S_i < k\}$ are probabilities and $\forall i \;\; 0 \leq z_i \leq 1$, the sequence $\{z_i\}_{i=1}^{\infty}$ is bounded from both sides.

This sequence is nonincreasing, we will prove it by induction. The first k terms are equal to 1: $z_0 = z_1 = z_2 = \cdots = z_{k-1} = 1$, because $S_0 = 0$ and if $i < k$ then $z_i = P\{S_i < k\} = 1$.

First of all we prove $z_k \leq z_{k-1}$ to have k inequalities for the induction

$$z_k = \alpha(1 + \beta + \beta^2 + \cdots + \beta^{k-1}) + \gamma = \alpha \frac{(1-\beta^k)}{(1-\beta)} + \frac{(1-\beta^k)(1-\alpha-\beta)}{(1-\beta)}$$

$$= 1 - \beta^k \leq 1 = z_{k-1}.$$

Let inequality $z_{i-1} \le z_{i-2}$ holds for all $i-1$ first terms of the sequence. We will take z_i and prove that $z_i \le z_{i-1}$. The difference between two terms of the sequence is

$$z_i = \alpha \cdot (z_{i-1} + \beta z_{i-2} + \beta^2 z_{i-2} + \cdots + \beta^{k-1} z_{i-k}) + \gamma$$

$$z_{i-1} = \alpha \cdot (z_{i-2} + \beta z_{i-3} + \beta^2 z_{i-4} + \cdots + \beta^{k-1} z_{i-k-1}) + \gamma$$

$$z_i - z_{i-1} =$$

$$= \alpha \cdot (z_{i-1} + (\beta - 1) z_{i-2} + \beta(\beta - 1) z_{i-2} + \cdots + \beta^{k-2}(\beta - 1) z_{i-k} - \beta^{k-1} z_{i-k-1}) \le$$

$$\le \alpha \cdot (z_{i-2} + (\beta - 1) z_{i-2} + \beta(\beta - 1) z_{i-2} + \cdots + \beta^{k-2}(\beta - 1) z_{i-k} - \beta^{k-1} z_{i-k-1}) =$$

$$= \alpha \cdot (\beta z_{i-2} + \beta(\beta - 1) z_{i-2} + \cdots + \beta^{k-2}(\beta - 1) z_{i-k} - \beta^{k-1} z_{i-k-1}) =$$

$$= \alpha \cdot (\beta^2 z_{i-2} + \beta^2(\beta - 1) z_{i-3} + \cdots + \beta^{k-2}(\beta - 1) z_{i-k} - \beta^{k-1} z_{i-k-1}) \le \cdots \le$$

$$\le \alpha(\beta^{k-1} z_{l-k} - \beta^{k-1} z_{l-k-1}) = \alpha\beta^{k-1}(z_{l-k} - z_{l-k-1}) \le 0,$$

because of $0 < \alpha < 1, 0 \le \beta < 1$, and $z_{i-k} \le z_{i-k-1}$.

So $z_i \le z_{i-1}$ and the sequence $\{z_i\}_{i=1}^{\infty}$ is monotonic (nonincreasing). If nonincreasing sequence is bounded from below, then this sequence converges [4], hence, the sequence $\{z_i\}_{i=1}^{\infty}$ has a limit $\lim_{i\to\infty} z_i = z$.

Let $i \to \infty$, then

$$z = \lim_{i\to\infty} z_i = \lim_{i\to\infty} \left[\alpha \sum_{s=i-k}^{i-1} \beta^{i-1-s} z_s + \gamma \right] = \alpha \sum_{s=i-k}^{i-1} \beta^{i-1-s} z + \gamma$$

$$= z\alpha \frac{1-\beta^k}{1-\beta} + \gamma$$

i.e.

$$z = \frac{(1-\beta)\gamma}{1-\beta-\alpha(1-\beta^k)}.$$

Rewriting it in the initial notations, we obtain

$$\lim_{i\to\infty} P\{S_i < k\} = \frac{\left(p_Q q_1^* + p_R\right)\left(1 - \left(p_Q p_1\right)^k\right)}{\left(p_Q q_1 + p_R\right) - p_Q\left(q_1 - q_1^*\right)\left(1 - \left(p_Q p_1\right)^k\right)}.$$

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в экономике, бизнесе и политике

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