Yangians and cohomology rings of Laumon spaces

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To our friend Sasha Shen on his 50th birthday

Abstract. Laumon moduli spaces are certain smooth closures of the moduli spaces of maps from the projective line to the flag variety of GL_n. We construct the action of the Yangian of $\mathfrak{sl}_n$ in the cohomology of Laumon spaces by certain natural correspondences. We construct the action of the affine Yangian (two-parametric deformation of the universal enveloping algebra of the universal central extension of $\mathfrak{sl}_n[s^{\pm 1}, t]$) in the cohomology of the affine version of Laumon spaces. We compute the matrix coefficients of the generators of the affine Yangian in the fixed point basis of cohomology. This basis is an affine analogue of the Gelfand-Tsetlin basis. The affine analogue of the Gelfand-Tsetlin algebra surjects onto the equivariant cohomology rings of the affine Laumon spaces. The cohomology ring of the moduli space $\mathfrak{M}_{n,d}$ of torsion free sheaves on the plane, of rank n and second Chern class d, trivialized at infinity, is naturally embedded into the cohomology ring of certain affine Laumon space. It is the image of the center Z of the Yangian of $\mathfrak{gl}_n$ naturally embedded into the affine Yangian. In particular, the first Chern class of the determinant line bundle on $\mathfrak{M}_{n,d}$ is the image of a noncommutative power sum in Z.

1. Introduction

1.1. Laumon spaces and Yangians

This note is a sequel to [7]. The moduli spaces $\mathfrak{Q}_{\underline{d}}$ were introduced by G. Laumon in [13] and [14]. They are certain partial compactifications of the moduli spaces of degree $\underline{d}$ based maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}_n$ of GL_n. In [7] we have studied the equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ where $\widetilde{T}$ is a Cartan torus of GL_n acting naturally on the target $\mathcal{B}_n$, and $\mathbb{C}^*$ acts as "loop rotations" on the source $\mathbb{P}^1$. The method of [7] was to introduce an action of

$U(\mathfrak{gl}_n)$ on $V = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$ by certain natural correspondences, and then to realize the cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ as a certain quotient of the Gelfand-Tsetlin subalgebra $\mathfrak{G} \subset U(\mathfrak{gl}_n)$.

In this note we adopt the following approach to the Gelfand-Tsetlin subalgebra going back to I. Cherednik. Namely, $\mathfrak{G}$ is the image of the maximal commutative subalgebra $\mathfrak{A}$ of the Yangian $Y(\mathfrak{gl}_n)$ (Gelfand-Tsetlin subalgebra) under the evaluation homomorphism to $U(\mathfrak{gl}_n)$ (see [16]). Composing the evaluation homomorphism $Y(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$ with the action of $U(\mathfrak{gl}_n)$ on V we obtain an action of $Y(\mathfrak{gl}_n)$ on V. The main observation of this note is that the "new Drinfeld generators" [5] of $Y(\mathfrak{sl}_n) \subset Y(\mathfrak{gl}_n)$ act on V by natural correspondences (Theorem 2.12). In fact they are very similar to the correspondences used by M. Varagnolo [25] to construct the action of Yangians in the equivariant cohomology of quiver varieties.

1.2. Affine Laumon spaces and affine Gelfand-Tsetlin bases

There is an affine version of the Laumon spaces, namely the moduli spaces $\mathcal{P}_{\underline{d}}$ of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, see [8]. The similar correspondences give rise to the action of the affine Yangian $\widehat{Y}$ (two-parametric deformation of the universal enveloping algebra of the universal central extension of $\mathfrak{sl}_n[s^{\pm 1}, t]$, see [12]) on the localized equivariant cohomology $M = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt))$ where the second copy of $\mathbb{C}^*$ acts by the loop rotation on the second copy of $\mathbb{P}^1$ (Theorem 3.19). We compute explicitly the action of Drinfeld generators of $\widehat{Y}$ in the fixed point basis of M (Theorem 3.20).

Since the fixed point basis of V corresponds to the Gelfand-Tsetlin basis of the universal Verma module over $U(\mathfrak{gl}_n)$, we propose to call the fixed point basis of M the *affine Gelfand-Tsetlin basis*. In particular, Conjecture 3.13 asserts that M is isomorphic to the universal Verma module over $U(\widehat{\mathfrak{gl}}_n)$. Moreover, we expect that the specialization of the affine Gelfand-Tsetlin basis gives rise to a basis in the integrable $\widehat{\mathfrak{gl}}_n$-modules (which we also propose to call the affine Gelfand-Tsetlin basis), see Conjecture 3.25. It seems likely that applying the Schur-Weyl functor of [12] to these integrable modules and then going to the limit $n \to \infty$ one obtains the "tableaux representations" (see [21]) of the trigonometric Cherednik algebra of type A. The set of affine Gelfand-Tsetlin patterns has a structure of $\widehat{\mathfrak{sl}}_n$-crystal of the integrable $\widehat{\mathfrak{gl}}_n$-module (Theorem 3.26), equivalent to that of cylindric plane partitions [22]. We expect that the action of $\widehat{Y}$ on the integrable $\widehat{\mathfrak{gl}}_n$-modules coincides with D. Uglov's Yangian action [24].

1.3. Cohomology ring of the Giesecker moduli space

We prove that the maximal commutative subalgebra of Cartan currents $\mathfrak{A}_{\mathrm{aff}} \subset \widehat{Y}$ (the affine Gelfand-Tsetlin algebra) surjects onto the cohomology ring of $\mathcal{P}_{\underline{d}}$ (Theorem 4.2). Furthermore, let $\mathfrak{M}_{n,d}$ denote the moduli space of torsion free sheaves of rank n and second Chern class d, trivialized at infinity.

The equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d})$ is naturally a subring of $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\dots,d})$. There is a natural embedding $Y(\mathfrak{gl}_n)\hookrightarrow\widehat{Y}$ which realizes the center $ZY(\mathfrak{gl}_n)$ as a subalgebra of $\mathfrak{A}_{\mathrm{aff}}$. This subalgebra surjects onto the cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d})\subset H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\dots,d})$. In particular, the first Chern class of the determinant line bundle $\mathcal{D}_0$ on $\mathfrak{M}_{n,d}$ is expressed as a certain noncommutative symmetric function (a power sum of the second kind, see [10]) $\Phi\in ZY(\mathfrak{gl}_n)$ (Theorem 5.7).

Our results are only proved when $n>2$; however we expect them to hold for $n=2$ as well, and it is instructive to compare them with the known results for $n=1$. In this case $\mathfrak{M}_{n,d}$ is the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^2)$. The first Chern class of the determinant line bundle on $\mathrm{Hilb}^d(\mathbb{A}^2)$ was computed by M. Lehn as a certain infinite cubic expression (Calogero-Sutherland operator) of the generators of the Heisenberg algebra acting by correspondences between Hilbert schemes. In our case the role of the Heisenberg algebra is played by $U(\widehat{\mathfrak{gl}}_n)$, and we were unable to express $c_1(\mathcal{D}_0)$ in terms of $U(\widehat{\mathfrak{gl}}_n)$, but there is an explicit formula for it in terms of $ZY(\mathfrak{gl}_n)$.

Finally, let us mention a trigonometric version of our note where the (affine) Yangian is replaced with the (toroidal) affine quantum group, and the equivariant cohomology is replaced with the equivariant K-theory. This is the subject of [23].

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2. Laumon spaces and $\mathfrak{sl}_n$-Yangians

2.1. Laumon spaces

We recall the setup of [7]. Let $\mathbf{C}$ be a smooth projective curve of genus zero. We fix a coordinate z on $\mathbf{C}$, and consider the action of $\mathbb{C}^*$ on $\mathbf{C}$ such that $v(z) = v^{-2}z$. We have $\mathbf{C}^{\mathbb{C}^*} = \{0, \infty\}$.

We consider an n-dimensional vector space W with a basis $w_1, \ldots, w_n$. This defines a Cartan torus $T \subset G = GL_n \subset Aut(W)$. We also consider its 2^n-fold cover, the bigger torus $\widetilde{T}$, acting on W as follows: for $\widetilde{T} \ni \underline{t} = (t_1, \ldots, t_n)$ we have $\underline{t}(w_i) = t_i^2 w_i$. We denote by $\mathcal{B}$ the flag variety of G.

Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1, \ldots, d_{n-1})$, we consider the Laumon's quasiflags' space $\mathcal{Q}_{\underline{d}}$, see [14], 4.2. It is the moduli space of flags of locally free subsheaves

$$0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathrm{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$.

It is known to be a smooth projective variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim \mathcal{B}$, see [13], 2.10.

$\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ (quasiflags based at $\infty \in \mathbf{C}$) formed by the flags

$$0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathcal{W}_i \subset \mathcal{W}$ is a vector subbundle in a neighbourhood of $\infty \in \mathbf{C}$, and the fiber of $\mathcal{W}_i$ at ∞ equals the span $\langle w_1, \ldots, w_i \rangle \subset W$.

It is known to be a smooth quasiprojective variety of dimension $2d_1 + \ldots + 2d_{n-1}$.

2.2. Fixed points

The group $G \times \mathbb{C}^*$ acts naturally on $\mathcal{Q}_{\underline{d}}$, and the group $\widetilde{T} \times \mathbb{C}^*$ acts naturally on $\mathfrak{Q}_{\underline{d}}$. The set of fixed points of $\widetilde{T} \times \mathbb{C}^*$ on $\mathfrak{Q}_{\underline{d}}$ is finite; we recall its description from [9], 2.11.

Let $\widetilde{\underline{d}}$ be a collection of nonnegative integers (d_{ij}), $i \geq j$, such that $d_i = \sum_{j=1}^{i} d_{ij}$, and for $i \geq k \geq j$ we have $d_{kj} \geq d_{ij}$. Abusing notation we denote by $\widetilde{\underline{d}}$ the corresponding $\widetilde{T} \times \mathbb{C}^*$-fixed point in $\mathfrak{Q}_{\underline{d}}$:

$\mathcal{W}_1 = \mathcal{O}_{\mathbf{C}}(-d_{11} \cdot 0) w_1,$

$\mathcal{W}_2 = \mathcal{O}_{\mathbf{C}}(-d_{21} \cdot 0) w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{22} \cdot 0) w_2,$

$\ldots \ldots \ldots,$

$\mathcal{W}_{n-1} = \mathcal{O}_{\mathbf{C}}(-d_{n-1,1} \cdot 0) w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,2} \cdot 0) w_2 \oplus \ldots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,n-1} \cdot 0) w_{n-1}.$

2.3. Correspondences

For $i \in \{1, \ldots, n-1\}$, and $\underline{d} = (d_1, \ldots, d_{n-1})$, we set $\underline{d} + i := (d_1, \ldots, d_i + 1, \ldots, d_{n-1})$. We have a correspondence $\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+i}$ formed by the pairs $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ such that for $j \neq i$ we have $\mathcal{W}_j = \mathcal{W}'_j$, and $\mathcal{W}'_i \subset \mathcal{W}_i$, see [9], 3.1. In other words, $\mathcal{E}_{\underline{d},i}$ is the moduli space of flags of locally free sheaves

$$0 \subset \mathcal{W}_1 \subset \ldots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \ldots \subset \mathcal{W}_{n-1} \subset \mathcal{W}$$

such that $\operatorname{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$, while $\operatorname{rank}(\mathcal{W}'_i) = i$, and $\deg(\mathcal{W}'_i) = -d_i - 1$.

According to [13], 2.10, $\mathcal{E}_{\underline{d},i}$ is a smooth projective algebraic variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim \mathcal{B} + 1$.

We denote by $\mathbf{p}$ (resp. $\mathbf{q}$) the natural projection $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}}$ (resp. $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}+i}$). We also have a map $\mathbf{r}: \ \mathcal{E}_{\underline{d},i} \to \mathbf{C}$,

$$(0 \subset \mathcal{W}_1 \subset \ldots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \ldots \subset \mathcal{W}_{n-1} \subset \mathcal{W}) \mapsto \operatorname{supp}(\mathcal{W}_i/\mathcal{W}'_i).$$

The correspondence $\mathcal{E}_{\underline{d},i}$ comes equipped with a natural line bundle $\mathcal{L}_i$ whose fiber at a point

$$(0 \subset \mathcal{W}_1 \subset \ldots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \ldots \subset \mathcal{W}_{n-1} \subset \mathcal{W})$$

equals $\Gamma(\mathbf{C}, \mathcal{W}_i/\mathcal{W}'_i)$.

Finally, we have a transposed correspondence ${}^{\mathsf{T}}\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}+i} \times \mathcal{Q}_{\underline{d}}$.

Restricting to $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ we obtain the correspondence $\mathfrak{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}} \times \mathfrak{Q}_{\underline{d}+i}$ together with line bundle $\mathfrak{L}_i$ and the natural maps $\mathbf{p}: \ \mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}}, \ \mathbf{q}: \ \mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}+i}, \ \mathbf{r}: \ \mathfrak{E}_{\underline{d},i} \to \mathbf{C} - \infty$. We also have a transposed correspondence ${}^{\mathsf{T}}\mathfrak{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}+i} \times \mathfrak{Q}_{\underline{d}}$. It is a smooth quasiprojective variety of dimension $2d_1 + \ldots + 2d_{n-1} + 1$.

2.4. Equivariant cohomology

We denote by ${}'V$ the direct sum of equivariant (complexified) cohomology: ${}'V = \oplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$. It is a module over $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt) = \mathbb{C}[\mathfrak{t} \oplus \mathbb{C}] = \mathbb{C}[x_1, \ldots, x_n, \hbar]$. Here $\mathfrak{t} \oplus \mathbb{C}$ is the Lie algebra of $\widetilde{T} \times \mathbb{C}^*$. We define $\hbar$ as twice the positive generator of $H^2_{\mathbb{C}^*}(pt, \mathbb{Z})$. Similarly, we define $x_i \in H^2_{\widetilde{T}}(pt, \mathbb{Z})$ in terms of the corresponding one-parametric subgroup. We define $V = {}'V \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \operatorname{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$.

We have an evident grading

$$V = \oplus_{\underline{d}} V_{\underline{d}}, \ V_{\underline{d}} = H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \operatorname{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)).$$

2.5. Universal Verma module

We denote by $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{gl}_n$ over the field $\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$. For $1 \le j, k \le n$ we denote by $E_{jk} \in \mathfrak{gl}_n \subset \mathfrak{U}$ the usual elementary matrix. The standard Chevalley generators are expressed as follows:

$$\mathfrak{e}_i := E_{i+1,i}, \ \mathfrak{f}_i := E_{i,i+1}, \ \mathfrak{h}_i := E_{i+1,i+1} - E_{ii}$$

(note that $\mathfrak{e}_i$ is represented by a *lower* triangular matrix). Note also that $\mathfrak{U}$ is generated by $E_{ii}, \ 1 \le i \le n, \ E_{i,i+1}, E_{i+1,i}, \ 1 \le i \le n-1$. We denote by $\mathfrak{U}_{\le 0}$ the subalgebra of $\mathfrak{U}$ generated by $E_{ii}, \ 1 \le i \le n, \ E_{i,i+1}, \ 1 \le i \le n-1$. It acts on the field $\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ as follows: $E_{i,i+1}$ acts trivially for any $1 \le i \le n-1$, and E_{ii} acts by multiplication by $\hbar^{-1}x_i + i - 1$. We define the *universal Verma module* $\mathfrak{V}$ over $\mathfrak{U}$ as $\mathfrak{U} \otimes_{\mathfrak{U}_{\le 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$. The universal Verma module $\mathfrak{V}$ is an irreducible $\mathfrak{U}$-module.

2.6. The action of generators

The grading and the correspondences ${}^{\mathsf{T}}\mathfrak{E}_{\underline{d},i}, \mathfrak{E}_{\underline{d},i}$ give rise to the following operators on V (note that though $\mathbf{p}$ is not proper, $\mathbf{p}_*$ is well defined on the localized equivariant cohomology due to the finiteness of the fixed point sets):

$E_{ii} = \hbar^{-1}x_i + d_{i-1} - d_i + i - 1 : \ V_{\underline{d}} \to V_{\underline{d}}$;

$\mathfrak{h}_i = \hbar^{-1}(x_{i+1} - x_i) + 2d_i - d_{i-1} - d_{i+1} + 1 : \ V_{\underline{d}} \to V_{\underline{d}}$;

$\mathfrak{f}_i = E_{i,i+1} = \mathbf{p}_*\mathbf{q}^* : \ V_{\underline{d}} \to V_{\underline{d}-i}$;

$\mathfrak{e}_i = E_{i+1,i} = -\mathbf{q}_*\mathbf{p}^* : \ V_{\underline{d}} \to V_{\underline{d}+i}$.

The following theorem is Theorem 2.7 of [7].

Theorem 2.7. *The operators $\mathfrak{e}_i = E_{i+1,i}, E_{ii}, \mathfrak{f}_i = E_{i,i+1}$ on V defined in* 2.6 *satisfy the relations in $\mathfrak{U}$, i.e. they give rise to the action of $\mathfrak{U}$ on V. There is a unique isomorphism Ψ of $\mathfrak{U}$-modules V and $\mathfrak{V}$ carrying $1 \in H^0_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_0) \subset V$ to the lowest weight vector $1 \in \mathbb{C}(\mathfrak{t} \oplus \mathbb{C}) \subset \mathfrak{V}$.*

2.8. Gelfand-Tsetlin basis of the universal Verma module

We will follow the notations of [15] on the Gelfand-Tsetlin bases in representations of $\mathfrak{gl}_n$. To a collection $\widetilde{\underline{d}} = (d_{ij}),\ n-1 \geq i \geq j$ we associate a *Gelfand-Tsetlin pattern* $\Lambda = \Lambda(\widetilde{\underline{d}}) := (\lambda_{ij}),\ n \geq i \geq j$ as follows: $\lambda_{nj} := \hbar^{-1}x_j + j - 1,\ n \geq j \geq 1$; $\lambda_{ij} := \hbar^{-1}x_j + j - 1 - d_{ij},\ n-1 \geq i \geq j \geq 1$. Now we define $\xi_{\widetilde{\underline{d}}} = \xi_\Lambda \in \mathfrak{V}$ by the formulas (2.9)–(2.11) of *loc. cit.* (where $\xi = \xi_0 = 1 \in \mathfrak{V}$). According to Theorem 2.7 of *loc. cit.*, the set $\{\xi_{\widetilde{\underline{d}}}\}$ (over all collections $\widetilde{\underline{d}}$) forms a basis of $\mathfrak{V}$.

According to the Thomason localization theorem, restriction to the $\widetilde{T} \times \mathbb{C}^*$-fixed point set induces an isomorphism

$$H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$$

The fundamental cycles $[\widetilde{\underline{d}}]$ of the $\widetilde{T} \times \mathbb{C}^*$-fixed points $\widetilde{\underline{d}}$ (see 2.2) form a basis in $\oplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$. The embedding of a point $\widetilde{\underline{d}}$ into $\mathfrak{Q}_{\underline{d}}$ is a proper morphism, so the direct image in the equivariant cohomology is well defined, and we will denote by $[\widetilde{\underline{d}}] \in V_{\underline{d}}$ the direct image of the fundamental cycle of the point $\widetilde{\underline{d}}$. The set $\{[\widetilde{\underline{d}}]\}$ forms a basis of V.

The following theorem is Theorem 2.11 and Proposition 2.9 of [7], cf. also [18] 8.2.

Theorem 2.9. *a) The isomorphism $\Psi : V \xrightarrow{\sim} \mathfrak{V}$ of Theorem* 2.7 *takes $[\widetilde{\underline{d}}]$ to $(-\hbar)^{|\underline{d}|}\xi_{\widetilde{\underline{d}}}$ where $|\underline{d}| = d_1 + \ldots + d_{n-1}$.*

b) The matrix coefficients of the operators $\mathfrak{e}_i, \mathfrak{f}_i$ in the basis $\{[\widetilde{\underline{d}}]\}$ are as follows:

$$\mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = -\hbar^{-1} \prod_{j\neq k\leq i} (x_j - x_k + (d_{i,k} - d_{i,j})\hbar)^{-1} \prod_{k\leq i-1} (x_j - x_k + (d_{i-1,k} - d_{i,j})\hbar)$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$;

$$\mathfrak{f}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}=\hbar^{-1}\prod_{j\neq k\leq i}(x_k-x_j+(d_{i,j}-d_{i,k})\hbar)^{-1}\prod_{k\leq i+1}(x_k-x_j+(d_{i,j}-d_{i+1,k})\hbar)$$

if $d'_{i,j}=d_{i,j}-1$ *for certain* $j\leq i$;

All the other matrix coefficients of $\mathfrak{e}_i,\mathfrak{f}_i$ *vanish.*

2.10. Yangian of $\mathfrak{sl}_n$

Let $(a_{kl})_{1\leq k,l\leq n-1}=A_{n-1}$ stand for the Cartan matrix of $\mathfrak{sl}_n$. The Yangian $Y(\mathfrak{sl}_n)$ is the free $\mathbb{C}[\hbar]$-algebra generated by $\mathbf{x}^{\pm}_{k,r},\mathbf{h}_{k,r},\ 1\leq k\leq n-1,\ r\in\mathbb{N}$, with the following relations:

$$[\mathbf{h}_{k,r},\mathbf{h}_{l,s}]=0,\ [\mathbf{h}_{k,0},\mathbf{x}^{\pm}_{l,s}]=\pm a_{kl}\mathbf{x}^{\pm}_{l,s},\tag{1}$$

$$2[\mathbf{h}_{k,r+1},\mathbf{x}^{\pm}_{l,s}]-2[\mathbf{h}_{k,r},\mathbf{x}^{\pm}_{l,s+1}]=\pm\hbar a_{kl}(\mathbf{h}_{k,r}\mathbf{x}^{\pm}_{l,s}+\mathbf{x}^{\pm}_{l,s}\mathbf{h}_{k,r}),\tag{2}$$

$$[\mathbf{x}^{+}_{k,r},\mathbf{x}^{-}_{l,s}]=\delta_{kl}\mathbf{h}_{k,r+s},\tag{3}$$

$$2[\mathbf{x}^{\pm}_{k,r+1},\mathbf{x}^{\pm}_{l,s}]-2[\mathbf{x}^{\pm}_{k,r},\mathbf{x}^{\pm}_{l,s+1}]=\pm\hbar a_{kl}(\mathbf{x}^{\pm}_{k,r}\mathbf{x}^{\pm}_{l,s}+\mathbf{x}^{\pm}_{l,s}\mathbf{x}^{\pm}_{k,r}),\tag{4}$$

$$[\mathbf{x}^{\pm}_{k,r},[\mathbf{x}^{\pm}_{k,p},\mathbf{x}^{\pm}_{l,s}]]+[\mathbf{x}^{\pm}_{k,p},[\mathbf{x}^{\pm}_{k,r},\mathbf{x}^{\pm}_{l,s}]]=0,\ k=l\pm1,\ \forall p,r,s\in\mathbb{N}.\tag{5}$$

For a formal variable u we introduce the generating series $\mathbf{h}_k(u):=1+\sum_{r=0}^{\infty}\mathbf{h}_{k,r}\hbar^{-r}u^{-r-1}$; $\mathbf{x}^{\pm}_k(u):=\sum_{r=0}^{\infty}\mathbf{x}^{\pm}_{k,r}\hbar^{-r}u^{-r-1}$. We can then rewrite the equations (2,4) in the following form

$$\partial_u\partial_v\mathbf{h}_k(u)\mathbf{x}^{\pm}_l(v)(2u-2v\mp a_{kl})=-\partial_u\partial_v\mathbf{x}^{\pm}_l(v)\mathbf{h}_k(u)(2v-2u\mp a_{kl})\tag{6}$$

$$\partial_u\partial_v\mathbf{x}^{\pm}_k(u)\mathbf{x}^{\pm}_l(v)(2u-2v\mp a_{kl})=-\partial_u\partial_v\mathbf{x}^{\pm}_l(v)\mathbf{x}^{\pm}_k(u)(2v-2u\mp a_{kl})\tag{7}$$

2.11. The action of Yangian generators

For any $0\leq i\leq n$ we will denote by $\underline{\mathcal{W}}_i$ the tautological i-dimensional vector bundle on $\mathfrak{Q}_{\underline{d}}\times\mathbf{C}$. Let $\pi:\mathfrak{Q}_{\underline{d}}\times(\mathbf{C}\setminus\infty)\to\mathfrak{Q}_{\underline{d}}$ denote the standard projection. Let q stand for the character of $\widetilde{T}\times\mathbb{C}^*:\ (\underline{t},v)\mapsto v^2$. We define the line bundle $\mathcal{L}'_k:=q^{\frac{1-k}{2}}\mathcal{L}_k$ on the correspondence $\mathcal{E}_{\underline{d},k}$, that is $\mathcal{L}'_k$ and $\mathcal{L}_k$ are isomorphic as line bundles but the equivariant structure of $\mathcal{L}'_k$ is obtained from the equivariant structure of $\mathcal{L}_k$ by the twist by the character $q^{\frac{1-k}{2}}$. Finally, for a vector bundle $\mathcal{V}$ and a formal variable x we denote by $c(\mathcal{V},x)$ the Chern polynomial.

Consider the operators:

$$\mathbf{a}_m(u)=u^m\cdot\mathbf{p}_*(c(\pi_*(\underline{\mathcal{W}}_m|_{\mathbf{C}\setminus\infty}),(-u\hbar)^{-1})\cdot\mathbf{q}^*):\ V_{\underline{d}}\to V_{\underline{d}}[[u^{-1}]][u]$$

$$\mathbf{x}^+_{k,r} := \mathbf{p}_*(c_1(\mathcal{L}'_k)^r \cdot \mathbf{q}^*) : \; V_{\underline{d}} \to V_{\underline{d}-k} \tag{8}$$

$$\mathbf{x}^-_{k,r} := -\mathbf{q}_*(c_1(\mathcal{L}'_k)^r \cdot \mathbf{p}^*) : \; V_{\underline{d}} \to V_{\underline{d}+k} \tag{9}$$

We consider the following generating series of operators on V:

$$\mathbf{h}_k(u) = 1 + \sum_{r=0}^{\infty} \mathbf{h}_{k,r} \hbar^{-r} u^{-r-1} :=$$
$$= \mathbf{a}_k(u + \frac{k+1}{2})^{-1} \mathbf{a}_k(u + \frac{k-1}{2})^{-1} \mathbf{a}_{k-1}(u + \frac{k-1}{2}) \mathbf{a}_{k+1}(u + \frac{k+1}{2}) :$$
$$V_{\underline{d}} \to V_{\underline{d}}[[u^{-1}]]; \tag{10}$$

$$\mathbf{x}^+_k(u) = \sum_{r=0}^{\infty} \mathbf{x}^+_{k,r} \hbar^{-r} u^{-r-1} : \; V_{\underline{d}} \to V_{\underline{d}-k}[[u^{-1}]] \tag{11}$$

$$\mathbf{x}^-_k(u) = \sum_{r=0}^{\infty} \mathbf{x}^-_{k,r} \hbar^{-r} u^{-r-1} : \; V_{\underline{d}} \to V_{\underline{d}+k}[[u^{-1}]] \tag{12}$$

Theorem 2.12. *The operators $\mathbf{h}_{k,r}, \mathbf{x}^{\pm}_{k,r}$ on V defined in* 2.11 *satisfy the relations in $Y(\mathfrak{sl}_n)$, i.e. they give rise to the action of $Y(\mathfrak{sl}_n)$ on V.*

Proof. We identify V with the universal Verma module $\mathfrak{V}$ via the isomorphism Ψ of Theorem 2.7. We consider the operators $\mathbf{A}_m(u), \mathbf{B}_m(u), \mathbf{C}_m(u) : \mathfrak{V} \to \mathfrak{V}[u]$, $1 \leq m \leq n-1$, and also $\mathbf{A}_0(u) = 1$, and $\mathbf{A}_n(u)$, introduced in [16] 5.3. The explicit formula for the (diagonal) action of $\mathbf{A}_m(u)$ in the Gelfand-Tsetlin basis $\{\xi_{\underline{\widetilde{d}}}\}$ of $\mathfrak{V}$ is given in Theorem 5.3.4 of [16]. It reads

$$\mathbf{A}_m(u)\xi_{\underline{\widetilde{d}}} = (u + \hbar^{-1}x_1 - d_{m1}) \ldots (u + \hbar^{-1}x_m - d_{mm})\xi_{\underline{\widetilde{d}}} \tag{13}$$

The explicit formula for the (diagonal) action of $\mathbf{a}_m(u)$ in the fixed point basis $\{[\underline{\widetilde{d}}]\}$ of V is given in the proof of Theorem 3.5 of [7]. Comparing the two formulas we see that the eigenvalues of $\mathbf{A}_m(u)$ in the Gelfand-Tsetlin basis coincide with the eigenvalues of $\mathbf{a}_m(u)$ in the fixed point basis. Now Theorem 2.9 implies that Ψ takes $\mathbf{a}_m(u)$ to $\mathbf{A}_m(u)$.

Recall that there is another "RTT" presentation of $Y(\mathfrak{sl}_n)$, see [16]. It is related to the "new Drinfeld presentation" 2.10 by the Drinfeld isomorphism (see [16] 5.3 and 3.1.8):

$$\mathbf{h}_k(u) = \mathbf{A}_k(u + \frac{k+1}{2})^{-1} \mathbf{A}_k(u + \frac{k-1}{2})^{-1} \mathbf{A}_{k-1}(u + \frac{k-1}{2}) \mathbf{A}_{k+1}(u + \frac{k+1}{2}) \tag{14}$$

$$\mathbf{x}^+_k(u) = \mathbf{A}_k(u + \frac{k-1}{2})^{-1} \mathbf{B}_k(u + \frac{k-1}{2}) \tag{15}$$

$$\mathbf{x}^-_k(u) = \mathbf{C}_k(u + \frac{k-1}{2}) \mathbf{A}_k(u + \frac{k-1}{2})^{-1} \tag{16}$$

According to [16], the operators $\mathbf{A}_k(u), \mathbf{B}_k(u), \mathbf{C}_k(u)$ arise from an action of $Y(\mathfrak{sl}_n)$ in the RTT presentation, so the LHS of equations (14,15,16) do satisfy the relations (1–5). So in order to prove the theorem, it remains to check that the isomorphism Ψ takes the generating series of the LHS of (11) (resp. of (12)) to the LHS of (15) (resp. of (16)). We consider the case of $\mathbf{x}_k^-(u)$, the case of $\mathbf{x}_k^+(u)$ being absolutely similar.

The character of $\widetilde{T} \times \mathbb{C}^*$ in the fiber of the line bundle $\mathcal{L}_i$ at a point $(\underline{\widetilde{d}}, \underline{\widetilde{d}}') \in \mathfrak{E}_{\underline{d},i}$ equals $-x_j + d_{ij}\hbar$ if $d'_{ij} = d_{ij} + 1$ for certain $j \leq i$. It follows from Theorem 2.9 b) that the matrix coefficients of $\mathbf{x}_{i,r}^-$ in the basis $\{[\underline{\widetilde{d}}]\}$ are given by

$$\mathbf{x}^-_{i,r[\underline{\widetilde{d}},\underline{\widetilde{d}}']} = (-x_j + (d_{ij} + \frac{1-i}{2})\hbar)^r \mathfrak{e}_{i[\underline{\widetilde{d}},\underline{\widetilde{d}}']} == -\hbar^{-1}(-x_j + (d_{ij} + \frac{1-i}{2})\hbar)^r \cdot \prod_{j\neq k\leq i} (x_j - x_k + (d_{i,k} - d_{i,j})\hbar)^{-1} \prod_{k\leq i-1} (x_j - x_k + (d_{i-1,k} - d_{i,j})\hbar) \quad (17)$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$, and all the other matrix coefficients vanish.

On the other hand, the matrix coefficients of $\mathbf{C}_i(u)\mathbf{A}_i(u)^{-1}$ in the Gelfand-Tsetlin basis are computed in Theorem 5.3.4 of [16]. Namely, if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$, then specializing $u = d_{ij} - \hbar^{-1}x_j$ we have

$$\mathbf{C}_i(d_{ij} - \hbar^{-1}x_j)\xi_{\underline{\widetilde{d}}} = \prod_{k=1}^{i-1}(\hbar^{-1}(x_k - x_j) - d_{i-1,k} + d_{ij})\xi_{\underline{\widetilde{d}}'} \quad (18)$$

Since $\deg \mathbf{C}_i(u) = i-1$, we can find the matrix coefficients of $\mathbf{C}_i(u)$, and then of $\mathbf{C}_i(u + \frac{i-1}{2})\mathbf{A}_i(u + \frac{i-1}{2})^{-1}$ by the Lagrange interpolation. The resulting formula coincides with the negative of (17). Since the isomorphism Ψ takes $[\underline{\widetilde{d}}]$ to $(-\hbar)^{|\underline{d}|}\xi_{\underline{\widetilde{d}}}$, the coincidence of the two formulas completes the proof of the theorem. □

2.13. Modified generators

We formulate a corollary which will be used in Section 3. For any $0 \leq m < i \leq n$ we will denote by $\underline{\mathcal{W}}_{mi}$ the quotient $\underline{\mathcal{W}}_i/\underline{\mathcal{W}}_m$ of the tautological vector bundles on $\mathfrak{Q}_{\underline{d}} \times \mathbf{C}$. Similarly to the above, we introduce the generating series:

$$\mathbf{a}_{mi}(u) = u^{i-m} \cdot \mathbf{p}_*(c(\pi_*(\underline{\mathcal{W}}_{mi}|_{\mathbf{C}\setminus\infty}), (-u\hbar)^{-1}) \cdot \mathbf{q}^*) : \ V_{\underline{d}} \to V_{\underline{d}}[[u^{-1}]][u]$$

Corollary 2.14. *The operator* $\mathbf{h}_i(u) = \mathbf{a}_{mi}(u + \frac{i-1}{2})^{-1}\mathbf{a}_{mi}(u + \frac{i+1}{2})^{-1}\mathbf{a}_{m,i-1}(u + \frac{i-1}{2})\mathbf{a}_{m,i+1}(u + \frac{i+1}{2})$ *does not depend on* m*, for any* $m < i$.

Proof. Let us denote the RHS of the corollary by $\mathbf{h}_{mi}(u)$. We have to prove that $\mathbf{h}_{mi}(u) = \mathbf{h}_i(u)$. Due to the relation (3) we have to check $\mathbf{h}_{mi,r+s} = [\mathbf{x}^+_{i,r}, \mathbf{x}^-_{i,s}]$. This is done by reduction to the moduli stack $\mathfrak{Z}_k$ introduced in section 3.11 of [3], absolutely similarly to *loc. cit.*

More precisely, we consider a k-dimensional vector space with a basis $\mathfrak{w}_1, \ldots, \mathfrak{w}_k$, and a torus $\mathfrak{T}$ acting on $\mathfrak{w}_l$ by the character τ_l^2. Let $\mathfrak{Z}_k$ be the moduli stack of flags of coherent sheaves $\mathfrak{W}_1 \subset \ldots \subset \mathfrak{W}_k$ on $\mathbf{C}$ locally free at $\infty \in \mathbf{C}$, equipped with compatible trivializations $\mathfrak{W}_l|_\infty = \langle \mathfrak{w}_1, \ldots, \mathfrak{w}_l \rangle$. Note that $\mathfrak{Z}_k$ has connected components numbered by the degrees of $\mathfrak{W}_l$. Absolutely similarly to 2.3 we introduce the correspondences between various connected components. They are equipped with line bundles similar to 2.3. This gives rise to the operators ${}^{\mathfrak{Z}}\mathbf{x}^{\pm}_{i,r},\ {}^{\mathfrak{Z}}\mathbf{a}_i(u),\ {}^{\mathfrak{Z}}\mathbf{h}_{i,r},\ 1 \leq i \leq k-1$, on the localized $\mathfrak{T} \times \mathbb{C}^*$-equivariant cohomology of $\mathfrak{Z}_k$.

For $k = n - m$, we have a map $\mathfrak{z}_k : \mathfrak{Q}_{\underline{d}} \to \mathfrak{Z}_k,\ (\mathcal{W}_\bullet) \mapsto (\mathcal{W}_{m+1}/\mathcal{W}_m \subset \ldots \subset \mathcal{W}_n/\mathcal{W}_m)$. The argument of [3, Sections 3.10,3.11] deduces the desired relation $\mathbf{h}_{mi,r+s} = [\mathbf{x}^+_{i,r}, \mathbf{x}^-_{i,s}]$ from a certain weak form of the identity ${}^{\mathfrak{Z}}\mathbf{h}_{i-m,r+s} = [\ {}^{\mathfrak{Z}}\mathbf{x}^+_{i-m,r},\ {}^{\mathfrak{Z}}\mathbf{x}^-_{i-m,s}]$. The (weak form of the) latter relation is in turn deduced in *loc. cit.* from $\mathbf{h}_{i,r+s} = [\mathbf{x}^+_{i,r}, \mathbf{x}^-_{i,s}]$.

Alternatively, the corollary can be proved by a direct calculation which is a rational version of the trigonometric calculation in the proof of [23, Corollary 2.16]. □

3. Parabolic sheaves and affine Yangians

In this section we generalize the previous results to the affine setting.

3.1. Parabolic sheaves

We recall the setup of section 3 of [3]. Let $\mathbf{X}$ be another smooth projective curve of genus zero. We fix a coordinate y on $\mathbf{X}$, and consider the action of $\mathbb{C}^*$ on $\mathbf{X}$ such that $c(x) = c^{-2}x$. We have $\mathbf{X}^{\mathbb{C}^*} = \{0_\mathbf{X}, \infty_\mathbf{X}\}$. Let $\mathbf{S}$ denote the product surface $\mathbf{C} \times \mathbf{X}$. Let $\mathbf{D}_\infty$ denote the divisor $\mathbf{C} \times \infty_\mathbf{X} \cup \infty_\mathbf{C} \times \mathbf{X}$. Let $\mathbf{D}_0$ denote the divisor $\mathbf{C} \times 0_\mathbf{X}$.

Given an n-tuple of nonnegative integers $\underline{d} = (d_0, \ldots, d_{n-1})$, we say that a *parabolic sheaf* $\mathcal{F}_\bullet$ of degree $\underline{d}$ is an infinite flag of torsion free coherent sheaves of rank n on $\mathbf{S}:\ \ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ such that:

(a) $\mathcal{F}_{k+n} = \mathcal{F}_k(\mathbf{D}_0)$ for any k;

(b) $ch_1(\mathcal{F}_k) = k[\mathbf{D}_0]$ for any k: the first Chern classes are proportional to the fundamental class of $\mathbf{D}_0$;

(c) $ch_2(\mathcal{F}_k) = d_i$ for $i \equiv k \pmod{n}$;

(d) $\mathcal{F}_0$ is locally free at $\mathbf{D}_\infty$ and trivialized at $\mathbf{D}_\infty:\ \mathcal{F}_0|_{\mathbf{D}_\infty} = W \otimes \mathcal{O}_{\mathbf{D}_\infty}$;

(e) For $-n \leq k \leq 0$ the sheaf $\mathcal{F}_k$ is locally free at $\mathbf{D}_\infty$, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-n},\ \mathcal{F}_0/\mathcal{F}_k$ (both supported at $\mathbf{D}_0 = \mathbf{C} \times 0_\mathbf{X} \subset \mathbf{S}$) are both locally free at the point $\infty_\mathbf{C} \times 0_\mathbf{X}$; moreover, the local sections of $\mathcal{F}_k|_{\infty_\mathbf{C} \times \mathbf{X}}$ are those sections of $\mathcal{F}_0|_{\infty_\mathbf{C} \times \mathbf{X}} = W \otimes \mathcal{O}_\mathbf{X}$ which take value in $\langle w_1, \ldots, w_{n+k} \rangle \subset W$ at $0_\mathbf{X} \in \mathbf{X}$.

The fine moduli space $\mathcal{P}_{\underline{d}}$ of degree $\underline{d}$ parabolic sheaves exists and is a smooth connected quasiprojective variety of dimension $2d_0 + \ldots + 2d_{n-1}$.

3.2. Fixed points

The group $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ acts naturally on $\mathcal{P}_{\underline{d}}$, and its fixed point set is finite. In order to describe it, we recall the well known description of the fixed point set of $\mathbb{C}^* \times \mathbb{C}^*$ on the Hilbert scheme of $(\mathbf{C} - \infty_{\mathbf{C}}) \times (\mathbf{X} - \infty_{\mathbf{X}})$. Namely, the fixed points are parametrized by the Young diagrams, and for a diagram $\lambda = (\lambda_0 \geq \lambda_1 \geq \ldots)$ (where $\lambda_N = 0$ for $N \gg 0$) the corresponding fixed point is the ideal $J_\lambda = \mathbb{C}[z] \cdot (\mathbb{C}y^0 z^{\lambda_0} \oplus \mathbb{C}y^1 z^{\lambda_1} \oplus \ldots)$. We will view J_λ as an ideal in $\mathcal{O}_{\mathbf{C}\times\mathbf{X}}$ coinciding with $\mathcal{O}_{\mathbf{C}\times\mathbf{X}}$ in a neighbourhood of infinity.

We say $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for any $i \geq 0$. We say $\lambda \widetilde{\supset} \mu$ if $\lambda_i \geq \mu_{i+1}$ for any $i \geq 0$.

We consider a collection $\boldsymbol{\lambda} = (\lambda^{kl})_{1\leq k,l\leq n}$ of Young diagrams satisfying the following inequalities:

$$\lambda^{11} \supset \lambda^{21} \supset \ldots \supset \lambda^{n1} \widetilde{\supset} \lambda^{11};\ \lambda^{22} \supset \lambda^{32} \supset \ldots \supset \lambda^{12} \widetilde{\supset} \lambda^{22};\ \ldots;$$
$$\lambda^{nn} \supset \lambda^{1n} \supset \ldots \supset \lambda^{n-1,n} \widetilde{\supset} \lambda^{nn} \qquad (19)$$

We set $d_k(\boldsymbol{\lambda}) = \sum_{l=1}^{n} |\lambda^{kl}|$, and $\underline{d}(\boldsymbol{\lambda}) = (d_0(\boldsymbol{\lambda}) := d_n(\boldsymbol{\lambda}), \ldots, d_{n-1}(\boldsymbol{\lambda}))$.

Given such a collection $\boldsymbol{\lambda}$ we define a parabolic sheaf $\mathcal{F}_\bullet = \mathcal{F}_\bullet(\boldsymbol{\lambda})$, or just $\boldsymbol{\lambda}$ by an abuse of notation, as follows: for $1 \leq k \leq n$ we set

$$\mathcal{F}_{k-n} = \bigoplus_{1\leq l\leq k} J_{\lambda^{kl}} w_l \oplus \bigoplus_{k<l\leq n} J_{\lambda^{kl}}(-\mathbf{D}_0) w_l \qquad (20)$$

Lemma 3.3. *The correspondence $\boldsymbol{\lambda} \mapsto \mathcal{F}_\bullet(\boldsymbol{\lambda})$ is a bijection between the set of collections $\boldsymbol{\lambda}$ satisfying (19) such that $\underline{d}(\boldsymbol{\lambda}) = \underline{d}$, and the set of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points in $\mathcal{P}_{\underline{d}}$.*

Proof. It is easy to see that flags of the form (20) are torus fixed points. Note that the condition $\lambda^{kk} \widetilde{\subset} \lambda^{k-1,k}$ appears because of the twists by $\mathbf{D}_0$ in (20). Conversely, any torus fixed point splits up as a sum of rank 1 torsion free sheaves:

$$\mathcal{F}_{k-n} = \bigoplus_{l} T_{kl} w_l,$$

just as in the well-known case of rank n torsion free sheaves on a surface. The rank 1 torsion free sheaves T_{kl} are just twisted ideal sheaves: $T_{kl} = J_{\lambda^{kl}}$ for $l \leq k$ and $T_{kl} = J_{\lambda^{kl}}(-\mathbf{D}_0)$ for $l > k$, as follows from the framing and c_1 conditions. Finally, the fact that the partitions λ^{kl} have to satisfy (19) is simply equivalent to the conditions $\mathcal{F}_{k-n} \subset \mathcal{F}_{k-1-n}$. □

3.4. Parabolic sheaves as orbifold sheaves

We will now introduce a different realization of parabolic sheaves, and another parameterization of the fixed point set which is very closely related to this new realization. We first learned of this construction from A. Okounkov, though it is already present in the work of Biswas [2]. Let $\sigma : \mathbf{C} \times \mathbf{X} \to \mathbf{C} \times \mathbf{X}$ denote the map $\sigma(z, y) = (z, y^n)$, and let $G = \mathbb{Z}/n\mathbb{Z}$. Then G acts on $\mathbf{C} \times \mathbf{X}$ by multiplying the coordinate on $\mathbf{X}$ with the $n-$th roots of unity.

A parabolic sheaf $\mathcal{F}_\bullet$ is completely determined by the flag of sheaves

$$\mathcal{F}_0(-\mathbf{D}_0) \subset \mathcal{F}_{-n+1} \subset ... \subset \mathcal{F}_0,$$

satisfying conditions 3.1.(a–e). To $\mathcal{F}_\bullet$ we can associate a single G-invariant sheaf $\tilde{\mathcal{F}}$ on $\mathbf{C} \times \mathbf{X}$:

$$\tilde{\mathcal{F}} = \sigma^* \mathcal{F}_{-n+1} + \sigma^* \mathcal{F}_{-n+2}(-\mathbf{D}_0) + ... + \sigma^* \mathcal{F}_0(-(n-1)\mathbf{D}_0).$$

This G invariant sheaf determines a parabolic sheaf if and only if it satisfies certain numeric and framing conditions that mimick conditions 3.1.(b–d):

(b′) $ch_1(\tilde{\mathcal{F}}) = -\frac{n(n-1)}{2}[\mathbf{D}_0]$;
(c′) $ch_2(\tilde{\mathcal{F}}) = d_1 + ... + d_n$;
(d′) $\tilde{\mathcal{F}}$ is locally free at $\mathbf{D}_\infty$ and trivialized there:

$$\tilde{\mathcal{F}}|_{\mathbf{D}_\infty} = \mathcal{O}_{\mathbf{D}_\infty} \oplus \mathcal{O}_{\mathbf{D}_\infty}(-\mathbf{D}_0) \oplus ... \oplus \mathcal{O}_{\mathbf{D}_\infty}(-(n-1)\mathbf{D}_0).$$

If $\mathcal{F}_\bullet$ is a $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ fixed parabolic sheaf corresponding to a collection $\boldsymbol{\lambda}$ as in the previous section, then we have

$$\tilde{\mathcal{F}} = \bigoplus_{l=1}^{n} J_{\lambda^l}(-(l-1)\mathbf{D}_0) w_l, \tag{21}$$

where $(\lambda^1, ..., \lambda^n)$ is a collection of partitions, given by

$$\lambda^l_{ni-n\lfloor \frac{k-l}{n} \rfloor + k - l} = \lambda^{kl}_i. \tag{22}$$

Here $\lfloor \frac{k-l}{n} \rfloor$ stands for the maximal integer smaller than or equal to $\frac{k-l}{n}$.

For $j \in \mathbb{Z}$, let $(j \bmod n)$ denote that element of $\{1, \ldots, n\}$ which is congruent to j modulo n. For $i \geq j \in \mathbb{Z}$, if we denote

$$d_{ij} = \lambda^{j \bmod n}_{i-j} \tag{23}$$

we obtain a collection $(d_{ij}) = \underline{\widetilde{d}} = \underline{\widetilde{d}}(\boldsymbol{\lambda})$ of non-negative integers with the properties that

$$d_{kj} \geq d_{ij} \ \forall i \geq k \geq j; \ d_{i+n,j+n} = d_{ij} \ \forall i \geq j; \ d_{ij} = 0 \ \text{for} \ i - j \gg 0. \tag{24}$$

For $1 \leq k \leq n$, let us write

$$d_k(\underline{\widetilde{d}}) = \sum_{j \leq k} d_{kj} = \sum_{l=1}^{n} \sum_{i \leq \lfloor \frac{k-l}{n} \rfloor} d_{k(l+ni)} =$$

$$= \sum_{l=1}^{n} \sum_{i \geq 0} \lambda^l_{ni-n\lfloor \frac{k-l}{n} \rfloor + k - l} = \sum_{l=1}^{n} \sum_{i \geq 0} \lambda^{kl}_i = d_k(\boldsymbol{\lambda}).$$

The collection $(d_1(\underline{\widetilde{d}}), \ldots, d_n(\underline{\widetilde{d}})) = (d_1(\boldsymbol{\lambda}), \ldots, d_n(\boldsymbol{\lambda}))$ will be denoted by $\|\underline{\widetilde{d}}\| = \|\boldsymbol{\lambda}\|$. Summarizing the above discussion, we have:

Lemma 3.5. *The correspondence* $\boldsymbol{\lambda} \mapsto \widetilde{\underline{d}}(\boldsymbol{\lambda})$ *is a bijection between the set of collections* $\boldsymbol{\lambda}$ *satisfying* (19)*, and the set* D *of collections* $\widetilde{\underline{d}}$ *satisfying* (24)*. We have* $\|\boldsymbol{\lambda}\| = \|\widetilde{\underline{d}}(\boldsymbol{\lambda})\|$.

By virtue of Lemmas 3.3 and 3.5 we will parametrize and sometimes denote the $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points in $\mathcal{P}_{\underline{d}}$ by collections $\widetilde{\underline{d}}$ such that $\underline{d} = \|\widetilde{\underline{d}}\|$.

3.6. Correspondences

If the collections $\underline{d}$ and $\underline{d}'$ differ at the only place $i \in I := \mathbb{Z}/n\mathbb{Z}$, and $d'_i = d_i + 1$, then we consider the correspondence $\mathsf{E}_{\underline{d},i} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}'}$ formed by the pairs $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ such that for $j \not\equiv i \pmod n$ we have $\mathcal{F}_j = \mathcal{F}'_j$, and for $j \equiv i \pmod n$ we have $\mathcal{F}'_j \subset \mathcal{F}_j$.

It is a smooth quasiprojective algebraic variety of dimension $1 + 2\sum_{i\in I} d_i$.

We denote by $\mathbf{p}$ (resp. $\mathbf{q}$) the natural projection $\mathsf{E}_{\underline{d},i} \to \mathcal{P}_{\underline{d}}$ (resp. $\mathsf{E}_{\underline{d},i} \to \mathcal{P}_{\underline{d}'}$). For $j \equiv i \pmod n$ the correspondence $\mathsf{E}_{\underline{d},i}$ is equipped with a natural line bundle L_j whose fiber at $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ equals $\Gamma(\mathbf{C}, \mathcal{F}_j/\mathcal{F}'_j)$. Finally, we have a transposed correspondence ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i} \subset \mathcal{P}_{\underline{d}'} \times \mathcal{P}_{\underline{d}}$.

3.7. Equivariant cohomology

We denote by $'M$ the direct sum of equivariant (complexified) cohomology: $'M = \oplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$. It is a module over $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt) = \mathbb{C}[\mathfrak{t} \oplus \mathbb{C} \oplus \mathbb{C}] = \mathbb{C}[x_1, \ldots, x_n, \hbar, \hbar']$. Here $\hbar'$ is twice the positive generator of $H^2_{\mathbb{C}^*}(pt, \mathbb{Z})$ for the second copy of $\mathbb{C}^*$. We define $M = {}'M \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt))$.

We have an evident grading

$$M = \oplus_{\underline{d}} M_{\underline{d}}, \ M_{\underline{d}} = H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt)).$$

3.8. The action of generators

The grading and the correspondences ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i}, \mathsf{E}_{\underline{d},i}$ give rise to the following operators on M (note that though $\mathbf{p}$ is not proper, $\mathbf{p}_*$ is well defined on the localized equivariant cohomology due to the finiteness of the fixed point set of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$):

$$\mathfrak{h}_i = \hbar^{-1}(x_{i+1} - x_i) + \delta_{i,0}\hbar^{-1}\hbar' + 2d_i - d_{i-1} - d_{i+1} + 1 : \ M_{\underline{d}} \to M_{\underline{d}}; \quad (25)$$

$$\mathfrak{f}_i = \mathbf{p}_*\mathbf{q}^* : \ M_{\underline{d}} \to M_{\underline{d}-i}; \quad (26)$$

$$\mathfrak{e}_i = -\mathbf{q}_*\mathbf{p}^* : \ M_{\underline{d}} \to M_{\underline{d}+i}. \quad (27)$$

Theorem 3.9. *For* $n > 2$*, the operators* $\mathfrak{e}_i, \mathfrak{h}_i, \mathfrak{f}_i$ *of* 3.8 *on* M *satisfy the relations of the Chevalley generators of the Kac-Moody algebra* $\widehat{\mathfrak{sl}}_n$.

The proof is entirely similar to the proof of Conjecture 3.7 of [3] in 3.8–3.10 of *loc. cit.* □

Remark 3.10. The paper [3] studies the equivariant K-theory of $\mathcal{P}_{\underline{d}}$, so the computations involved in the proof of Theorem 3.9 are the rational (simpler) versions of the trigonometric computations of [3]. We believe Theorem 3.9 holds true for $n = 2$ (as well as Conjecture 3.7 of [3]). However, our method of verification of the relations between the Chevalley generators consists in reduction to the finite case $d_0 = 0$ considered in Section 2. Since the Cartan matrix of $\widehat{\mathfrak{sl}}_2$ contains the off-diagonal entries -2, contrary to the Cartan matrix of $\mathfrak{sl}_n$, this method fails for $n = 2$. Alternatively, Theorem 3.9 can be proved by direct calculations using the explicit matrix coefficients of the Chevalley generators given in Theorem 3.11 below. These calculations, again, are the rational versions of the trigonometric calculations of [23].

Similarly to Theorem 2.9, it is possible to compute the matrix coefficients of $\mathfrak{e}_i, \mathfrak{f}_i$ in the basis formed by the fundamental classes of the $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points $[\widetilde{\underline{d}}] \in M$. To this end let us assign to a collection $\widetilde{\underline{d}}$ a collection of weights $p_{ij} := -x_{j \pmod n} + d_{ij}\hbar + \lfloor \frac{-j}{n} \rfloor \hbar'$.

Theorem 3.11. *The matrix coefficients of the operators $\mathfrak{e}_i, \mathfrak{f}_i$ in the basis $\{[\widetilde{\underline{d}}]\}$ are as follows:*

$$\mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = -\hbar^{-1} \frac{p_{i-1,j} - p_{ij}}{p_{ii} - p_{ij}} \prod_{j \neq k \leq i-1} \frac{p_{i-1,k} - p_{ij}}{p_{ik} - p_{ij}}$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$ (note that almost all factors in this product are equal to 1 due to the condition (24); also, in case $i = j$ the factor $\frac{p_{i-1,j} - p_{ij}}{p_{ii} - p_{ij}}$ is set to be 1);

$$\mathfrak{f}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = \hbar^{-1}(p_{i+1,j} - p_{ij})(p_{i+1,i+1} - p_{ij}) \prod_{j \neq k \leq i} \frac{p_{i+1,k} - p_{ij}}{p_{ik} - p_{ij}}$$

if $d'_{i,j} = d_{i,j} - 1$ for certain $j \leq i$;

All the other matrix coefficients of $\mathfrak{e}_i, \mathfrak{f}_i$ vanish.

The proof will be given in section 4.22 below.

3.12. Universal Verma module over $\widehat{\mathfrak{gl}}_n$

Contrary to the case of Laumon spaces (Theorem 2.7), M is not isomorphic to the universal Verma module over $\widehat{\mathfrak{sl}}_n$. We conjecture that M is isomorphic to the universal Verma module over $\widehat{\mathfrak{gl}}_n$. Let us introduce some more notations and correspondences in order to formulate the conjecture more precisely.

Note that the centre of $\widehat{\mathfrak{sl}}_n$ is spanned by the element $C = \mathfrak{h}_0 + \ldots + \mathfrak{h}_{n-1}$ which acts on M as $n + \hbar'\hbar^{-1}$. We denote by $\mathfrak{H}$ the Heisenberg Lie algebra with generators $a_i,\ i \in \mathbb{Z}$, and C', and relations $[a_p, C'] = 0,\ [a_p, a_q] = \delta_{p,-q} p C'$. We denote by $\widehat{\mathfrak{gl}}_n$ the quotient $(\mathfrak{H} \oplus \widehat{\mathfrak{sl}}_n)/(C' - nC)$. The universal Verma module over $\widehat{\mathfrak{gl}}_n$ is the tensor product of the universal Verma module over $\widehat{\mathfrak{sl}}_n$ and the Fock module over $\mathfrak{H}$.

For $\underline{d} = (d_1, \ldots, d_n)$ and $m \geq 1$ we set $\underline{d} + m\delta := (d_1 + m, \ldots, d_n + m)$. Let $\mathfrak{E}_{\underline{d},m\delta} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}+m\delta}$ be the correspondence formed by all the pairs $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ such that $\mathcal{F}_\bullet \supset \mathcal{F}'_\bullet$, and the quotient $\mathcal{F}_\bullet/\mathcal{F}'_\bullet$ is supported at a single point in $\mathbf{C}$. In this case the collection of quotients $(\mathcal{F}_1/\mathcal{F}'_1, \ldots, \mathcal{F}_n/\mathcal{F}'_n)$ can be organized into a nilpotent representation $T_\bullet$ of the cyclic quiver $\widetilde{A}_{n-1}$, see [8] 7.4. Let $\mathsf{E}^{\circ}_{\underline{d},m\delta} \subset \mathfrak{E}_{\underline{d},m\delta}$ be the locally closed subset formed by all the pairs $\mathcal{F}_\bullet \supset \mathcal{F}_\bullet$ such that the corresponding $\widetilde{A}_{n-1}$-representation is indecomposable. According to Proposition 7.8 of [8], $\mathsf{E}^{\circ}_{\underline{d},m\delta}$ is a union of n middle-dimensional irreducible components. We denote by $\mathsf{E}_{\underline{d},m\delta}$ the closure of $\mathsf{E}^{\circ}_{\underline{d},m\delta}$.

The correspondence $\mathsf{E}_{\underline{d},m\delta} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}+m\delta}$ gives rise to the operator $M_{\underline{d}} \to M_{\underline{d}+m\delta}$ which we denote by a_m. The transpose correspondence gives rise to the operator $M_{\underline{d}+m\delta} \to M_{\underline{d}}$ which we denote by a_{-m}. We set $a_0 = n + \frac{\hbar'}{\hbar}$. The following conjecture was proposed by A. Kuznetsov a few years ago.

Conjecture 3.13. *a) The operators a_m, $m \in \mathbb{Z}$ and the operators $\mathfrak{e}_i, \mathfrak{h}_i, \mathfrak{f}_i$ of (*25,26,27*) satisfy the relations of the Lie algebra $\widehat{\mathfrak{gl}}_n$, and equip M with a structure of $\widehat{\mathfrak{gl}}_n$-module.*

b) M is isomorphic to the universal Verma module over $\widehat{\mathfrak{gl}}_n$.

Remark 3.14. V. Baranovsky [1] has defined the action of Heisenberg algebra on the cohomology of Giesecker moduli spaces (of torsion free sheaves on surfaces) via certain correspondences $\mathsf{B}_{d,m}$. More precisely, let $\mathfrak{M}_{n,d}$ be the moduli space of torsion free sheaves on $\mathbf{C} \times \mathbf{X}$ of rank n and second Chern class d, trivialized at $\mathbf{C} \times \infty_{\mathbf{X}} \cup \infty_{\mathbf{C}} \times \mathbf{X}$ (see [17], section 2). We have an evident morphism $\eta:\ \mathcal{P}_{\underline{d}} \to \mathfrak{M}_{n,d_0}$ (forgetting the flag). The correspondence $\mathsf{B}_{d,m} \subset \mathfrak{M}_{n,d} \times \mathfrak{M}_{n,d+m}$ is formed by the pairs $(\mathcal{F}, \mathcal{F}')$ such that $\mathcal{F} \supset \mathcal{F}'$, and the quotient $\mathcal{F}/\mathcal{F}'$ is supported at a single point x of $\mathbb{A}^2$. We consider the open piece $\overset{\circ}{\mathsf{B}}_{d,m}$ given by the condition that x lies out of $\mathbf{C} \times 0_{\mathbf{X}}$. The inverse image of $\overset{\circ}{\mathsf{B}}_{d_0,m}$ under η gives a correspondence $\overset{\circ}{\mathsf{B}}_{\underline{d},m\delta} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}+m\delta}$. It seems likely that as x tends to $\mathbf{C} \times 0_{\mathbf{X}}$, the correspondence $\overset{\circ}{\mathsf{B}}_{\underline{d},m\delta}$ tends to $\mathsf{E}_{\underline{d},m\delta}$.

3.15. Affine Yangian

From now on we impose the restriction $n > 2$. Let $(a_{kl})_{1\le k,l\le n} = \widehat{A}_{n-1}$ stand for the Cartan matrix of $\widehat{\mathfrak{sl}}_n$. The Yangian $Y(\widehat{\mathfrak{sl}}_n)$ is the free $\mathbb{C}[\hbar]$-algebra generated by $\mathbf{x}^{\pm}_{k,r}, \mathbf{h}_{k,r},\ 1 \le k \le n,\ r \in \mathbb{N}$, with the relations (1–5) where k, l are understood as residues modulo n, so that for instance if $k = n$ then $k + 1 = 1$.

The affine Yangian $\widehat{Y}$ of type $\widehat{A}_{n-1}$ is the free $\mathbb{C}[\hbar, \hbar']$-algebra generated by $\mathbf{x}^{\pm}_{k,r}, \mathbf{h}_{k,r},\ 1 \le k \le n,\ r \in \mathbb{N}$ with the same relations as in $Y(\widehat{\mathfrak{sl}}_n)$ except for relations (2,4) for the pairs $(k, l) = (n, 1), (1, n)$. These relations are modified as follows. We introduce the shifted generating series $\mathbf{h}_n(u - \frac{\hbar'}{\hbar} - \frac{n}{2}) =: {}'\mathbf{h}_n(u) := 1 + \sum_{r=0}^{\infty} {}'\mathbf{h}_{n,r}\hbar^{-r}u^{-r-1}$; $\mathbf{x}^{\pm}_n(u - \frac{\hbar'}{\hbar} - \frac{n}{2}) =: {}'\mathbf{x}^{\pm}_n(u) := \sum_{r=0}^{\infty} {}'\mathbf{x}^{\pm}_{k,r}\hbar^{-r}u^{-r-1}$. Now the new relations read

$$2[{}'\mathbf{h}_{n,r+1}, \mathbf{x}^{\pm}_{1,s}] - 2[{}'\mathbf{h}_{n,r}, \mathbf{x}^{\pm}_{1,s+1}] = \mp\hbar({}'\mathbf{h}_{n,r}\mathbf{x}^{\pm}_{1,s} + \mathbf{x}^{\pm}_{1,s}{}'\mathbf{h}_{n,r}), \tag{28}$$

$$2[\mathbf{h}_{1,r+1}, {}'\mathbf{x}^{\pm}_{n,s}] - 2[\mathbf{h}_{1,r}, {}'\mathbf{x}^{\pm}_{n,s+1}] = \mp\hbar(\mathbf{h}_{1,r}{}'\mathbf{x}^{\pm}_{n,s} + {}'\mathbf{x}^{\pm}_{n,s}\mathbf{h}_{1,r}), \tag{29}$$

$$2[{}'\mathbf{x}^{\pm}_{n,r+1}, \mathbf{x}^{\pm}_{1,s}] - 2[{}'\mathbf{x}^{\pm}_{n,r}, \mathbf{x}^{\pm}_{1,s+1}] = \mp\hbar({}'\mathbf{x}^{\pm}_{n,r}\mathbf{x}^{\pm}_{1,s} + \mathbf{x}^{\pm}_{1,s}{}'\mathbf{x}^{\pm}_{n,r}). \tag{30}$$

Thus we have $Y(\widehat{\mathfrak{sl}}_n) = \widehat{Y}/(\hbar' + \frac{n\hbar}{2})$.

Remark 3.16. It is possible to define the Yangian $Y(\widehat{\mathfrak{sl}}_2)$ modifying the Serre relations (5). However, the above definition of the affine Yangian $\widehat{Y}$ of type $\widehat{A}_{n-1}$ makes no sense for $n = 2$. We do not know any reasonable definition of $\widehat{Y}$ for $n = 2$.

Note that $\widehat{Y}$ is isomorphic to $\widehat{\mathbf{Y}}_{\beta,\lambda}$ introduced in [12] Definition 3.3. Indeed, the Yangian $\widehat{\mathbf{Y}}_{\beta,\lambda}$ is generated by $X^{\pm}_{k,r}$ and $H^{\pm}_{k,r}$, satisfying the relations (1,3,5) and some modification of the relations (2,4). These modified relations can be rewritten in terms of the generating series $X^{\pm}_k(u) = \sum_{r=0}^{\infty} X^{\pm}_{k,r}\lambda^{-r}u^{-1-r}$, $H_k(u) = \sum_{r=0}^{\infty} H^{\pm}_{k,r}\lambda^{-r}u^{-1-r}$. The series $X^{\pm}_k(u)$, $H_l(u)$ satisfy the relations (6,7) except that for the pairs $(k,l) = (1,n),(n,n-1)$ the relations are modified in the following way:

$$\partial_u\partial_v H_k(u)X^{\pm}_l(v)(2u-2v\pm1-\frac{2\beta}{\lambda}+1) = -\partial_u\partial_v X^{\pm}_l(v)H_k(u)(2v-2u\pm1+\frac{2\beta}{\lambda}-1) \tag{31}$$

$$\partial_u\partial_v H_l(u)X^{\pm}_k(v)(2u-2v\pm1+\frac{2\beta}{\lambda}-1) = -\partial_u\partial_v X^{\pm}_k(v)H_l(u)(2v-2u\pm1-\frac{2\beta}{\lambda}+1) \tag{32}$$

$$\partial_u\partial_v X^{\pm}_k(u)X^{\pm}_l(v)(2u-2v\pm1-\frac{2\beta}{\lambda}+1) = -\partial_u\partial_v X^{\pm}_l(v)X^{\pm}_k(u)(2v-2u\pm1+\frac{2\beta}{\lambda}-1) \tag{33}$$

The isomorphism $\widehat{Y} \to \widehat{\mathbf{Y}}_{\beta,\lambda}$ takes $\hbar$ to λ, and $\hbar'$ to $-\frac{n\lambda}{2} + 2\beta - \lambda$. It is defined on the generating series as

$$\mathbf{x}^{\pm}_k(u) \mapsto X^{\pm}_k(u),\ \mathbf{h}_k(u) \mapsto H_k(u) \text{ for } k \neq n, \tag{34}$$

$$\mathbf{x}^{\pm}_n(u - \frac{\hbar'}{2\hbar} - \frac{n}{4}) \mapsto X^{\pm}_n(u),\ \mathbf{h}_n(u - \frac{\hbar'}{2\hbar} - \frac{n}{4}) \mapsto H_n(u). \tag{35}$$

3.17. The action of the affine Yangian generators

For any $m \leq i \in \mathbb{Z}$ we will denote by $\underline{\mathcal{W}}_{mi}$ the quotient $\underline{\mathcal{F}}_i/\underline{\mathcal{F}}_m$ of the tautological vector bundles, living on $\mathcal{P}_{\underline{d}} \times \mathbf{C} \subset \mathcal{P}_{\underline{d}} \times \mathbf{S}$. Once again, $\pi : \mathcal{P}_{\underline{d}} \times (\mathbf{C} \setminus \infty) \to \mathcal{P}_{\underline{d}}$ denotes the standard projection. Let us consider:

$$\mathbf{a}_{mi}(u) = u^{i-m} \cdot \mathbf{p}_*(c(\pi_*(\underline{\mathcal{W}}_{mi}|_{\mathbf{C}\setminus\infty}), (-u\hbar)^{-1}) \cdot \mathbf{q}^*) : \ M_{\underline{d}} \to M_{\underline{d}}[[u^{-1}]][u] \tag{36}$$

Corollary 3.18. *The expression* $\mathbf{h}_i(u) := \mathbf{a}_{mi}(u + \frac{i-1}{2})^{-1}\mathbf{a}_{mi}(u + \frac{i+1}{2})^{-1}\mathbf{a}_{m,i-1}(u + \frac{i-1}{2})\mathbf{a}_{m,i+1}(u + \frac{i+1}{2}) : \ M_{\underline{d}} \to M_{\underline{d}}[[u^{-1}]]$ *is independent of* $m < i$.

The proof follows from Corollary 2.14 by reduction to the stack $\mathfrak{Z}_N$, cf. [3, Section 3.11], and the proof of Corollary 2.14. Alternatively, it can be proved by a direct computation which is a rational version of the trigonometric computation in the proof of [23, Corollary 4.12]. □

Recall that q stands for the character of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^* : (\underline{t}, v, c) \mapsto v^2$. We define the line bundle $\mathsf{L}'_k := q^{\frac{1-k}{2}} \mathsf{L}_k$ on the correspondence $\mathsf{E}_{\underline{d},k}$, that is L'_k and L_k are isomorphic as line bundles but the equivariant structure of L'_k is obtained from the equivariant structure of L_k by the twist by the character $q^{\frac{1-k}{2}}$.

For $1 \le k \le n$ we consider the following generating series of operators on M:

$$\mathbf{h}_k(u) =: 1 + \sum_{r=0}^{\infty} \mathbf{h}_{k,r} \hbar^{-r} u^{-r-1} : \ M_{\underline{d}} \to M_{\underline{d}}; \tag{37}$$

$$\mathbf{x}_k^{\pm}(u) =: \sum_{r=0}^{\infty} \mathbf{x}_{k,r}^{\pm} \hbar^{-r} u^{-r-1} : \ M_{\underline{d}} \to M_{\underline{d} \mp k}[[u^{-1}]], \tag{38}$$

where

$$\mathbf{x}_{k,r}^{+} := \mathbf{p}_*(c_1(\mathsf{L}'_k)^r \cdot \mathbf{q}^*) : \ M_{\underline{d}} \to M_{\underline{d}-k}; \tag{39}$$

$$\mathbf{x}_{k,r}^{-} := -\mathbf{q}_*(c_1(\mathsf{L}'_k)^r \cdot \mathbf{p}^*) : \ M_{\underline{d}} \to M_{\underline{d}+k}. \tag{40}$$

Theorem 3.19. *For $n > 2$ the operators $\mathbf{h}_{k,r}, \mathbf{x}_{k,r}^{\pm}$ on M defined in (37,39,40) satisfy the relations in $\widehat{Y}$, i.e. they give rise to the action of $\widehat{Y}$ on M.*

Proof. For arbitrary $k \in \mathbb{Z}$ we define $\mathbf{x}_{k,r}^{\pm} : M_{\underline{d}} \to M_{\underline{d} \mp k}$ by the same formulas (39,40).

Let q' stand for the character of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^* : (\underline{t}, v, c) \mapsto c^2$. Then we have $\underline{\mathcal{W}}_{m-n,i-n} = q' \underline{\mathcal{W}}_{mi}$, that is $\underline{\mathcal{W}}_{m-n,i-n}$ and $\underline{\mathcal{W}}_{mi}$ are isomorphic as vector bundles but the equivariant structure of $\underline{\mathcal{W}}_{m-n,i-n}$ is obtained from the equivariant structure of $\underline{\mathcal{W}}_{mi}$ by the twist by the character q'. It follows that $\mathbf{a}_{m-n,i-n}(u) = \mathbf{a}_{mi}(u - \frac{\hbar'}{\hbar})$, and hence $\mathbf{h}_{k-n}(u) = \mathbf{h}_k(u - \frac{\hbar'}{\hbar} - \frac{n}{2})$.

Similarly, for the equivariant line bundles on the correspondence $\mathsf{E}_{\underline{d},k}$ we have $\mathsf{L}_{k-n} = q \mathsf{L}_k$, and hence $\mathbf{x}_{k-n}^{\pm}(u) = \mathbf{x}_k^{\pm}(u - \frac{\hbar'}{\hbar} - \frac{n}{2})$. In particular, $\mathbf{x}_0^{\pm}(u) = \mathbf{x}_n^{\pm}(u - \frac{\hbar'}{\hbar} - \frac{n}{2}) = {}'\mathbf{x}_n(u)$.

Now the relations (28,29,30) follow again from Theorem 2.12 by reduction to the stack $\mathfrak{Z}_N$, cf. section 3.11 of [3]. Alternatively, the desired relations can be proved by direct computations which are rational versions of the trigonometric computations in the proof of [23, Theorem 4.13]. □

Recall the weights $p_{ij} := -x_{j \pmod n} + d_{ij}\hbar + \lfloor \frac{-j}{n} \rfloor \hbar'$ introduced before Theorem 3.11.

Theorem 3.20. *The matrix coefficients of the operators $\mathbf{x}^{\pm}_{i,r}$ in the fixed point basis $\{[\widetilde{\underline{d}}]\}$ of M are as follows:*

$$\mathbf{x}^{-}_{i,r[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = (p_{ij} + \frac{1-i}{2}\hbar)^r \mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} =$$
$$-\hbar^{-1}(p_{ij} + \frac{1-i}{2}\hbar)^r \frac{p_{i-1,j} - p_{ij}}{p_{ii} - p_{ij}} \prod_{j\neq k\leq i-1} \frac{p_{i-1,k} - p_{ij}}{p_{ik} - p_{ij}}$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$ (note that almost all factors in this product are equal to 1 due to the condition (24); also, in case $i = j$ the factor $\frac{p_{i-1,j}-p_{ij}}{p_{ii}-p_{ij}}$ is set to be 1);

$$\mathbf{x}^{+}_{i,r[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = (p_{ij} - \frac{1+i}{2}\hbar)^r \mathfrak{f}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} =$$
$$\hbar^{-1}(p_{ij} - \frac{1+i}{2}\hbar)^r (p_{i+1,j} - p_{ij})(p_{i+1,i+1} - p_{ij}) \prod_{j\neq k\leq i} \frac{p_{i+1,k} - p_{ij}}{p_{ik} - p_{ij}}$$

if $d'_{i,j} = d_{i,j} - 1$ for certain $j \leq i$;

All the other matrix coefficients of $\mathbf{x}^{\pm}_{i,r}$ vanish.

The eigenvalue of $\mathbf{h}_i(u)$ on $[\widetilde{\underline{d}}]$ equals

$$\prod_{j\leq i}(u+\frac{i+1}{2}-p_{ij})^{-1}(u+\frac{i-1}{2}-p_{ij})^{-1}(u+\frac{i+1}{2}-p_{i+1,j+1})(u+\frac{i-1}{2}-p_{i-1,j-1})$$

Proof. Follows immediately from Theorem 3.11 and the definition of $\mathbf{x}^{\pm}_{i,r}$. The formula for $\mathbf{h}_i(u)$ follows from the fact that the eigenvalue of $\mathbf{a}_{mi}(u)$ on $[\widetilde{\underline{d}}]$ is $\prod_{j\leq i}(u - p_{ij}) \prod_{k\leq m}(u - p_{mk})^{-1}$. □

Theorem 3.21. *M is an irreducible $\widehat{Y}$-module.*

Proof. According to Theorem 3.20, the Gelfand-Tsetlin subalgebra of $\widehat{Y}$ generated by $\mathbf{h}_{i,r}$ acts diagonally in the basis $\{[\widetilde{\underline{d}}]\}$ with pairwise distinct joint eigenvalues. Therefore it suffices to check the following two things:

1. for each $[\widetilde{\underline{d}}]$ there is an index i such that $\mathbf{x}^{-}_{i,0}[\widetilde{\underline{d}}] \neq 0$;
2. for each $[\widetilde{\underline{d}}] \neq [\widetilde{\underline{0}}]$ there is an index i such that $\mathbf{x}^{+}_{i,0}[\widetilde{\underline{d}}] \neq 0$.

Both follow directly from Theorem 3.20. □

3.22. Specialization of Gelfand-Tsetlin base

We fix a positive integer K (a level). We consider an n-tuple $\mu = (\mu_{1-n}, \ldots, \mu_0) \in \mathbb{Z}^n$ such that $\mu_0 + K \geq \mu_{1-n} \geq \mu_{2-n} \geq \ldots \geq \mu_{-1} \geq \mu_0$. We view μ as a dominant (integrable) weight of $\widehat{\mathfrak{gl}}_n$ of level K. We extend μ to a nonincreasing sequence $\widetilde{\mu} = (\widetilde{\mu}_i)_{i\in\mathbb{Z}}$ setting $\widetilde{\mu}_i := \mu_{i \pmod n} + \lfloor \frac{-i}{n} \rfloor K$.

We define a subset $D(\mu)$ (*affine Gelfand-Tsetlin patterns*) of the set D of all collections $\widetilde{\underline{d}}$ satisfying the conditions (24) as follows:

$$\widetilde{\underline{d}} \in D(\mu) \text{ iff } d_{ij} - \widetilde{\mu}_j \leq d_{i+l,j+l} - \widetilde{\mu}_{j+l} \; \forall \; j \leq i, \; l \geq 0. \tag{41}$$

We specialize the values of $x_1, \dots, x_n, \hbar, \hbar'$ so that

$$\hbar = 1,\ \hbar' = -K - n,\ x_j = \widetilde{\mu}_j - j + 1. \tag{42}$$

We define the renormalized vectors

$$\langle \widetilde{\underline{d}} \rangle := C_{\widetilde{\underline{d}}}^{-1} [\widetilde{\underline{d}}] \tag{43}$$

where $C_{\widetilde{\underline{d}}}$ is the product $\prod_{w \in T_{\widetilde{\underline{d}}} \mathcal{P}_{\underline{d}}} w$ of the weights of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ in the tangent space to $\mathcal{P}_{\underline{d}}$ at the point $\widetilde{\underline{d}}$. The explicit formula for the multiset $\{w\}$ is given in Proposition 4.15 below. Namely, one has to plug $\widetilde{\underline{d}}' = \widetilde{\underline{d}}$ into the formula of Proposition 4.15, then expand it as a sum of Laurent monomials in $\underline{t}, q, q'$ with positive integer coefficients, and then replace each monomial $q^b (q')^c \prod_{i=1}^n t_i^{2a_i}$ with the corresponding weight $w = b\hbar + c\hbar' + \sum_{i=1}^n a_i x_i$ (with multiplicity given by the corresponding positive integer coefficient).

We define $V(\mu)$ as the $\mathbb{C}$-linear span of the vectors $\langle \widetilde{\underline{d}} \rangle$ for $\widetilde{\underline{d}} \in D(\mu)$.

Theorem 3.23. *The formulas of Theorem* 3.20 *give rise to the action of* $\widehat{Y}/(\hbar - 1, \hbar' + K + n)$ *in* $V(\mu)$.

Proof. It follows from (61) below that for the matrix coefficients in the renormalized basis $\{\langle \widetilde{\underline{d}} \rangle\}$ we have $\mathfrak{e}_{i \langle \widetilde{\underline{d}}, \widetilde{\underline{d}}' \rangle} = -\mathfrak{f}_{i [\widetilde{\underline{d}}', \widetilde{\underline{d}}]}$, $\mathfrak{f}_{i \langle \widetilde{\underline{d}}, \widetilde{\underline{d}}' \rangle} = -\mathfrak{e}_{i [\widetilde{\underline{d}}', \widetilde{\underline{d}}]}$. We have to check two things:

a) for $\widetilde{\underline{d}} \in D(\mu)$ the denominators of the matrix coefficients $\mathbf{x}^{\pm}_{i,r \langle \widetilde{\underline{d}}, \widetilde{\underline{d}}' \rangle}$ do not vanish;

b) for $\widetilde{\underline{d}} \in D(\mu)$, $\widetilde{\underline{d}}' \notin D(\mu)$ the numerators of the matrix coefficients $\mathbf{x}^{\pm}_{i,r \langle \widetilde{\underline{d}}, \widetilde{\underline{d}}' \rangle}$ do vanish. Both are straightforward. □

Remark 3.24. It seems likely that applying the Schur-Weyl functor of [12] to $V(\mu)$ and then going to the limit $n \to \infty$ one obtains the "tableaux representations" (see section 3 of [21]) of the trigonometric Cherednik algebra of type A.

Restricting $V(\mu)$ to $U(\hat{\mathfrak{sl}}_n) \subset \widehat{Y}$ we obtain the same named $\hat{\mathfrak{sl}}_n$-module. Recall the embedding $\hat{\mathfrak{sl}}_n \subset \hat{\mathfrak{gl}}_n$ of section 3.12.

Conjecture 3.25. *The* $\hat{\mathfrak{sl}}_n$*-module* $V(\mu)$ *extends to the irreducible integrable* $\hat{\mathfrak{gl}}_n$*-module with highest weight* μ.

Though we do not know how to define the action of Heisenberg algebra on $V(\mu)$, we can prove the following weak version of Conjecture 3.25.

Theorem 3.26. *The character of* $V(\mu)$ *equals the character of the irreducible integrable* $\hat{\mathfrak{gl}}_n$*-module with highest weight* μ.

Proof. We will use a combinatorial model for integrable $\widehat{\mathfrak{sl}}_n$-crystals introduced by P. Tingley in [22], namely, the cylindric plane partitions model of section 4 of *loc. cit.* We will denote by $\mathfrak{B}_\mu$ the set of cylindric plane partitions with boundary μ (*loc. cit.*, Definition 4.6). P. Tingley introduces a structure of $\widehat{\mathfrak{sl}}_n$-crystal on $\mathfrak{B}_\mu$ (section 4.2 of *loc. cit.*), and proves (Theorem 4.16 of *loc. cit.*) that this crystal is isomorphic to the $\widehat{\mathfrak{sl}}_n$-crystal of the irreducible integrable $\hat{\mathfrak{gl}}_n$-module with highest weight μ.

Thus it remains to construct a weight-preserving bijection between $D(\mu)$ and $\mathfrak{B}_\mu$ (adding a fourth combinatorial model to Tingley's list). Namely, given a cylindric plane partition $\pi \in \mathfrak{B}_\mu$ we view it in the three dimensional representation of Figure 13 of *loc. cit.* We shift the boxes in each row $x = x_0,\ z = z_0$ in y-direction so that the boundary (the infinitely high wall) of π becomes the plane $y = 0$. We will denote the resulting configuration of boxes by π_{red}. Note that it is not a plane partition anymore: the heights of piles are not necessarily nonincreasing in x-direction, though the heights of piles in y-direction are still nonincreasing. Still we can reconstruct π from π_{red} shifting back by $\widetilde{\mu}$ in y-direction.

We will denote by π^*_{red} the three dimensional picture of π_{red} reflected in the plane $z = y$. We denote the height of pile of π^*_{red} at (x, y) by $h(\pi)_{x,y}$. Here x and y are integers, and y is nonnegative. Finally, we set $d(\pi)_{ij} := h(\pi)_{-j,i-j}$. It is easy to see that the condition of π being a cylindric plane partition with boundary μ is equivalent to the condition $d(\pi) \in D(\mu)$. Thus we have constructed the desired bijection. This completes the proof of the theorem. □

4. Cohomology rings of affine Laumon spaces

4.1. Affine Gelfand-Tsetlin subalgebra

We extend the scalars in the $\mathbb{C}[\hbar, \hbar']$-algebra $\widehat{Y}$ to $\mathbb{C}(\hbar, \hbar')[x_1, \ldots, x_n]$, and denote the resulting algebra by $\widetilde{Y}$. We define the affine Gelfand-Tsetlin subalgebra $\mathfrak{A}_{\mathrm{aff}} \subset \widetilde{Y}$ as the subalgebra generated by all the elements $\mathbf{h}_{i,r}$. In this section we construct a surjective homomorphism from the affine Gelfand-Tsetlin subalgebra $\mathfrak{A}_{\mathrm{aff}} \subset \widetilde{Y}$ to the localized equivariant cohomology ring ${}^{\mathrm{loc}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}) := H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{\mathbb{C}[\hbar,\hbar']} \mathbb{C}(\hbar, \hbar')$. Namely, let $1_{\underline{d}} \in H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$ denote the unit element of the cohomology ring. For $a \in \mathfrak{A}_{\mathrm{aff}}$ we set $\psi(a) := a(1_{\underline{d}})$ (the action of Theorem 3.19).

Theorem 4.2. *The homomorphism* $\psi :\ \mathfrak{A}_{\mathrm{aff}} \to {}^{\mathrm{loc}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$ *is surjective.*

The proof occupies the rest of the section.

4.3. Modified generators

First we are going to express the coefficients of $\mathbf{a}_{0i}(u)$ as polynomials in $\mathbf{h}_{k,r}$. Note that $\mathbf{a}_{00}(u) = 1$ and

$$\mathbf{a}_{0,i+n}(u)\mathbf{a}_{0,n}(u)^{-1} = \mathbf{a}_{n,i+n}(u) = \mathbf{a}_{0i}(u + \frac{\hbar'}{\hbar}), \tag{44}$$

and hence it is sufficient to express $a_{0i}(u)$ for $i = 1, \ldots, n$. We have

$$\mathbf{a}_{0i}(u + \frac{i-1}{2})^{-1}\mathbf{a}_{0i}(u + \frac{i+1}{2})^{-1}\mathbf{a}_{0,i-1}(u + \frac{i-1}{2})\mathbf{a}_{0,i+1}(u + \frac{i+1}{2}) = \mathbf{h}_i(u). \tag{45}$$

By induction we obtain

$$\mathbf{a}_{0i}(u) = \prod_{j=0}^{i-1} \mathbf{a}_{01}(u - j) \prod_{j=1}^{i-1}\prod_{l=1}^{i-j} \mathbf{h}_j(u - l - \frac{j-1}{2}). \tag{46}$$

Thus it remains to express $\mathbf{a}_{01}(u)$. From (44) for $i = 1$ and (46) for $i = n,\ n+1$, we have

$$\mathbf{a}_{01}(u - n) \prod_{j=1}^{n} \mathbf{h}_j(u - n + \frac{j-1}{2}) = \mathbf{a}_{01}(u + \frac{\hbar'}{\hbar}). \tag{47}$$

For $\frac{\hbar'}{\hbar} \neq -n$ this equation uniquely determines $\mathbf{a}_{01}(u)$.

Remark 4.4. The "critical" value $\frac{\hbar'}{\hbar} = -n$ corresponds to the level 0 Verma module, and, on the other hand, to the specialization of the affine Yangian $\widehat{Y}$ to $\beta = -\frac{n\lambda}{4} + \frac{\lambda}{2}$. This corresponds to the degenerate trigonometric DAHA $\mathbf{H}_{0,c}$ by the Schur-Weyl duality (see [12]). There is an additional relation on $\mathbf{h}_i(u)$ for $\frac{\hbar'}{\hbar} = -n$, namely, $\prod_{j=1}^{n} \mathbf{h}_j(u - n + \frac{j-1}{2}) = 1$.

4.5. Künneth components of characteristic classes

Now we are going to express the Künneth components of characteristic classes of the tautological bundles $\underline{\mathcal{F}}_i$ on $\mathcal{P}_{\underline{d}} \times \mathbf{C} \times \mathbf{X}$ in terms of the coefficients of $\mathbf{a}_{0k}(u)$ (for all k). We start by noting that our definition (36) is equivalent to the following: under the Künneth decomposition, one writes $c_j(\underline{\mathcal{W}}_{mi}) =: c_j^{(j)}(\underline{\mathcal{W}}_{mi}) \otimes 1 + c_j^{(j-1)}(\underline{\mathcal{W}}_{mi}) \otimes \tau$ where $c_j^{(j)}(\underline{\mathcal{W}}_{mi}) \in H^{2j}_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$, and $c_j^{(j-1)}(\underline{\mathcal{W}}_{mi}) \in H^{2j-2}_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$. By equivariant localization on $\mathbf{C} \setminus \infty \cong \mathbf{A}^1$, it is not hard to prove:

$$\mathbf{a}_{mi}(u) = u^{i-m} + \sum_{r=1}^{\infty} (-\hbar)^{-r} \left(c_r^{(r)}(\underline{\mathcal{W}}_{mi}) - \hbar c_r^{(r-1)}(\underline{\mathcal{W}}_{mi}) \right) u^{i-m-r}. \tag{48}$$

Let τ and τ' stand for the first Chern class of the line bundle $\mathcal{O}(1)$ on $\mathbf{C}$ and $\mathbf{X}$, respectively. Then we have $c_j(\underline{\mathcal{F}}_i) =: c_j^{(j)}(\underline{\mathcal{F}}_i) \otimes 1 + c_j^{(j-1)}(\underline{\mathcal{F}}_i) \otimes \tau + c_j^{(j-1)'}(\underline{\mathcal{F}}_i) \otimes \tau' + c_j^{(j-2)}(\underline{\mathcal{F}}_i) \otimes \tau\tau'$ for the Künneth components $c_j^{(j)}(\underline{\mathcal{F}}_i), c_j^{(j-1)}(\underline{\mathcal{F}}_i), c_j^{(j-1)'}(\underline{\mathcal{F}}_i), c_j^{(j-2)}(\underline{\mathcal{F}}_i)$. In order to prove that all the Künneth components just defined are expressible in terms of the coefficients of all $\mathbf{a}_{0k}(u)$, it suffices to do this for just one tautological bundle $\underline{\mathcal{F}}_0$.

For an equivariant vector bundle $\mathcal{F}$ on $\mathcal{P}_{\underline{d}} \times \mathbf{C} \times \mathbf{X}$ let $X^{\widetilde{\underline{d}},0,0}(\mathcal{F})$ denote the multiset of characters of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ at the fixed point $(\widetilde{\underline{d}}, 0, 0) \in \mathcal{P}_{\underline{d}} \times \mathbf{C} \times \mathbf{X}$. Similarly, we define the multisets

$X^{\underline{\widetilde{d}},0,\infty}(\mathcal{F}), X^{\underline{\widetilde{d}},\infty,0}(\mathcal{F}), X^{\underline{\widetilde{d}},\infty,\infty}(\mathcal{F})$. Let $e_j^{\underline{\widetilde{d}},0,0}(\mathcal{F})$ stand for the sum of products of j distinct elements of the multiset $X^{\underline{\widetilde{d}},0,0}(\mathcal{F})$. Similarly, we define $e_j^{\underline{\widetilde{d}},0,\infty}(\mathcal{F}), e_j^{\underline{\widetilde{d}},\infty,0}(\mathcal{F}), e_j^{\underline{\widetilde{d}},\infty,\infty}(\mathcal{F}) \in \mathbb{C}[\hbar,\hbar',x_1,\ldots,x_n]$. Let $e_j^{0,0}(\mathcal{F})$ be the diagonal operator in the basis $\{[\underline{\widetilde{d}}]\}$ with eigenvalues $e_j^{\underline{\widetilde{d}},0,0}(\mathcal{F})$. Similarly, we define $e_j^{0,\infty}(\mathcal{F}), e_j^{\infty,0}(\mathcal{F}), e_j^{\infty,\infty}(\mathcal{F})$.

We have

$$\begin{aligned}
e_j^{0,0}(\mathcal{F}) &= c_j^{(j)}(\mathcal{F}) - \hbar c_j^{(j-1)}(\mathcal{F}) - \hbar' c_j^{(j-1)'}(\mathcal{F}) + \hbar\hbar' c_j^{(j-2)}(\mathcal{F})\\
e_j^{0,\infty}(\mathcal{F}) &= c_j^{(j)}(\mathcal{F}) - \hbar c_j^{(j-1)}(\mathcal{F}) + \hbar' c_j^{(j-1)'}(\mathcal{F}) - \hbar\hbar' c_j^{(j-2)}(\mathcal{F})\\
e_j^{\infty,0}(\mathcal{F}) &= c_j^{(j)}(\mathcal{F}) + \hbar c_j^{(j-1)}(\mathcal{F}) - \hbar' c_j^{(j-1)'}(\mathcal{F}) - \hbar\hbar' c_j^{(j-2)}(\mathcal{F})\\
e_j^{\infty,\infty}(\mathcal{F}) &= c_j^{(j)}(\mathcal{F}) + \hbar c_j^{(j-1)}(\mathcal{F}) + \hbar' c_j^{(j-1)'}(\mathcal{F}) + \hbar\hbar' c_j^{(j-2)}(\mathcal{F}) \qquad (49)
\end{aligned}$$

For an equivariant vector bundle $\mathcal{G}$ on $\mathcal{P}_{\underline{d}} \times \mathbf{C}$ we spare the reader the bulk of the similar self-explaining notation but just note that we have

$$e_j^0(\mathcal{G}) = c_j^{(j)}(\mathcal{G}) - \hbar c_j^{(j-1)}(\mathcal{G}), \quad e_j^\infty(\mathcal{G}) = c_j^{(j)}(\mathcal{G}) + \hbar c_j^{(j-1)}(\mathcal{G}) \qquad (50)$$

whence

$$c_j^{(j)}(\mathcal{G}) = \frac{1}{2}(e_j^\infty(\mathcal{G}) + e_j^0(\mathcal{G})), \quad c_j^{(j-1)}(\mathcal{G}) = \frac{1}{2\hbar}(e_j^\infty(\mathcal{G}) - e_j^0(\mathcal{G})) \qquad (51)$$

If we have $\mathcal{G} = \mathcal{F}|_{\mathcal{P}_{\underline{d}}\times\mathbf{C}\times 0}$ then $e_j^{0,0}(\mathcal{F}) = e_j^0(\mathcal{G})$, that is

$$c_j^{(j)}(\mathcal{F}) - \hbar c_j^{(j-1)}(\mathcal{F}) - \hbar' c_j^{(j-1)'}(\mathcal{F}) + \hbar\hbar' c_j^{(j-2)}(\mathcal{F}) = c_j^{(j)}(\mathcal{G}) - \hbar c_j^{(j-1)}(\mathcal{G}). \qquad (52)$$

The equality (52) makes sense in the equivariant K-groups of $\mathcal{P}_{\underline{d}} \times \mathbf{C} \times \mathbf{X}$ and $\mathcal{P}_{\underline{d}} \times \mathbf{C}$, so it holds true for any coherent sheaf $\mathcal{F}$ on $\mathcal{P}_{\underline{d}} \times \mathbf{C} \times \mathbf{X}$ and its (left derived) restriction $\mathcal{G}$ to $\mathcal{P}_{\underline{d}} \times \mathbf{C} \times 0$. Let us take $\mathcal{F} = \underline{\mathcal{F}}_0$, so that $\mathcal{G} = \underline{\mathcal{W}}_{-n,0}$, and then (48) shows that the RHS of (52) is just a coefficient of the series $\mathbf{a}_{-n,0}(u)$. Note that $e_j^{0,\infty}(\underline{\mathcal{F}}_n), e_j^{\infty,0}(\underline{\mathcal{F}}_n)$, and $e_j^{\infty,\infty}(\underline{\mathcal{F}}_0)$ are all equal to the j-th elementary symmetric function of the variables $x_1,\ldots,x_n$. Now it is easy to see that the system (49) can be solved over the ring $\mathbb{C}(\hbar,\hbar')[x_1,\ldots,x_n]$ to express the Künneth components $c_j^{(j)}(\underline{\mathcal{F}}_0), c_j^{(j-1)}(\underline{\mathcal{F}}_0), c_j^{(j-1)'}(\underline{\mathcal{F}}_0), c_j^{(j-2)}(\underline{\mathcal{F}}_0)$ in terms of the coefficients of the series $\mathbf{a}_{-n,0}(u)$ (or equivalently, in terms of the coefficients of the series $\mathbf{a}_{0n}(u)$).

4.6. Okounkov's vector bundle E

Now it remains to prove that ${}^{\mathrm{loc}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$ is generated by the Künneth components of the Chern classes of the tautological bundles $\underline{\mathcal{F}}_i$. By Theorem 2.1 in [6], it is enough to show that the cohomology class of the diagonal $\Delta \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$ is of the form

$$[\Delta] = \sum_i \alpha_i \boxtimes \beta_i, \qquad (53)$$

where $\alpha_i, \beta_i \in {}^{\mathrm{loc}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$ are generated by the Künneth components of the Chern classes of the tautological bundles. The following sections are concerned with proving this fact: we will compute the class $[\Delta]$ and show that it is of the form (53).

Recall the setup of subsection 3.4, where we realized a parabolic sheaf $\mathcal{F}_\bullet$ as a single, $G = \mathbb{Z}/n\mathbb{Z}-$invariant sheaf $\tilde{\mathcal{F}}$. Following [4], consider the sheaf E on $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$, whose fiber above $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ is

$$E|_{(\mathcal{F}_\bullet,\mathcal{F}'_\bullet)} = \mathrm{Ext}^1_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty))$$

In the above, Ext^1_G denotes the $G-$invariant part of the vector space in question. By the Kodaira-Spencer theorem, the restriction of E to the diagonal $\Delta \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$ coincides with the tangent bundle of $\Delta \cong \mathcal{P}_{\underline{d}}$.

Proposition 4.7. *The sheaf E is a vector bundle of rank $2(d_1 + \cdots + d_n)$ on $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$. Note that the rank of E equals half of the dimension of the base space.*

Proof. Since our sheaves are on the surface $\mathbf{S}$, all Ext^i groups vanish for $i > 2$. However, in our particular case we can say more. Because $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ have the same trivialization at $\mathbf{D}_\infty$, they have the same first Chern class and thus $\mathrm{Hom}(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty)) = 0$. Moreover, Serre duality implies that $\mathrm{Ext}^2(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty)) \cong \mathrm{Hom}(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}'(-\mathbf{D}_\infty))$, which vanishes for the same reason. This suggests that instead of Ext^1 one should consider the following functor of sheaves on $\mathbf{S}$:

$$\chi(\mathcal{S}', \mathcal{S}) := \mathrm{Hom}(\mathcal{S}', \mathcal{S}) - \mathrm{Ext}^1(\mathcal{S}', \mathcal{S}) + \mathrm{Ext}^2(\mathcal{S}', \mathcal{S}).$$

Then the above discussion implies that

$$E|_{(\mathcal{F}_\bullet,\mathcal{F}'_\bullet)} = \mathrm{Ext}^1_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty)) = -\chi_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty)),$$

where χ_G denotes the $G-$invariant part of the vector space in question. Therefore, Corollary 7.9.9 of [11] implies that the dimension of the fibers of E is constant, and therefore E is a vector bundle. The fact that its rank equals $2(d_1 + \cdots + d_n)$ follows from Remark 4.16. □

4.8. A section of E

Now we will construct a regular section of E that vanishes on the diagonal Δ. The following construction was proposed by A. Kuznetsov. For any $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$, consider the short exact sequence

$$0 \to \tilde{\mathcal{F}}(-\mathbf{D}_\infty) \to \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}|_{\mathbf{D}_\infty} \to 0$$

Apply the functor $\mathrm{Hom}_G(\tilde{\mathcal{F}}', \cdot)$ to the above short exact sequence, and we will obtain the following long exact sequence

$$\cdots \to \mathrm{Hom}_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}) \to \mathrm{Hom}_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}|_{\mathbf{D}_\infty}) \to \mathrm{Ext}^1_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty)) \to \cdots \quad (54)$$

Since our parabolic sheaves are framed at $\mathbf{D}_\infty$, then there is a fixed isomorphism $\tilde{\mathcal{F}}|_{\mathbf{D}_\infty} \cong \tilde{\mathcal{F}}'|_{\mathbf{D}_\infty}$. Therefore, the morphism which restricts $\tilde{\mathcal{F}}'$ to $\tilde{\mathcal{F}}'|_{\mathbf{D}_\infty} \cong \tilde{\mathcal{F}}|_{\mathbf{D}_\infty}$ is an element of the middle Hom space in the above exact sequence. Push this element forward to $\mathrm{Ext}^1_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}(-\mathbf{D}_\infty))$, and by definition this will be the value of our section $s(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$.

Proposition 4.9. *The zero locus of s is precisely the diagonal $\Delta \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$.*

Proof. The fact that $s(\mathcal{F}_\bullet, \mathcal{F}_\bullet) = 0$ easily follows from the long exact sequence (54). Indeed, in this case the element of the middle Hom space which we push forward to obtain $s(\mathcal{F}_\bullet, \mathcal{F}_\bullet)$ simply comes from the identity element in the left Hom space. Since the composition of two successive maps in a long exact sequence is zero, this implies that $s(\mathcal{F}_\bullet, \mathcal{F}_\bullet) = 0$.

Conversely, suppose $s(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) = 0$. By the exactness of (54), it follows that the natural restriction morphism in the middle Hom space comes from the left Hom space. This implies that there exists a morphism $\phi \in \mathrm{Hom}_G(\tilde{\mathcal{F}}', \tilde{\mathcal{F}})$ which restricts to the identity on $\mathbf{D}_\infty$. Since the sheaves $\tilde{\mathcal{F}}'$ and $\tilde{\mathcal{F}}$ have the same framing at $\mathbf{D}_\infty$, such a ϕ can exist only if $\tilde{\mathcal{F}}' \subset \tilde{\mathcal{F}} \Leftrightarrow \mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ (in which case ϕ can only be the natural inclusion). Since $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ have the same first and second Chern classes, this implies that $\mathcal{F}'_\bullet = \mathcal{F}_\bullet$. □

The zero-locus $[\Delta]$ is irreducible (since it is isomorphic to $\mathcal{P}_{\underline{d}}$), and by Proposition 4.7 its codimension in $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$ precisely equals the rank of the bundle E. Therefore we have

$$e(E) = k \cdot [\Delta], \tag{55}$$

where $k \in \mathbb{N}$ is the order of vanishing of the section s at its zero-locus.

Remark 4.10. We conjecture that the section s is transversal, and therefore $k = 1$. However, we will not need this technical result.

4.11. The class of E

In this section, we will compute the class of E in the $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant K-theory of $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$. In the following, we will abuse notation and denote by $\chi(\mathcal{F}'_{k-1-n}, \mathcal{F}_{k-n}(-\mathbf{D}_\infty))$ and $\chi(\mathcal{F}'_{k-n}, \mathcal{F}_{k-n}(-\mathbf{D}_\infty))$ the sheaves over $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$ whose fibers above $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ are the χ spaces in question. By the argument of Proposition 4.7, these sheaves are vector bundles. Recall the character q' of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$.

Proposition 4.12. *We have the following equality of K-theory classes:*

$$[E] = \frac{\sum_{k=1}^{n} [\chi(\mathcal{F}'_{k-n}, \mathcal{F}_{k-n}(-\mathbf{D}_\infty))] - \sum_{k=1}^{n} [\chi(\mathcal{F}'_{k-1-n}, \mathcal{F}_{k-n}(-\mathbf{D}_\infty))]}{q' - 1}. \tag{56}$$

Remark 4.13. The equivariant K-theory of $\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}$ is an algebra over the ring of scalars $K_0 := K_{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)$. Relation (56) only makes sense in the localized equivariant K-theory algebra, i.e. localized with respect to $q - 1 \in K_0$.

Proof. By the Thomason localization theorem, restriction to the fixed point locus produces the following isomorphism of vector spaces

$$K_{\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}}) \bigotimes_{K_0} \mathrm{Frac}(K_0) \xrightarrow{\sim}$$

$$K_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}\times\mathcal{P}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*})\bigotimes_{K_0}\mathrm{Frac}(K_0).$$

Therefore, to prove the equality (56) it is enough to show that equality holds when restricting to every torus fixed point of $\mathcal{P}_{\underline{d}}\times\mathcal{P}_{\underline{d}}$. But the fixed points are isolated, and thus at each fixed point will we just have to check an isomorphism of $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*-$representations. Two representations of a torus are isomorphic if and only if their characters are equal. Thus, we only need to compute the characters of $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$ in the fixed fibers of the left hand side and right hand side of (56), and show that they are equal.

Recall from sections 3.2 and 3.4 that a fixed point $(\mathcal{F}_\bullet,\mathcal{F}'_\bullet)\in\mathcal{P}_{\underline{d}}\times\mathcal{P}_{\underline{d}}$ is given by two collections of partitions $(\boldsymbol{\lambda},\boldsymbol{\lambda}')$, or alternatively by two vectors of positive integers $(\widetilde{\underline{d}},\widetilde{\underline{d}}')$. The relation between $\boldsymbol{\lambda}$ and $\widetilde{\underline{d}}$ is given by relations (22) and (23): if $1\le k,l\le n$, then

$$\lambda_s^{kl}=d_{ns-n\lfloor\frac{k-l}{n}\rfloor+k,l}$$

The computation of the characters we need is based on the following lemma:

Lemma 4.14. *Let* $\mathbf{D}_0=\mathbf{C}\times 0_{\mathbf{X}}$ *and* $\mathbf{D}_1=0_{\mathbf{C}}\times\mathbf{X}$. *Let us consider two partitions* λ,λ' *and the associated rank 1 torsion free sheaves* $J_\lambda,J_{\lambda'}$ *on* $\mathbf{S}$. *Then the torus character of* $\mathbb{C}^*\times\mathbb{C}^*$ *in* $\chi(J_{\lambda'},J_\lambda(-\mathbf{D}_\infty+\alpha\mathbf{D}_0+\beta\mathbf{D}_1))$ *equals*

$$\mathrm{char}_{\alpha\beta}(\lambda',\lambda):=-\sum_{i=0}^{\infty}\sum_{i'=0}^{\infty}q^{\beta+1}\frac{(q^{\lambda'_{i'}}-1)(q^{-\lambda_i}-1)}{q-1}\cdot q'^{\alpha+i'-i}(q'-1)+$$

$$+\sum_{i=0}^{\infty}q^{\beta+1}\frac{q^{-\lambda_i}-1}{q-1}\cdot q'^{\alpha-i}-\sum_{i'=0}^{\infty}q^{\beta+1}\frac{q^{\lambda'_{i'}}-1}{q-1}\cdot q'^{\alpha+i'+1}+qq'\frac{q^\beta-1}{q-1}\cdot\frac{q'^\alpha-1}{q'-1}$$

Proof. Let us begin by noting that the character of $\mathbb{C}^*\times\mathbb{C}^*$ in $\chi(\mathcal{O}_{\mathbf{S}},\mathcal{O}_{\mathbf{S}}(-\mathbf{D}_\infty+a\mathbf{D}_0+b\mathbf{D}_1))$ equals

$$\mathrm{char}_{a,b}:=qq'\cdot\frac{q^b-1}{q-1}\cdot\frac{q'^a-1}{q'-1}\tag{57}$$

Recall that $J_\lambda=y^0z^{\lambda_0}+...+y^tz^{\lambda_t}$, where t is large enough such that $\lambda_t=0$. Then a resolution of J_λ is naturally given by

$$0\to\bigoplus_{i=0}^{t-1}\mathcal{O}_{\mathbf{S}}(-(i+1)\mathbf{D}_0-\lambda_i\mathbf{D}_1)\to\bigoplus_{i=0}^{t}\mathcal{O}_{\mathbf{S}}(-i\mathbf{D}_0-\lambda_i\mathbf{D}_1)\to J_\lambda\to 0$$

A similar resolution holds for $J_{\lambda'}$. Since both the functor χ and the character of a representation are additive in exact sequences, we have

$$\mathrm{char}_{\alpha,\beta}(\lambda',\lambda)=\sum_{i=0}^{t}\sum_{i'=0}^{t'}\mathrm{char}_{i'-i+\alpha,\lambda'_{i'}-\lambda_i+\beta}-\sum_{i=0}^{t}\sum_{i'=0}^{t'-1}\mathrm{char}_{i'-i+1+\alpha,\lambda'_{i'}-\lambda_i+\beta}-$$

$$-\sum_{i=0}^{t-1}\sum_{i'=0}^{t'}\mathrm{char}_{i'-i-1+\alpha,\lambda'_{i'}-\lambda_i+\beta}+\sum_{i=0}^{t-1}\sum_{i'=0}^{t'-1}\mathrm{char}_{i'-i+\alpha,\lambda'_{i'}-\lambda_i+\beta}$$

The desired formula for $\mathrm{char}_{\alpha\beta}(\lambda',\lambda)$ in the statement of the lemma is obtained by plugging (57) into the above expression, and performing some predictable computational manipulations. □

Recall that our fixed parabolic sheaves $\tilde{\mathcal{F}}'$ and $\tilde{\mathcal{F}}$ have the form (21). Thus one can compute the torus character in $\chi(\tilde{\mathcal{F}}',\tilde{\mathcal{F}}(-\mathbf{D}_\infty))$ by applying the above Lemma. Then, to obtain the torus character in the $G-$invariant part of this representation, we must keep only the terms of the form ${q'}^{nx}\cdot\ldots$, and replace them by ${q'}^{x}\cdot\ldots$. In this way, we obtain the following result. Let t_i stand for the following character of $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$: $(t_1,\ldots,t_i,\ldots,t_n,v,c)\mapsto t_i$.

Proposition 4.15. *The character of* $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$ *in* $E|_{(\mathcal{F}_\bullet,\mathcal{F}'_\bullet)} = -\chi_G(\tilde{\mathcal{F}}',\tilde{\mathcal{F}}(-\mathbf{D}_\infty))$ *equals*

$$\sum_{k=1}^{n}\sum_{l\le k}^{l'\le k-1}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{(q^{d'_{(k-1)l'}}-1)(q^{-d_{kl}}-1)}{q-1}\cdot {q'}^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}+$$

$$+\sum_{k=1}^{n}\sum_{l'\le k-1}\frac{t_k^2}{t_{l'}^2}\cdot q\frac{q^{d'_{(k-1)l'}}-1}{q-1}\cdot {q'}^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{-k}{n}\rfloor}-$$

$$-\sum_{k=1}^{n}\sum_{l\le k}^{l'\le k}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{(q^{d'_{kl'}}-1)(q^{-d_{kl}}-1)}{q-1}\cdot {q'}^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}-$$

$$-\sum_{k=1}^{n}\sum_{l\le k}\frac{t_l^2}{t_k^2}\cdot q\frac{q^{-d_{kl}}-1}{q-1}\cdot {q'}^{\lfloor\frac{-k}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}$$

where $t_l := t_{l \bmod n}$.

Remark 4.16. By letting $q,q',t_l\to 1$ in the above expression, one obtains that the dimension of the fiber of E equals $\sum_{i\ge j} d_{ij}+\sum_{i\ge j} d'_{ij} = 2(d_1+\cdots+d_n) = \dim(\mathcal{P}_{\underline{d}})$.

Remark 4.17. As was noted above, the restriction of E to the diagonal is isomorphic to the tangent bundle of $\mathcal{P}_{\underline{d}}$. Therefore, the torus character in a fixed tangent space to $\mathcal{P}_{\underline{d}}$ is obtained by setting $\widetilde{\underline{d}} = \widetilde{\underline{d}'}$ in the above expression.

In a similar way, one computes the character in the right hand side of (56).

Proposition 4.18. *The character of* $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$ *in* $\chi(\mathcal{F}'_{k-1-n},\mathcal{F}_{k-n}(-\mathbf{D}_\infty))$ *equals*

$$-\sum_{l\le k}^{l'\le k-1}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{(q^{d'_{(k-1)l'}}-1)(q^{-d_{kl}}-1)}{q-1}\cdot {q'}^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}(q'-1)+$$

$$+\sum_{l'=1}^{n}\sum_{l\le k}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{q^{-d_{kl}}-1}{q-1}\cdot {q'}^{\lfloor\frac{l'-k}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}-$$

$$-\sum_{l=1}^{n}\sum_{l'\le k-1}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{q^{d'_{(k-1)l'}}-1}{q-1}\cdot q'^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{l-k-1}{n}\rfloor+1}$$

Similarly, the character of $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$ *in* $\chi(\mathcal{F}'_{k-n},\mathcal{F}_{k-n}(-\mathbf{D}_\infty))$ *equals*

$$-\sum_{l\le k}^{l'\le k}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{(q^{d'_{kl'}}-1)(q^{-d_{kl}}-1)}{q-1}\cdot q'^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}(q'-1)+$$

$$+\sum_{l'=1}^{n}\sum_{l\le k}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{q^{-d_{kl}}-1}{q-1}\cdot q'^{\lfloor\frac{l'-k-1}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}-$$

$$-\sum_{l=1}^{n}\sum_{l'\le k}\frac{t_l^2}{t_{l'}^2}\cdot q\frac{q^{d'_{kl'}}-1}{q-1}\cdot q'^{\lfloor\frac{-l'}{n}\rfloor-\lfloor\frac{l-k-1}{n}\rfloor+1}$$

The above two propositions allow us to compare the left and right hand sides of (56) at the level of characters in the fixed fibers, and note that they are equal. As was mentioned at the beginning of the proof, this is enough to ensure that equality (56) holds in equivariant K-theory. □

4.19. Relation to Künneth components

The vector bundles $\chi(\mathcal{F}'_k,\mathcal{F}_k(-\mathbf{D}_\infty))$, $\chi(\mathcal{F}'_{k-1},\mathcal{F}_k(-\mathbf{D}_\infty))$ can be linked to the Künneth components of the Chern classes of the universal sheaves $\underline{\mathcal{F}}_k$ through a universal procedure that we illustrate below. Let $\pi:\mathcal{P}_{\underline{d}}\times\mathbf{S}\to\mathcal{P}_{\underline{d}}$ denote the standard projection. The Beilinson spectral sequence (Theorem 3.1.3 of [20]) for the universal sheaf $\underline{\mathcal{F}}_k$ has E_1 term given by

$$E_1^{pq}=R^q\pi_*\underline{\mathcal{F}}_k(\dots)\boxtimes\Omega_{\mathbf{S}}^{-p}(\dots),$$

where $\dots$ denote twists by line bundles on $\mathbf{S}$, which do not depend on the integer q. This spectral sequence converges, and the E_∞ term has $E_\infty^{pq}=0$ for $p+q\neq 0$, while $E_\infty^{p,-p}$ gives a filtration of $\underline{\mathcal{F}}_k$ itself. As a general fact about spectral sequences, the K-theory class of the alternating sum $(-1)^{p+q}E_r^{pq}$ is the same for all r. Equating these K-theory classes for $r=1$ and $r=\infty$ gives us the following identity in the K-theory ring of $\underline{\mathcal{F}}_k\times\mathbf{S}$:

$$[\underline{\mathcal{F}}_k]=\sum_p(-1)^p\left(\sum_q(-1)^q[R^q\pi_*\underline{\mathcal{F}}_k(\dots)]\right)\boxtimes[\Omega_{\mathbf{S}}^{-p}(\dots)].\qquad(58)$$

Consider the following complex of sheaves $\mathcal{G}$ on $\mathcal{P}_{\underline{d}}\times\mathcal{P}_{\underline{d}}\times\mathbf{S}$:

$$\mathcal{G}:=R\mathcal{H}om_{\mathbf{S}}(\underline{\mathcal{F}}'_k,\underline{\mathcal{F}}_k(-\mathbf{D}_\infty))\cong\underline{\mathcal{F}}_k(-\mathbf{D}_\infty)\overset{L}{\otimes}_{\mathcal{O}_{\mathbf{S}}}\underline{\mathcal{F}}'^{\vee}_k,$$

where $\underline{\mathcal{F}}'^{\vee}_k$ stands for the dual sheaf $R\mathcal{H}om(\underline{\mathcal{F}}'_k,\mathcal{O})$. Let $\tilde{\pi}:\mathcal{P}_{\underline{d}}\times\mathcal{P}_{\underline{d}}\times\mathbf{S}\to\mathcal{P}_{\underline{d}}\times\mathcal{P}_{\underline{d}}$ denote the standard projection. Since the functor χ is the derived functor of Hom, we have the obvious identity in K-theory:

$$[\chi(\mathcal{F}'_k,\mathcal{F}_k(-\mathbf{D}_\infty))]=\sum_q(-1)^qR^q\tilde{\pi}_*[\mathcal{G}].\qquad(59)$$

However, the class of $\mathcal{G}$ is clearly given by

$$[\mathcal{G}] = [\underline{\mathcal{F}}_k(-\mathbf{D}_\infty)] \otimes [\underline{\mathcal{F}}_k^\vee],$$

where we suppress the notation for the obvious pull-back morphisms. We can use (58) to evaluate the above:

$$[\mathcal{G}] = \sum_{p_1}\sum_{p_2}\left(\sum_q (-1)^q [R^q\pi_*\underline{\mathcal{F}}_k(\dots)]\right) \boxtimes \left(\sum_q (-1)^q [R^q\pi_*\underline{\mathcal{F}}_k(\dots)^\vee]\right) \boxtimes$$

$$\left([\Omega_{\mathbf{S}}^{-p_1}(\dots)] \otimes [\Omega_{\mathbf{S}}^{-p_2}(\dots)^\vee]\right)$$

Plugging this into (59) gives us

$$[\chi(\mathcal{F}'_k, \mathcal{F}_k(-\mathbf{D}_\infty))] = \sum_{p_1}\sum_{p_2}\left(\sum_q (-1)^q [R^q\pi_*\underline{\mathcal{F}}_k(\dots)]\right) \boxtimes \left(\sum_q (-1)^q [R^q\pi_*\underline{\mathcal{F}}_k(\dots)^\vee]\right) \cdot Z, \quad (60)$$

where $Z = \chi(\Omega_{\mathbf{S}}^{-p_2}(\dots), \Omega_{\mathbf{S}}^{-p_1}(\dots))$ is a constant in the K-theory algebra. By the Grothendieck-Riemann-Roch theorem applied to the projection map $\pi : \mathcal{P}_{\underline{d}} \times \mathbf{S} \to \mathcal{P}_{\underline{d}}$, we obtain

$$\mathrm{ch}\left(\sum_q (-1)^q R^q\pi_*\underline{\mathcal{F}}_k(\dots)\right) = \pi_*\mathrm{ch}(\underline{\mathcal{F}}_k \cdot \dots),$$

where $\dots$ again denotes a class on $\mathbf{S}$. Each graded piece of the right hand side is precisely a certain polynomial in the Künneth components of the tautological bundle $\underline{\mathcal{F}}_k$. Thus, applying the Chern character to (60), we see that the Chern classes of the bundle $\chi(\mathcal{F}'_k, \mathcal{F}_k(-\mathbf{D}_\infty))$ are exterior products of polynomials in the Künneth components of the tautological bundle $\underline{\mathcal{F}}_k$. This is exactly the form we want for the expression in the right hand side of (53). By a similar argument, the Chern classes of $\chi(\mathcal{F}'_{k-1}, \mathcal{F}_k(-\mathbf{D}_\infty))$ will also be exterior products of polynomials in the Künneth components of the Chern classes of the tautological bundles.

Now let us look at relation (56). Since $q' = e^{\hbar'}$, this relation allows us to inductively express the Chern classes of E in terms of the Chern classes of the bundles $\chi(\mathcal{F}'_k, \mathcal{F}_k(-\mathbf{D}_\infty)), \chi(\mathcal{F}'_{k-1}, \mathcal{F}_k(-\mathbf{D}_\infty))$. More precisely, the Chern classes of E will be polynomials in the Chern classes of the χ bundles (divided by polynomials in the constant $\hbar'$, but this is allowed since we tensor all our cohomology rings by $\mathbb{C}(\hbar, \hbar')$). Therefore the Chern classes of E all have the form in the right hand side of (53). By (55), $[\Delta]$ will also have that form. This completes the proof of Theorem 4.2.

4.20. Torus eigenvalues in the tangent spaces

In this section, we will use torus character computations as in section 4.11 in order to prove Theorem 3.11. In other words, we want to compute the matrix coefficients of $\mathfrak{e}_i, \mathfrak{f}_i$ in the basis $[\widetilde{\underline{d}}]$. By equivariant localization, this comes down to computing the torus character in the tangent spaces to $\mathsf{E}_{\underline{d},i}$ at the torus fixed points. If we let $d'_j = d_j + \delta^i_{j \bmod n}$, recall that the correspondence $\mathsf{E}_{\underline{d},i} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}'}$ consists of flags $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ such that $\mathcal{F}'_j = \mathcal{F}_j$ for $i \neq j \bmod n$, while $\mathcal{F}'_j \subset \mathcal{F}_j$ for $i = j \bmod n$.

Let $\eta : \mathbf{S} \to \mathbf{S}$ be given by $\eta(x,y) = (x, y^{n+1})$. Let $H = \mathbb{Z}/(n+1)\mathbb{Z}$, which acts on $\mathbf{S} = \mathbf{C} \times \mathbf{X}$ by multiplying $\mathbf{X}$ with the roots of unity of order $n+1$. As in section 3.4, to any point $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in \mathsf{E}_{\underline{d},i}$ we can associate the single $H-$invariant sheaf

$$\mathcal{H} := \eta^*\mathcal{F}_{1-n} + \eta^*\mathcal{F}_{2-n}(-\mathbf{D}_0) + \cdots +$$

$$\eta^*\mathcal{F}'_{i-n}(-(i-1)\mathbf{D}_0) + \eta^*\mathcal{F}_{i-n}(-i\mathbf{D}_0) + \cdots + \eta^*\mathcal{F}_0(-n\mathbf{D}_0).$$

This gives a realization of $\mathsf{E}_{\underline{d},i}$ as the moduli space of $H-$invariant sheaves on $\mathbf{S}$ satisfying a certain framing at $\mathbf{D}_\infty$ and certain numerical conditions. Moreover, the tangent space to this moduli space at the point $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ is given by:

$$T_{(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)}\mathsf{E}_{\underline{d},i} = \mathrm{Ext}^1_H(\mathcal{H}, \mathcal{H}(-\mathbf{D}_\infty)).$$

Just like in the proof of Proposition 4.7, the corresponding Hom and Ext^2 spaces vanish, and therefore we have

$$T_{(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)}\mathsf{E}_{\underline{d},i} = -\chi_H(\mathcal{H}, \mathcal{H}(-\mathbf{D}_\infty)).$$

We want now to compute the character of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$ in this tangent space, when $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in \mathsf{E}_{\underline{d},i}$ is a torus fixed point. Recall that $\mathcal{F}_\bullet, \mathcal{F}'_\bullet$ are given by collections of indices $\widetilde{\underline{d}}, \widetilde{\underline{d}'}$, and that their components break up into direct sums as in (20). Then we can use Lemma 4.14 to compute the torus character in $\chi(\mathcal{H}, \mathcal{H}(-\mathbf{D}_\infty))$. To obtain the $H-$invariant part of this, we keep only those terms of the form $q'^{(n+1)x} \cdot \ldots$ and replace them with $q'^{x} \cdot \ldots$. Carrying this computation through, we obtain:

Proposition 4.21. *The torus character in the tangent space to $\mathsf{E}_{\underline{d},i}$ above the torus fixed point given by indices $\widetilde{\underline{d}}, \widetilde{\underline{d}'}$ equals*

$$\sum_{k=1}^{n} \sum_{l\leq k}^{l'\leq k-1} \frac{t_l^2}{t_{l'}^2} \cdot q \frac{(q^{d_{(k-1)l'}} - 1)(q^{-d_{kl}} - 1)}{q-1} \cdot q'^{\lfloor \frac{-l'}{n} \rfloor - \lfloor \frac{-l}{n} \rfloor} +$$

$$+ \sum_{k=1}^{n} \sum_{l'\leq k-1} \frac{t_k^2}{t_{l'}^2} \cdot q \frac{q^{d_{(k-1)l'}} - 1}{q-1} \cdot q'^{\lfloor \frac{-l'}{n} \rfloor - \lfloor \frac{-k}{n} \rfloor} -$$

$$- \sum_{k=1}^{n} \sum_{l\leq k}^{l'\leq k} \frac{t_l^2}{t_{l'}^2} \cdot q \frac{(q^{d_{kl'}} - 1)(q^{-d_{kl}} - 1)}{q-1} \cdot q'^{\lfloor \frac{-l'}{n} \rfloor - \lfloor \frac{-l}{n} \rfloor} -$$

$$-\sum_{k=1}^{n}\sum_{l\le k}\frac{t_l^2}{t_k^2}\cdot q\frac{q^{-d_{kl}}-1}{q-1}\cdot q'^{\lfloor\frac{-k}{n}\rfloor-\lfloor\frac{-l}{n}\rfloor}+$$

$$+q-q^{-d_{ij}+d_{(i-1)j}}+\frac{t_j^2}{t_i^2}\cdot q^{-d_{ij}+d_{ii}}\cdot q'^{\lfloor\frac{-i}{n}\rfloor-\lfloor\frac{-j}{n}\rfloor}+$$

$$+\sum_{j\neq k\le i-1}\frac{t_j^2}{t_k^2}\cdot q^{-d_{ij}}\cdot\left(q^{d_{ik}}-q^{d_{(i-1)k}}\right)\cdot q'^{\lfloor\frac{-k}{n}\rfloor-\lfloor\frac{-j}{n}\rfloor}$$

if $d'_{ij}=d_{ij}+1$ *for certain* $j\le i$.

4.22. Proof of Theorem 3.11

Proposition 4.21 enables us to prove Theorem 3.11. Indeed, by Theorem 3.7 in [19], we have

$$\mathfrak{e}_i[\widetilde{\underline{d}}]=-\mathbf{q}_*\mathbf{p}^*[\widetilde{\underline{d}}]=-\sum_{\widetilde{\underline{d}}'}[\widetilde{\underline{d}}']\cdot\frac{\prod_{w\in T_{\widetilde{\underline{d}}}\mathcal{P}_{\underline{d}}}w}{\prod_{w\in T_{(\widetilde{\underline{d}},\widetilde{\underline{d}}')}\mathsf{E}_{\underline{d},i}}w},\tag{61}$$

where the notation $\prod_{w\in T_{\dots}}$ denotes the product of the torus weights in the respective tangent spaces. The sum above runs over all $\widetilde{\underline{d}}'$ such that $d'_{ij}=d_{ij}+1$ for a certain j. The coefficient $\mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}$ is precisely the ratio of products of weights in the above expression. To compute this ratio, note that Remark 4.17 and Proposition 4.21 imply that the character in $T_{\widetilde{\underline{d}}}\mathcal{P}_{\underline{d}}$ minus the character in $T_{(\widetilde{\underline{d}},\widetilde{\underline{d}}')}\mathsf{E}_{\underline{d},i}$ is precisely

$$-q+q^{-d_{ij}+d_{(i-1)j}}-\frac{t_j^2}{t_i^2}\cdot q^{-d_{ij}+d_{ii}}\cdot q'^{\lfloor\frac{-i}{n}\rfloor-\lfloor\frac{-j}{n}\rfloor}+$$

$$\sum_{j\neq k\le i-1}\frac{t_j^2}{t_k^2}\cdot\left(q^{d_{(i-1)k}-d_{ij}}-q^{d_{ik}-d_{ij}}\right)\cdot q'^{\lfloor\frac{-k}{n}\rfloor-\lfloor\frac{-j}{n}\rfloor}$$

The above expression is a sum of terms of the form $\pm e^w$, where w runs over the weights in the ratio we want to compute. This easily shows that the matrix coefficient of $\mathfrak{e}_i$ has the form claimed in Theorem 3.11. The matrix coefficients of $\mathfrak{f}_i$ are computed in the same way, and thus Theorem 3.11 is proved.

5. Cohomology ring of $\mathfrak{M}_{n,d}$

5.1. Giesecker moduli space

$\mathfrak{M}_{n,d}$ is the moduli space of torsion free sheaves on $\mathbf{C}\times\mathbf{X}$ of rank n and second Chern class d, trivialized at $\mathbf{C}\times\infty_{\mathbf{X}}\cup\infty_{\mathbf{C}}\times\mathbf{X}$ (see [17], section 2). We have an evident morphism $\eta:\ \mathcal{P}_{\underline{d}}\to\mathfrak{M}_{n,d_0}$ (forgetting the flag). It induces the morphism on cohomology $\eta^*:\ H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d_0})\to H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}})$.

Lemma 5.2. *Assume* $\underline{d} = (d,\dots,d)$*. Then* $\eta^*: H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d}) \to H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\dots,d})$ *is an embedding.*

Proof. Embedding the source and the target of η^* into their localizations and applying the Localization Theorem, it suffices to check that η is surjective on the sets of torus-fixed points. This is clear. □

5.3. Yangian of $\mathfrak{gl}_n$.

We will view $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d})$ as a subring of $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\dots,d})$. Our next task is to exhibit a subalgebra $ZY(\mathfrak{gl}_n)\subset\mathfrak{A}_{\mathrm{aff}}$ such that $\psi(ZY(\mathfrak{gl}_n)) = H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d})$.

The elements $\mathbf{h}_{i,r},\ i=1,\dots,n$, together with $\mathbf{x}^\pm_{i,r},\ i=1,\dots,n-1$, generate a copy of $Y(\mathfrak{gl}_n)$ inside $\widehat{Y}$. According to [16], the Fourier components of $\mathbf{a}_{0,n}(u)$ generate the center $ZY(\mathfrak{gl}_n)$ of $Y(\mathfrak{gl}_n)$. The eigenvalue of $\mathbf{a}_{0,n}(u)$ on the basis vector $[\widetilde{\underline{d}}]$ is $\prod_{j\le 0}(u-p_{0j}+\hbar')(u-p_{0j})^{-1}$.

Theorem 5.4. *Assume* $\underline{d} = (d,\dots,d)$*. Then* $\psi(ZY(\mathfrak{gl}_n)\otimes\mathbb{C}(\hbar,\hbar')[x_1,\dots,x_n]) = \eta^*({}^{\mathrm{loc}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d}))$*.*

Proof. Follows from the proof of Theorem 4.2. The fact that the cohomology ring of $\mathfrak{M}_{n,d}$ is generated by the Künneth components of the Chern classes of the universal bundle $\underline{\mathcal{F}}$ on $\mathfrak{M}_{n,d}\times\mathbf{C}\times\mathbf{X}$ follows from the resolution of diagonal of $\mathfrak{M}_{n,d}$ constructed in section 4 of [17]. □

5.5. Determinant line bundle

We consider the line bundle $\mathcal{D}_0$ on $\mathcal{P}_{\underline{d}}$ whose fiber at the point $(\mathcal{F}_\bullet)$ equals $\det R\Gamma(\mathbf{C}\times\mathbf{X},\mathcal{F}_0)$.

Lemma 5.6. $\mathcal{D}_0$ *is a* $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$*-equivariant line bundle, and the character of* $\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$ *acting in the fiber of* $\mathcal{D}_0$ *at a point* $\widetilde{\underline{d}}=(d_{ij})$ *equals* $\sum_{j=1}^n x_j(1-\sum_{k=j \mod n} d_{0k}) + \frac{1}{2}\sum_{j\le 0} d_{0j}(d_{0j}-1)\hbar + \sum_{j\le 0} d_{0j}\lfloor\frac{-j}{n}\rfloor\hbar'$*.*

Proof. Straightforward. □

Let $\Phi_n(u)=\sum_{r\ge 0}\Phi_{n,r}u^{-r-1}=\partial_u\log\mathbf{a}_{0,n}(u)$. This is the *noncommutative power sum of the second kind* of Gelfand et al. (see [10] and [16] 7.3 and Corollary 1.11.8). Let N be the minimal number such that $d_{0N}\ne 0$. The eigenvalue of $\Phi_n(u)$ on $[\widetilde{\underline{d}}]$ then reads

$$\sum_{j\le 0}(u-p_{0j}+\hbar')^{-1}-(u-p_{0j})^{-1}=$$

$$=\sum_{N\le j\le 0}-\hbar' u^{-2}(1-(p_{0j}-\hbar')u^{-1})^{-1}(1-p_{0j}u^{-1})^{-1}+$$

$$+\sum_{N-n\le j\le N-1}u^{-1}(1-(p_{0j}-\hbar')u^{-1})^{-1}.$$

In particular, we have

$$\Phi_{n,1} = -\sum_{j=1}^{n} x_j - n\hbar',$$

$$\Phi_{n,2} = \sum_{j=1}^{n}(x_j + \hbar')^2 - 2\hbar\hbar' \sum d_{0j},$$

and

$$\Phi_{n,3} = -\sum_{j=1}^{n}(x_j+\hbar')^3+3\hbar\hbar'(2\sum_{j=1}^{n} x_j \sum_{k=j \mod n} d_{0k}-\sum d_{0j}^2\hbar-\sum d_{0j}(2\lfloor\frac{-j}{n}\rfloor-1)\hbar').$$

Theorem 5.7. *The operator of multiplication by the first Chern class of the determinant line bundle* $\mathcal{D}_0$ *acts as*

$$\frac{-2\Phi_{n,3} + 3(\hbar - \hbar')\Phi_{n,2} - 2\sum_{j=1}^{n}(x_j + \hbar')^3 - 3(\hbar - \hbar')\sum_{j=1}^{n}(x_j + \hbar')^2}{12\hbar\hbar'} + \sum_{j=1}^{n} x_j.$$

Proof. Due to Lemma 5.6, the eigenvalue of $c_1(\mathcal{D}_0)$ on $[\widetilde{\underline{d}}]$ reads

$$\sum_{j=1}^{n} x_j(1 - \sum_{k=j \mod n} d_{0k}) + \frac{1}{2}\sum_{j\leq 0} d_{0j}(d_{0j} - 1)\hbar + \sum_{j\leq 0} d_{0j}\lfloor\frac{-j}{n}\rfloor\hbar' = \sum_{j=1}^{n} x_j -$$

$$-\frac{1}{2}\left(2\sum_{j=1}^{n} x_j \sum_{k=j \mod n} d_{0k} - \sum d_{0j}^2\hbar - \sum d_{0j}2\lfloor\frac{-j}{n}\rfloor\hbar' + \sum d_{0j}(\hbar - \hbar')\right) =$$

$$= \frac{-2\Phi_{n,3} + 3(\hbar - \hbar')\Phi_{n,2} - 2\sum_{j=1}^{n}(x_j + \hbar')^3 - 3(\hbar - \hbar')\sum_{j=1}^{n}(x_j + \hbar')^2}{12\hbar\hbar'} + \sum_{j=1}^{n} x_j.$$

□

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