An Efficient Method for Market Risk Management under Multivariate Extreme Value Theory Approach

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Moscow Finance Conference
November 18-19, 2011
Introduction

The Method

Data and Empirical Results
- Data
- Full-Sample Estimates
- Backtesting

Summary
Market Risk Assessment

Typically relies on quantile-based measures of risk

- Value at Risk (VaR)
- Expected Shortfall (ES)

Conventional methods of VaR and ES estimation in practice:

- Historical simulation
- Analytical method
- Monte Carlo simulation
Market Risk Assessment

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- Historical simulation
- Analytical method
- Monte Carlo simulation
Problem with conventional methods:
- Single parametric family fitted to the whole distribution.
- Trying to reconstruct the entire distribution of returns.
- However, extremely large (but rare) losses are the ones that matter the most!

Extreme Value Theory (EVT) approach:
- Characterizes the tail behavior of distribution of returns.
- Focuses on extreme losses rather than interior.
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- Trying to reconstruct the entire distribution of returns.
- However, extremely large (but rare) losses are the ones that matter the most!

Extreme Value Theory (EVT) approach:
- Characterizes the tail behavior of distribution of returns.
- Focuses on extreme losses rather than interior.
VaR and ES estimates obtained by using EVT-based models outperform the ones based on standard methods:

- McNeil (1997)
- Nyström & Skoglund (2002)
- Harmantzis, Chien & Miao (2005)
- Marinelli, d’Addona & Rachev (2007)
- Lai & Wu (2010)
Introduction

The Method

Data and Empirical Results

Summary

Univariate EVT

Theorem

Picklands (1975): Let \( \{X_i\}_{i=1}^n \) be a set of \( n \) independent and identically distributed random variables with distribution function \( F \), and \( F_u \) the distribution of excesses of \( X \) over the threshold \( u \). Let \( x_F \) be the end of the upper tail of \( F \) (possibly \( +\infty \)). Then, there are constants \( \xi \in \mathbb{R} \) and \( \beta \in \mathbb{R}_+ \) such that

\[
\lim_{u \to x_F} \sup_{u < X < x_F} |F_u(X) - G_{\xi,\beta}(X - u)| = 0,
\]

where

\[
G_{\xi,\beta}(y) := 1 - \left( 1 + \frac{\xi y}{\beta} \right)^{-1/\xi}_+
\]

is known as the generalized Pareto (GP) distribution.
Multivariate EVT?

Problems with univariate EVT:
- Useful for portfolio-level analysis only.
- Cannot capture behavior of correlations during extreme market movements.

Problems with available multivariate alternatives:
- Copulas
  - Too complicated for practitioners.
  - More difficult to implement than EVT.
  - Introduce additional “model risk”.
- Multidimensional limiting relations
  - See e.g. Smith (2000) or Dupuis & Jones (2007).
  - Model complexity increases greatly with the number of risk factors.
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Basic Idea

- Use series of returns on individual assets in portfolio.
- Construct $n$ orthogonal series that are approximately i.i.d.
- Use separate estimations of univariate EVT.
Estimation of the Univariate EVT

- Choose a reasonable threshold \( u \).
- Use one of the conventional methods to fit the interior of the distribution (e.g., historical simulation).
- Fit the upper and lower tail separately with the GP distribution.
Estimation of the Univariate EVT

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Market risk management under multivariate EVT
Principal Components

In $n$ dimensions, use the principal components of the unconditional VCV matrix:

$$
\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)
$$

$$
V_\infty = P \Lambda P'
$$

$$
L := P \Lambda^{1/2}
$$

$$
z_t = L^{-1} \varepsilon_t
$$

$$
E(z_t) = 0
$$

$$
\text{var}(z_t) = 1_n
$$

This gives a set of (at most) $n$ orthogonal standardized coordinates.
GARCH Filtering

- Conditional variance: GJR-GARCH\((p, q)\)

\[
V_t = \Omega + \sum_{s=1}^{p} A_s E_{t-s} + \sum_{s=1}^{p} \Theta_s I_{t-s} E_{t-s} + \sum_{s=1}^{q} B_s V_{t-s}
\]

\[
E_t := \varepsilon_t \varepsilon_t'
\]

\[
I_t := \text{diag}(\text{sgn}(-\varepsilon_{t,1}), \text{sgn}(-\varepsilon_{t,2}), \ldots, \text{sgn}(-\varepsilon_{t,n}))
\]

- Make transformation from returns to principal components.
- Estimate \(n\) separate univariate GJR-GARCH\((p, q)\) models:

\[
\hat{V}_{t,i} = \hat{\Omega}_i + \sum_{s=1}^{p} \hat{A}_{s,i} \hat{E}_{t-s,i} + \sum_{s=1}^{p} \hat{\Theta}_{s,i} \hat{I}_{t-s,i} \hat{E}_{t-s,i} + \sum_{s=1}^{q} \hat{B}_{s,i} \hat{V}_{t-s,i}
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\]

\[
E_t := \varepsilon_t \varepsilon'_t
\]

\[
I_t := \text{diag}(\text{sgn}(-\varepsilon_{t,1})_+, \text{sgn}(-\varepsilon_{t,2})_+, \ldots, \text{sgn}(-\varepsilon_{t,n})_+)
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\]
Forecasting

- Confidence interval for the value of $i$-th principal component, $h$ steps ahead:

$$z_{t+h,i}^\pm = F_i^{-1}(q_\pm) \sqrt{\hat{V}_{t+h,i}}$$

- Forecast of multivariate VaR:

$$S_t^{\pm} := \text{diag} \left[ (z_{t',1}^\pm)^2 (z_{t',2}^\pm)^2 \ldots (z_{t',m}^\pm)^2 \right]$$

$$Q_t^{\pm} := L S_t^{\pm} L'$$

$$\text{VaR}^{\pm} = a' \mu_{t'} \pm \sqrt{a' Q_t^{\pm} a}$$

- For ES just replace $F_i^{-1}$ by $(F_i^{-1} + \beta_\pm - \xi_\pm u_\pm)/(1 - \xi_\pm)$. 

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Market risk management under multivariate EVT
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$$Q_{t'}^\pm := LS_{t'}^\pm L'$$

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Market risk management under multivariate EVT
Data

- Daily returns computed from adjusted prices of Dow Jones stocks.
- DJIA composition on December 31, 2010 was used to choose the 30 constituents of the equally-weighted portfolio.
- 2401 observations in each of the series (June 14, 2001 – December 31, 2010).
Data and Empirical Results

Summary
## Parameters of Univariate GP Distributions

December 31, 2010

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\xi}_+$</td>
<td>$-0.1458$</td>
<td>$0.0369$</td>
<td>$0.1179$</td>
<td>$0.0829$</td>
</tr>
<tr>
<td></td>
<td>$(0.0588)$</td>
<td>$(0.0703)$</td>
<td>$(0.0719)$</td>
<td>$(0.0564)$</td>
</tr>
<tr>
<td>$\hat{\beta}_+$</td>
<td>$0.5492$</td>
<td>$0.6055$</td>
<td>$0.4755$</td>
<td>$0.5598$</td>
</tr>
<tr>
<td></td>
<td>$(0.0479)$</td>
<td>$(0.0582)$</td>
<td>$(0.0461)$</td>
<td>$(0.0481)$</td>
</tr>
<tr>
<td>$\hat{\xi}_-$</td>
<td>$-0.0089$</td>
<td>$0.0859$</td>
<td>$0.0670$</td>
<td>$0.0207$</td>
</tr>
<tr>
<td></td>
<td>$(0.0482)$</td>
<td>$(0.0679)$</td>
<td>$(0.0669)$</td>
<td>$(0.0701)$</td>
</tr>
<tr>
<td>$\hat{\beta}_-$</td>
<td>$0.6222$</td>
<td>$0.5080$</td>
<td>$0.5917$</td>
<td>$0.6354$</td>
</tr>
<tr>
<td></td>
<td>$(0.0503)$</td>
<td>$(0.0478)$</td>
<td>$(0.0553)$</td>
<td>$(0.0609)$</td>
</tr>
</tbody>
</table>
Fitting the Tails

First principal component

![Graph showing lower and upper tails of standardized residuals with various distributions](image)
Fitting the Tails

Second principal component

Lower Tail of Standardized Residuals

Upper Tail of Standardized Residuals
Fitting the Tails

Third principal component

Lower Tail of Standardized Residuals

Upper Tail of Standardized Residuals

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Fitting the Tails

Fourth principal component

- Lower Tail of Standardized Residuals
- Upper Tail of Standardized Residuals

Graphs showing fitted Generalized Pareto CDF, Normal CDF, t CDF, and Empirical CDF for both tails of standardized residuals.
### December 31, 2010

#### Upper tail

<table>
<thead>
<tr>
<th>CL</th>
<th>0.900</th>
<th>0.950</th>
<th>0.990</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>0.6431</td>
<td>0.8288</td>
<td>1.1975</td>
<td>1.3334</td>
<td>1.6058</td>
</tr>
<tr>
<td>ES</td>
<td>0.8901</td>
<td>1.0534</td>
<td>1.3789</td>
<td>1.4994</td>
<td>1.7428</td>
</tr>
</tbody>
</table>

#### Lower tail

<table>
<thead>
<tr>
<th>CL</th>
<th>0.900</th>
<th>0.950</th>
<th>0.990</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>−0.6173</td>
<td>−0.8366</td>
<td>−1.3416</td>
<td>−1.5574</td>
<td>−2.0546</td>
</tr>
<tr>
<td>ES</td>
<td>−0.9322</td>
<td>−1.1500</td>
<td>−1.6515</td>
<td>−1.8659</td>
<td>−2.3601</td>
</tr>
</tbody>
</table>
Backtesting Setup

- Last 1000 observations are chosen as out-of-the-sample data.
- Expanding-window estimation of the model for each day.
- Tomorrow’s VaR forecasts compared with tomorrow’s actual return on the portfolio.
### Number of VaR Violations by Quantile: Upper Tail

<table>
<thead>
<tr>
<th>Method</th>
<th>0.900</th>
<th>0.950</th>
<th>0.990</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>HS</td>
<td>374</td>
<td>307</td>
<td>195</td>
<td>143</td>
<td>89</td>
</tr>
<tr>
<td>mv Normal</td>
<td>97</td>
<td>42</td>
<td>13</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>mv Student</td>
<td>106</td>
<td>42</td>
<td>9</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>mv EVT</td>
<td>104</td>
<td>45</td>
<td>13</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Expected</td>
<td>100</td>
<td>50</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
### Number of VaR Violations by Quantile: Lower Tail

<table>
<thead>
<tr>
<th>Method</th>
<th>0.900</th>
<th>0.950</th>
<th>0.990</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL</td>
<td>0.900</td>
<td>0.950</td>
<td>0.990</td>
<td>0.995</td>
<td>0.999</td>
</tr>
<tr>
<td>HS</td>
<td>321</td>
<td>279</td>
<td>202</td>
<td>163</td>
<td>108</td>
</tr>
<tr>
<td>mv Normal</td>
<td>106</td>
<td>68</td>
<td>26</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>mv Student</td>
<td>105</td>
<td>68</td>
<td>23</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>mv EVT</td>
<td>110</td>
<td>68</td>
<td>13</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Expected</td>
<td>100</td>
<td>50</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

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### Pearson’s Test

<table>
<thead>
<tr>
<th>Method</th>
<th>Lower tail</th>
<th>Upper tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS</td>
<td>12420.2</td>
<td>9029.60</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>mv Normal</td>
<td>37.4700</td>
<td>6.5850</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.2534)</td>
</tr>
<tr>
<td>mv Student</td>
<td>25.0828</td>
<td>6.6350</td>
</tr>
<tr>
<td></td>
<td>(0.0001)</td>
<td>(0.2492)</td>
</tr>
<tr>
<td>mv EVT</td>
<td>8.8161</td>
<td>4.2878</td>
</tr>
<tr>
<td></td>
<td>(0.1166)</td>
<td>(0.5088)</td>
</tr>
</tbody>
</table>

(p-values in parentheses.)
Introduction

The Method

Data and Empirical Results

Summary

Summary of the Method

Original multivariate time series

PCA

Orthogonal coordinates
[and reduced dimensionality]

Univariate GARCH

Orthogonal and i.i.d. coordinates
[and reduced dimensionality]

Univariate EVT

Historical simulation

Model parameters

QML

Closed-form expressions

ML

VaR and ES forecasts

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Market risk management under multivariate EVT
The proposed approach employs the notion that some key results of the univariate EVT can be applied separately to a set of orthogonal i.i.d. random variables.

Such random variables can be constructed from the principal components of GARCH conditional residuals of a multivariate return series.

The approach yields more precise VaR and ES forecasts than conventional methods based on historical simulation, conditional normality or conditional t-distribution, without losing efficiency.