Phase interaction of short vector solitons

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Abstract

An interaction of vector solitons in the frame of coupled third-order nonlinear Schrödinger equations taking into account third-order linear dispersion, nonlinear dispersion, and cross-phase modulation terms is considered. Phase nature of the solitons’ interaction is shown. In particular, dependence of solitons’ trajectories on initial distance between solitons is shown. Conditions of reflection and propagation of solitons through each other are obtained.

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1. Introduction

Interest to solitons is conditioned because of their possibility to propagate on considerable distance keeping the form and transporting the energy and information without losses. Soliton solutions are considered in many different nonlinear models in various areas of physics for investigation of intensive wave fields in dispersive media propagation: optical pulses in fibers, electromagnetic waves in plasma, surface waves on deep water, nonlinear left-handed metamaterials [1–4].

Propagation of high frequency wave packets of rather big extension is described by the second-order nonlinear dispersive wave theory. In isotropic media basic equation of the theory is nonlinear Schrödinger equation (NSE) [5,6], considering both second-order linear dispersion and cubic nonlinearity. Soliton solution in this case arises as a balance of dispersion dilatation and nonlinear compression of wave packet. In anisotropic media basic equations are coupled nonlinear Schrödinger equations (CNSE) [7–10], considering also cross-phase modulation. Interaction of vector solitons in the frame of CNSE is described in details in [11–13]. Dynamics of polarizing wave packets (including solitons interaction) in the frame of the CNSE-2 particularly studied in [14]. The possibility to use the CNSE as a basic equation is also discussed in that work.

Another model in anisotropic media is coupled Ginzburg–Landau equations (CGL) [15–17], considering third-order linear dispersion (TOLD) and loss.

For the nonlinear wave packets of low extension (several wavelengths) and high intensity, the second approximation of dispersion theory doesn’t work. For deep water waves experimentally in [18,19] shows that for wave packets containing 2, 5 wavelengths, on the distance more than 5 wave packet length from the source self-steeping effects, which can’t be described by classical NSE of second order appear. Comparing results of full-scale and numerical experiments in [19] it was shown that those effects are described by TNSE. The same result was confirmed also for electromagnetic wave packets [20–25].

In [26] the impossibility to describe self-steeping effects in the frame of second order classical NSE and the necessity to consider in the basic equation third order nonlinear dispersion terms are discussed. It is the third-order nonlinear Schrödinger equation (TNSE) that is suggested in [23,29] for the description of the effects which are impossible to consider in the classical NSE of second order.

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Dynamics of high frequency wave packets of low extension is described by the third-order nonlinear dispersive wave theory, taking into account third-order terms: nonlinear dispersion [27], stimulated Raman-scattering (SRS) [28] and TOLD. In isotropic media basic equation is TNSE [29–36]. In particular, for nonlinear left-handed metamaterials the basic equation is TNSE without SRS [37].

In [33,37] soliton solution in the frame of TNSE without SRS is...
found. Such soliton exists as balance of TOLD and nonlinear dispersion. In [38] stationary kink-waves are found in the frame of TNSE without TOLD. This soliton exists as a balance of nonlinear dispersion and SRS.

In anisotropic media dynamics of vector wave packets of low extension is described by coupled third-order nonlinear Schrödinger equations (CTNSE) taking into account also third-order cross nonlinear terms [39–42]. In the frame of CTNSE without SRS short vector solitons’ solution is found [41]. In all investigation described above interaction of short vector solitons is considered for small space distance between their centers and with absence of phase effects [41].

In this Letter short vector solitons’ interaction in the frame of CTNSE without SRS for arbitrary initial soliton distance and with taking into account phase effects is studied. Optical (electromagnetic) wave packets of complex envelope function (of the following form) in anisotropic media are considered. Analytical in the frame of the adiabatic approximation and numerical approaches are considered. It is shown that taking into account phase effects leads to short vector soliton’s interaction character change.

2. Basic equations

Let’s consider the dynamics of the vector wave field \( \mathbf{E} = \mathbf{e}(\xi, t) \exp(i(\omega t - k_0 \xi)) + \mathbf{e}_2(\xi, t) \exp(i(\omega t - k_0 \xi)) \) in the frame of CTNSE without SRS. Basic equations CTNSE (1), for example, can be derived from Maxwell’s equation, as it is shown at [43] for electromagnetic (including optical) wave packets. While deriving equations the proximity of polarizing components’ frequencies \(|\omega_0 - \omega_0| \ll \omega_0, \omega_1 \) was considered and linear and nonlinear media sensitivity parameters for wave components of different polarization were neglected. In more general case parameters of the same type in CTNSE (and CNSE as well) are considered to be different: \( q_0 \neq q_1, \gamma_0 \neq \gamma_1 \), and so on. But this general case usually is not considered because of technical difficulties in calculation process.

\[
\begin{align*}
2i \left[ \frac{\partial U}{\partial t} + \frac{\partial (|U|^2 U) + \partial W^2 U^*}{2 \partial \xi} \right] + \frac{q_0^2 U}{\partial \xi} + 2\alpha(|U|^2 + |W|^2)U + \alpha \sigma W^2 U^* + i\gamma \frac{\partial^2 U}{\partial \xi^2} &= 0, \\
2i \left[ \frac{\partial W}{\partial t} + \frac{\partial (|W|^2 W + |U|^2 U)}{2 \partial \xi} + \beta \frac{\partial (U^2 W^*)}{\partial \xi} \right] + \frac{q_1^2 W}{\partial \xi} + 2\alpha(|W|^2 + |U|^2)W + \alpha \sigma U^2 W^* + i\gamma \frac{\partial^2 W}{\partial \xi^2} &= 0,
\end{align*}
\]

where in consequence of nonlinear dispersion law \( \omega = \omega(k, |U|^2, |W|^2) \) the following notation is used: \( q = -\partial^2 \omega(\partial k^2) \) is the second-order linear dispersion of \( U \) and \( W \) components, \( \alpha = \partial \omega(\partial |U|^2) + \partial \omega(\partial |W|^2) \) is the self-phase modulation, \( \sigma \) is the cross-phase modulation, \( \gamma = -\partial^3 \omega(3\partial k^2) \) is the TOLD, \( \beta \) is the nonlinear dispersion, \( U^* \) and \( W^* \) are the complex conjugate for \( U \) and \( W \).

System of Eqs. (1) and (2) with the zero conditions on infinity \( U(\xi, t)|_{\xi \rightarrow \pm \infty} \rightarrow 0 \) has the following integrals:

- the conservation of energy law

\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \left( |U|^2 + |W|^2 \right) d\xi = 0.
\]

First three terms in the right side of (4) correspond to cross-nonlinear dispersion effects, the last term – to cross-phase modulation effects. First term describes solitons’ amplitude interaction; others depend phase interaction and interaction phase effects.

The system (1)-(2) has two-component (mutually orthogonal) short vector solitons of different polarization as it is shown in [41]

\[
\begin{align*}
U(\xi, t) &= \frac{A_1}{\cos(\Omega_1^2 \xi - V_1 t) + \exp(i\Omega_1 t + iK\xi)}, \\
W &= 0,
\end{align*}
\]

and

\[
\begin{align*}
W(\xi, t) &= \frac{A_2}{\cos(\Omega_2^2 \xi - V_2 t) + \exp(i\Omega_2 t + iK\xi)}, \\
U &= 0,
\end{align*}
\]

where \( \Omega_{1,2} \) are amplitudes of solitons of different polarization, \( \gamma = \sqrt{\beta/\gamma} \), \( K = (q_1 - q_0)/2\gamma \) is the additional wave number, \( \Omega_{1,2} = \frac{2}{\alpha^2 + K^2} \) are additional frequencies of solitons of different polarization, \( V_{1,2} = (\beta A_1^2 + Kq - 3\gamma K^2/2)2 \) are velocities of solitons of different polarization. Such solitons exist under following condition: \( \gamma > 0 \).

Let’s consider the following initial problem. Suppose, in the time moment \( t = 0 \) two mutually orthogonal vector solitons of different polarization, different amplitude and with distance \( \xi_0 \) between their centers appear in anisotropic medium,

\[
U(\xi, t = 0) = \frac{A_1(0)}{\cos(\Omega_1^2 \xi - \xi_0) + \exp(i\Omega_1 \xi)},
\]

\[
W(\xi, t = 0) = \frac{A_2(0)}{\cos(\Omega_2^2 \xi - \xi_0) + \exp(i\Omega_2 \xi)}.
\]

3. Adiabatic approximation

Consider that solitons’ parameters vary slowly, then the solution for \( t > 0 \) can be found in the adiabatic approximation,

\[
U(\xi, t) = \frac{A_1(t)}{\cos(\Omega_1^2 \xi - \xi_0 - i\int_0^t V_1(\tilde{t}) d\tilde{t})} \exp\left(i \int_0^t \Omega_1(\tilde{t}) d\tilde{t} + iK\xi\right),
\]
3.1. Interaction without phase effects

Substituting (8) to the conservation of energy law (3), for solitons' amplitudes will be obtained

\[ A_1(t) + A_2(t) = A_1(0) + A_2(0) = C_0. \]

In this investigation let's consider solitons with small difference of their amplitudes \(|\Delta A| \ll C_0 \approx 2A_0\). Differentiating (9) over time and substituting (9) into (4), the system of equations for solitons' moving trajectory will be obtained

\[
\begin{align*}
\frac{d\rho}{dt} &= a, \\
\frac{da}{d\tau} &= 6(3\rho - 3\alpha \rho - \rho \cdot \tan^2 \rho) - 2\alpha \rho - \rho \cdot \tan^2 \rho, \\
\end{align*}
\]

where \(\alpha = \frac{A_0}{C_0}, \rho_0 = A_0 \xi_0, \tau = t \cdot \alpha \sqrt{\rho_0}, a = \frac{\Delta A}{\alpha \rho}, p = \frac{\alpha \rho_0}{\alpha \rho} \). First term in the right side of (11) corresponds to amplitude type of the interaction, second and third depend on initial distance \(\rho_0\) and correspond to phase effects of the interaction.

3.2. Interaction with phase effects

When \(p = 0\), the following equation for trajectories in explicit form will be obtained from system (10)-(11):

\[
a^2 + 12 \frac{\rho - \tan \rho}{\cosh^2 \rho \cdot \tan^3 \rho} = a^2_{\infty}. \tag{12}
\]

where \(a_{\infty} \) - initial solitons' amplitudes difference on considerable large distance \(|\rho| \gg 1\). In Fig. 1 there is a phase-plane of (12). Curves 1 describe passing solitons though each other. Curves 2 - separatrix, corresponding to the critical difference of initial solitons' amplitudes. Solitons' amplitudes difference \(a_0 = a_{\infty}\) for trajectories 2 (reflection interval) is the following: \(a_0 = 2\). Curves 3 describe solitons' pushing off from each other.

3.2. Phase effects

When \(p \neq 0\) solitons' trajectories depend on initial distance \(\rho_0\). In Figs. 2(a)-(d) there are phase-planes of (10)-(11) for \(p = 1\) and different \(\rho_0\): Fig. 2(a) - \(\rho_0 = \pi k\), Fig. 2(b) - \(\rho_0 = \frac{1}{2}\pi + \pi k\), Fig. 2(c) - \(\rho_0 = \pi k + \frac{3}{2}\), and Fig. 2(d) - \(\rho_0 = \pi - \pi k\).

When \(\rho_0 = \pi k \pm \frac{3}{2}\) phase-plane (Fig. 2(b)) has three balance states: in point \(\rho = 0\) - saddle, in points \(\rho = \pm \frac{3}{2}\) - saddle-center. The reflection interval in this case \(|a| \leq 2\) is equal to the reflection interval without phase effects for \(p = 0\).

When \(\rho_0 = \pi k \pm \frac{3}{2}\) phase-plane (Fig. 2(c)) has three balance states. Left and right saddle's positions are \((\rho_0)_{\pm} = \pm 1\). In those points solitons' amplitudes on separatrix 1, 2 coincide. Solitons' amplitudes difference for trajectories 1, 2 in Fig. 2(c) for \(|\rho| \gg 1\) (reflection interval) \(|a| \leq k = 2\) is smaller then the reflection interval without phase effects.

When \(\rho_0 = \pi k + \frac{3}{2}\) phase-plane (Fig. 2(d)) corresponds to phase-plane \(\rho_0 = \frac{3}{2}\pi + \pi k\) (Fig. 2(b)) after transformation \(\rho 
\rightarrow -\rho\). Increasing parameter \(p\) leads to more difficult soliton trajectories.

Analytical investigation is made for a small difference of solitons' amplitudes \(|\Delta A|/C_0 \ll 1\). In particular, evaluate the reflection interval by the relation \(|a| \approx 2\) \(\approx 1\). For more detail analysis of the validity of the adiabatic approximation let's consider numerically initial problem of solitons' interaction (7) in the frame of (1)-(2).

4. Numerical results

Let's consider numerically the dynamic of wave packets (7) in the frame of (1)-(2) with \(\alpha = q = k = 1\) and different \(\tau\). Analytical investigation is made for a small difference of solitons' amplitudes \(|\Delta A|/C_0 \ll 1\). In particular, evaluate the reflection interval by the relation \(|a| \approx 2\) \(\approx 1\). For more detail analysis of the validity of the adiabatic approximation let's consider numerically initial problem of solitons' interaction (7) in the frame of (1)-(2).

In Figs. 4 and 5 numerical solution for relative difference of polarization components maximum \(a_{\sum} = \frac{2}{3} \max(|U|+|W|) \approx 2(A_1-A_2)/(A_1+A_2)\) as a function of space distance \(\Delta \xi = \xi_{\max} - \xi_{\max} \) between them is shown, for the following amplitude parameter

\[
W(\xi, t) = \frac{A_2(t)}{\cosh[A_2(t)\xi(t) - \int_0^t V_2(t) d\tau]} \\
\times \exp\{i \int_0^t \Omega_2(t) d\tau + i K \xi\},
\]

where \(V_1 = (\beta A_2^2 + K q - 3\gamma K^2/2)/2\) are solitons' velocity at the time moment \(t, \Omega_1 = \frac{\beta}{2} A_1^2 + \frac{\gamma}{2} K^2\). Distance between solitons varies with the following law

\[
\Delta \xi = \xi_0 + \int_0^t (V_1(\tau) - V_2(\tau)) d\tau = \xi_0 + \frac{\beta C_0}{2} \int_0^t \Delta A(t) d\tau.
\]

Solitons' phases difference is the following

\[
\phi_w - \psi_w = \int (\Omega_1(\tau) - \Omega_2(\tau)) d\tau = \int \Delta A(\tau) d\tau = \xi_0 + \frac{\beta C_0}{2} \int_0^t \Delta A(t) d\tau.
\]
considering a separatrix regime considering curve 3 describes soliton's pushing off from each other considering the adiabatic approximation for which is not much differ from the value obtained in the frame of the adiabatic approximation for \(\rho_0 = \pi k + \pi/4\) (Fig. 2(c)).

In Fig. 4 curve 1 describes passing though each other solitons considering \(A_1(0) = 1.75\) (\(\alpha_{\text{num}}(-\pi) = 2.18\)); curve 2 describes separatrix regime considering \(A_1(0) = 1.65\) (\(\alpha_{\text{num}}(-\pi) = 1.93\)) and curve 3 describes soliton's pushing off from each other considering \(A_1(0) = 1.55\) (\(\alpha_{\text{num}}(-\pi) = 1.73\)).

Fig. 4 corresponds to phase-plane of system (10)-(11) considering \(\rho_0 = \pi k\) (Fig. 2(a) in adiabatic approximation). The maximum difference of soliton’s amplitude for \(\rho_0 = \pi k\) in case of reflection obtained numerically is \(\alpha_{\text{num}}(-\pi) \approx 1.93\) (curve 2 for \(\Delta \xi = -\pi\)) which is not much differ from the value obtained in the frame of the adiabatic approximation for \(\rho_0 = \pi k\) (curve 2 in Fig. 2(a)). This distinction is the result of radiation fields appearance along soliton interaction.

In Fig. 5 \(\xi_0 = -\pi/4\) curve 1 describes passing though each other solitons considering \(A_1(0) = 1.55\) (\(\alpha_{\text{num}}(-\pi) = 1.73\)); curve 2 describes separatrix regime considering \(A_1(0) = 1.43\) (\(\alpha_{\text{num}}(-\pi) = 1.47\)) and curve 3 describes soliton’s pushing off from each other considering \(A_1(0) = 1.35\) (\(\alpha_{\text{num}}(-\pi) = 1.19\)).

Fig. 5 corresponds to phase-plane of the system (10)-(11) for \(\rho_0 = \pi k + \pi/4\) (Fig. 2(c) in adiabatic approximation). Obtained numerically maximum difference of solitons’ amplitude for \(\rho_0 = \pi k + \pi/4\) in case of reflection is \(\alpha_{\text{num}}(-\pi) \approx 1.43\) (curve 2 for \(\Delta \xi = -\pi/2\)) which is not much differ from the value obtained in the frame of the adiabatic approximation for \(\rho_0 = \pi k + \pi/4\) (curve 2 in Fig. 2(c)).
It leads to greater difference between analytical and numerical action but gives more considerable variation of soliton amplitudes. The difference of solitons' amplitude obtained numerically in case approaching each other asymptotically are found. Increasing parameter $\sigma$ keeps soliton-like character of the interaction but gives more considerable variation of soliton amplitudes. It leads to greater difference between analytical and numerical results. In particular, for $\sigma = \frac{1}{4}$ and initial distance $\xi_0 = -\frac{\pi}{2}$ maximum difference of solitons' amplitude obtained numerically in case of reflection is $(a_{\text{num}})_{c} \approx 1.8$ and $(a_{\text{num}})_{c} \approx 1.3$ for initial distance $\xi_0 = -\frac{\pi}{2}$. For greater $\sigma$ detail comparison analytical and numerical results is impossible.

This distinction is the result of radiation fields appearance along soliton interaction.

5. Conclusion

Phase effects of vector solitons’ interaction in the frame of CTNSE without SRS are analysed. Investigation of solitons’ interaction is made both analytically in the adiabatic approximation, and numerically. Analytical and numerical results are in a good agreement for small cross-phase modulation parameter $\sigma \ll 1$. Regimes of passing through each other, pushing off though each other and trajectories in adiabatic approximation are found in an explicit form. For $\sigma \neq 0$ solitons’ trajectories depend on initial distance between solitons.

Tacking into account SRS gives down shift spectrum of short solitons to the long wave region [28]. In the same time arising radiation fields lead to soliton acceleration and up shift spectrum to the short wave region. Balance of those effects lead to stabilisation of solitons propagation with present SRS [44]. Another way for stabilisation of solitons propagation with taking into account SRS is inhomogeneous parameter of second-order linear dispersion $q$ [45]. These effects will be considering next step.

References