Maps that take lines to conics

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MI RAS

Some classical theorems

Theorem (Möbius, 1827)

Suppose that $f : \mathbb{RP}^n \to \mathbb{RP}^n$ *is a continuous one-to-one map taking all straight lines to straight lines. Then* f *is a projective transformation.*

Theorem (Möbius, 1820s)

Suppose that $f : S^n \to S^n$ *is a continuous one-to-one map taking all circles to circles. Then* f *is a Möbius transformation.*

By definition, a Möbius transformation is an element of the group generated by inversions.

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August Möbius 1790–1868

Theorem (von Staudt)

The continuity assumption in theorems of Möbius is superfluous.

Theorem (local version)

If $U \subset \mathbb{RP}^n$ *is a connected open set, and* $f : U \to \mathbb{RP}^n$ *is a continuous one-to-one map such that, for every line* L*, the set* $f(U \cap L)$ *is a subset of a line, then* f *is the restriction of some projective transformation.*

There is also a local version of Möbius' theorem about circles.

Karl Georg Christian von Staudt 1798 - 1867

Classical geometric structures

Definition

A projective structure on a manifold M of dimension n is a family of curves $\mathcal{C}$ in M that is locally modeled by the family of all lines in $\mathbb{R}^n$.

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Describe all (sufficiently smooth) (one-to-one) maps from an open subset of $\mathbb{RP}^n$ *to an open subset of* S^n *that take all lines to circles.*

Definition

We say that a map $f : U \subset \mathbb{RP}^n \to V \subset S^n$ takes all lines to circles if the image of each straight line segment contained in U is an arc of Euclidean circle contained in V.

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Motivations

- Geometrization: how to define transition functions between charts carrying different geometric structures?
- Nomography: how to transform nomograms with aligned points into circular nomograms?
- Blaschke's problem: how to describe all hexagonal 3-webs of circles?
- Architecture: how to build free-form structures from arcs of circles? (M. Skopenkov, H. Pottmann, L. Shi, Th. Nilov, Ph. Grohs)
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Wilhelm Blaschke 1885 - 1962

Some other classical theorems

Theorem (Beltrami, 1880s)

Let g be a Riemannian metric on an open subset of $\mathbb{RP}^n$ such that all geodesics are straight line segments. Then g is a classical metric, i.e. has constant sectional curvature.

Theorem (Segre, 1950s)

Let g be a Riemannian metric on an open subset of S^2 such that all geodesics are arcs of circles. Then g is a classical metric, i.e. has constant Gaussian curvature.

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Eugenio Beltrami 1835–1900

Beniamino Segre 1903–1977

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Note that the theorem of Segre is only about dimension 2!

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Motivation: Nomography

Nomograms:

A nomogram is a planar picture representing a function of many variables. Usually, it consists of several curves equipped with scalings. One uses a straightedge or a compass to read the output.

Compass vs Straightedge:

Compass is more accurate than a straightedge, because even a deformed compass draws round circles. Thus circular nomograms are more practical, while nomograms with aligned points (those using a straightedge) are easier theoretically.

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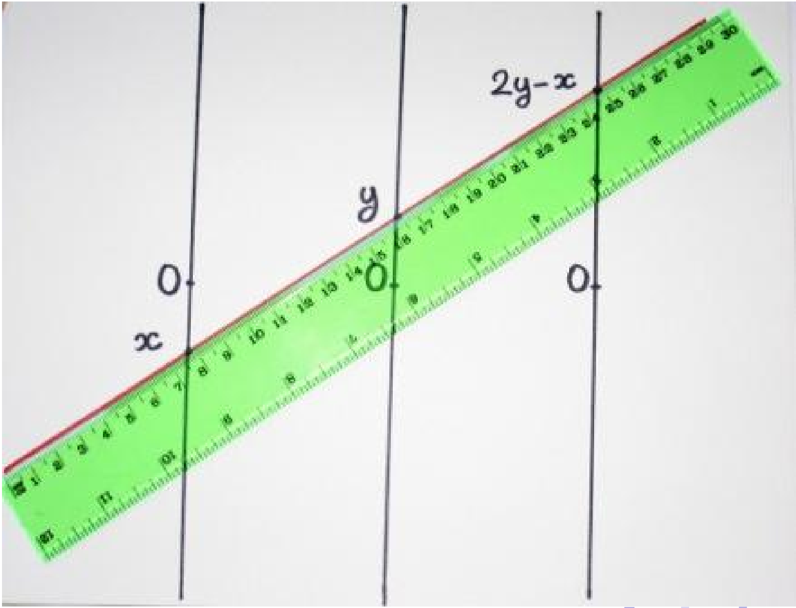
A simple nomogram computing $(x, y) \mapsto 2y - x$



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Results in dimensions 2 and 3

Theorem (Khovanskii, 70s)

Suppose that a diffeomorphism $f : U \subset \mathbb{RP}^2 \to V \subset S^2$ *takes all lines to circles. Then* $f = M \circ \phi \circ P$, *where* $P : \mathbb{RP}^2 \to \mathbb{RP}^2$ *is a projective transformation,* $M : S^2 \to S^2$ *is a Möbius transformation, and* $\phi \in \{\phi_{Euc}, \phi_{sph}, \phi_{hyp}\}$ *corresponds to a classical model of a classical geometry (i.e. Euclidean, spherical or hyperbolic geometry).*

Theorem (Izadi,2003)

The same result is true for diffeomorphisms $f : U \subset \mathbb{RP}^3 \to V \subset S^3$.

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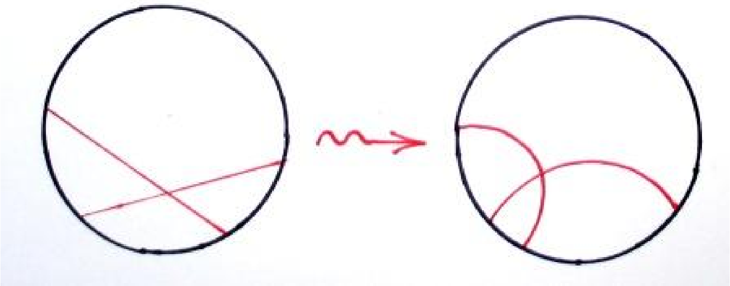
The maps ϕ_{Euc}, ϕ_{sph}, ϕ_{hyp}

$$\phi_{Euc} = id, \quad \phi_{sph}^{-1}(x,y) = \frac{(x,y)}{1+x^2+y^2}, \quad \phi_{hyp}^{-1}(x,y) = \frac{(x,y)}{1-x^2-y^2}.$$

Example

The map ϕ_{hyp} establishes an isomorphism between the Klein model and the Poincaré model of the classical hyperbolic geometry.



Quaternionic Hopf fibrations

Definition

Consider the map

$$\mathbb{H}^2 - 0 \to \mathbb{HP}^1, \quad (q_1, q_2) \mapsto [q_1 : q_2],$$

where $\mathbb{HP}^1$ is the (left or right) quaternionic projective line. Note that $\mathbb{HP}^1 = S^4$. This map factors through the real projectivization

$$\mathbb{H}^2 - 0 \to \mathbb{RP}^7$$

to give a map

$$\mathbb{RP}^7 \to S^4$$

called a quaternionic Hopf fibration.

Results in dimension 4

Theorem

Let $f : U \subset \mathbb{RP}^4 \to V \subset S^4$ *be a diffeomorphism taking all lines to circles. Then*

- *either* $f = M \circ \phi \circ P$, *where* $P : \mathbb{RP}^4 \to \mathbb{RP}^4$ *is projective,* $M : S^4 \to S^4$ *is Möbius and* $\phi \in \{\phi_{Euc}, \phi_{sph}, \phi_{hyp}\}$
- *or* f *is of the form* $\mathbb{RP}^4 \hookrightarrow \mathbb{RP}^7 \to S^4$, *where the first arrow is a projective embedding, and the second is a quaternionic Hopf fibration.*

There are much more maps of the second kind.

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Results in dimension 4

Theorem

Let g be a Kähler metric on an open subset of $\mathbb{C}^2$ such that all geodesics are arcs of circles (or straight segments). Then g has constant holomorphic sectional curvature, i.e. g is a "complexification" of one of the classical geometries.

These are: Euclidean geometry, Fubini–Study metric on $\mathbb{C}P^2$, complex hyperbolic geometry.

Higher dimensions

In higher dimensions, the problem is still OPEN, but there are remarkable relations with classical problems in algebra, including:

- Hurwitz problem on sums of squares,
- quadratic maps between spheres,
- fractional quadratic parameterizations of quadrics

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Results in higher dimensions

Theorem

Suppose that $f : U \subset \mathbb{RP}^n \to V \subset S^n$ *takes all lines passing through a particular point* $p \in U$ *to circles. Also, let* f *be differentiable sufficiently many times and satisfy* $\mathrm{rank}(d_p f) > 1$*. Then there is a fractional quadratic map* $Q : \mathbb{RP}^n \dashrightarrow S^m$ *such that* $f(l) = Q(l)$ *for all lines* $l \ni p$*.*

Open problems in algebra

Problem

Describe all fractional quadratic *maps* $\mathbb{RP}^n \dashrightarrow S^m$.

Remark:

This problem is very difficult. A special case of it is the following

Problem (Hurwitz, 1898)

Describe all triples of integers (r, s, n) *such that*

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2,$$

where z_i *are some bilinear combinations of* x_j *and* y_k.

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(2,2,2) = multiplication of complex numbers.

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Fractional quadratic transformations in terms of Hurwitz formulas

Set

$$X = (x_1, \ldots, x_r), \quad Y = (y_1, \ldots, y_s), \quad Z = (z_1, \ldots, z_n).$$

Then

$$Q[X, Y] = \left(\frac{2Z}{|X|^2 + |Y|^2}, \frac{|X|^2 - |Y|^2}{|X|^2 + |Y|^2} \right)$$

is a fractional quadratic map from $\mathbb{RP}^{r+s-1}$ to S^n.

Some similar problems

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Describe all maps that take one nice class of curves to another nice class of curves.

E.g.

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Planarizations

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Let $U \subset \mathbb{RP}^2$ be an open subset, and $F : U \to \mathbb{RP}^3$ a sufficiently smooth map such that $F(U \cap L)$ lies in a plane for every line $L \subset \mathbb{RP}^2$. Then F is called a planarization.

Theorem

For every planarization $F : U \to \mathbb{RP}^3$, there exists an open subset $U' \subset U$ such that one of the following:

- *$F|_{U'}$ is trivial ($F(U')$ lies in a line)*
- *$F|_{U'}$ is co-trivial (planes containing $F(U' \cap L)$ have a common point)*
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Linear webs of conics

Let $\mathcal{L}$ be a linear system of conics of dimension 3 (a linear web of conics).

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Suppose that $f : U \to \mathbb{RP}^2$ is a sufficiently smooth map such that $f(U \cap L)$ is contained in a conic from $\mathcal{L}$ for every line L. Then there is an open subset $U' \subset U$ such that one of the following:

- *$f(U')$ lies in a conic from $\mathcal{L}$*
- *$f|_{U'}$ is a quadratic rational map*
- *$f|_{U'}$ is a local inverse of a quadratic rational map*
- *$f|_{U'} = \Phi^{-1} \circ F$, where F and Φ are fractional quadratic maps to an irreducible quadric $S \subset \mathbb{RP}^3$*

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Rectifiable pencils of conics

Theorem

Suppose that a local analytic diffeomorphism $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *takes all lines through 0 to conics and satisfies a minor non-degeneracy assumption. Then, for all lines* $l \ni 0$*, the conic* $f(l)$ *has 3 points of tangency with a curve of class 3 (i.e. dual curve to a cubic).*