On the Cayley degree
of an algebraic group

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Abstract

A connected linear algebraic group \( G \) is called a Cayley group if the Lie algebra of \( G \) endowed with the adjoint \( G \)-action and the group variety of \( G \) endowed with the conjugation \( G \)-action are birationally \( G \)-isomorphic. In particular, the classical Cayley map

\[
X \mapsto (I_n - X)(I_n + X)^{-1}
\]

between the special orthogonal group \( \text{SO}_n \) and its Lie algebra \( \mathfrak{so}_n \), shows that \( \text{SO}_n \) is a Cayley group. In an earlier paper we classified the simple Cayley groups defined over an algebraically closed field of characteristic zero. Here we consider a new numerical invariant of \( G \), the Cayley degree, which “measures” how far \( G \) is from being Cayley, and prove upper bounds on Cayley degrees of some groups.

1. Introduction

Let \( G \) be a connected linear algebraic group and let \( \mathfrak{g} \) be its Lie algebra. We say that \( G \) is a Cayley group if there is a birational isomorphism

\[
\varphi : G \dashrightarrow \mathfrak{g}
\]

which is equivariant with respect to the conjugation action of \( G \) on itself and the adjoint action of \( G \) on \( \mathfrak{g} \); see [3, Definition 1.5]. In particular, the classical Cayley map [1]

\[
X \mapsto (I_n - X)(I_n + X)^{-1}
\]

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References


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