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# HOUSEHOLD BEHAVIOR AND INDIVIDUAL AUTONOMY: AN EXTENDED LINDAHL MECHANISM

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We present a comprehensive model of household economic decision covering both fully cooperative and non-cooperative cases as well as semi-cooperative cases, varying with income distribution and a parameter vector θ representing degrees of individual autonomy with respect to the public goods. In this model, the concept of "household θ-equilibrium" is introduced through the reformulation of the Lindahl equilibrium for Nash-implementation and its extension to semi-cooperation. Existence is proved and some generic properties derived. An example is given to illustrate. Finally, a particular decomposition of the pseudo-Slutsky matrix is derived and the testability of the various models discussed.

*JEL codes*: D10, C72, H41.
*Keywords*: Intra-household allocation, household financial management, degree of autonomy, Lindahl prices, local income pooling, separate spheres.

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Мы представляем весьма общую модель принятия экономических решений в домохозяйствах, включая полностью кооперативные и некооперативные случаи, а также полукооперативные случаи, основанные на изменениях распределения доходов и вектора-параметра θ общественных благ. В этой модели понятие θ-равновесия домохозяйства вводится через переформулировку равновесия Линдалла для модели имплементации по Нэшу и ее обобщения на полукооперативный случай. Доказана теорема о существовании и изучены свойства соответствующих равновесий. Дан иллюстративный пример. Наконец, получена специальная декомпозиция псевдоматрицы Слуцкого и обсуждены возможности верификации различных моделей.

*Ключевые слова*: распределение в домохозяйстве, финансовый менеджмент в домохозяйстве, степень автономности, цены по Линдаллу, локальное объединение дохода, различные сферы.

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# Household behavior and individual autonomy: An extended Lindahl mechanism*

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**Abstract**

We present a comprehensive model of household economic decision covering both fully cooperative and non-cooperative cases as well as semi-cooperative cases, varying with income distribution and a parameter vector $\theta$ representing degrees of individual autonomy with respect to the public goods. In this model, the concept of "household $\theta$-equilibrium" is introduced through the reformulation of the Lindahl equilibrium for Nash-implementation and its extension to semi-cooperation. Existence is proved and some generic properties derived. An example is given to illustrate. Finally, a particular decomposition of the pseudo-Slutsky matrix is derived and the testability of the various models discussed.

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Keywords: Intra-household allocation, household financial management, degree of autonomy, Lindahl prices, local income pooling, separate spheres.

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# 1 Introduction

Most economic models of household behavior, both theoretical and empirical, have assumed that households act cooperatively, implying that binding marriage agreements under full information and perfect communication are feasible at no (or low) transaction costs. This cooperative approach includes the traditional, but empirically contested, "unitary models" viewing the household as a single decision unit, *e.g.* Samuelson's consensus model and Becker's altruist model. It also includes models based on axiomatic bargaining theory, as well as the more recent "collective" models, exploring the restrictions on observable household behavior implied by the assumption of Pareto efficiency without explicitly referring to some decision making process (Chiappori, 1988, 1992, Browning and Chiappori, 1998).

However, some theoretical models have formalised the possibility for a household agreement to be non-cooperative and in particular to be an equilibrium in a game of voluntary contributions to public goods. In these models (e.g. Ulph, 1988, Lundberg and Pollak, 1993, Chen and Woolley, 2001), the noncooperative equilibrium is introduced as the threat (or disagreement) point of a bargaining model[1] and, under some conditions, as a "separate spheres" equilibrium where each spouse is responsible for a distinct set of goods and services within the household, the partition being "defined in terms of traditional gender roles and gender roles expectations" (Lundberg and Pollak, 1993, p.1007).[2] When binding agreements are impossible or too costly to enforce, the non-cooperative agreement may even become the acceptable default option in a continuing relationship. In this context, the concept of noncooperative equilibrium for the household has been further analysed by Lechene and Preston (2005, 2011) and by Browning, Chiappori and Lechene (2010) who confirm previous results by

[1] This is different from earlier models where the disagreement point was taken to be divorce (Manser and Brown, 1980, and McElroy and Horney, 1981).

[2] Indeed, "specialization in the provision of such goods reduces the need for complex patterns of coordination, and traditional gender roles serve as a focal point for tacit division of responsibilities" (Lundberg and Pollak, 1993, p.993).

showing that there are generically only two possible types of noncooperative equilibria, "separate spheres" and "separate spheres up to one public good", the latter meaning that spouses do not contribute jointly to more than one public good.

Such a conclusion, though, seems to be too clearcut to give account of the variety of household arrangements. In reality, the definition, and hence the division, of tasks and responsibilities to provide some goods and services within the household is ambiguous and may vary over husband-wife pairs. "Child care" or "housekeeping" might be divided in many sub-tasks that can be differently shared. The perfect partition according to gender roles is a limit case that does not hold for most households in most contemporary societies. Also, arrangements within the household might change along life path (with the number of children, the income level, the geographical location, etc.). Good intentions erode with time and a genuine cooperative agreement might gradually evolve towards some more traditional division of responsibilities. The way in which the household organizes its finances, in particular whether each spouse has its own account and/or shares some joint account, as well as the type of marriage contract may influence this evolution.[3] As emphasized by Lechene and Preston (2005): "neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending behaviour and analysis of such extreme cases can be seen as a first step towards understanding of a more adequate model" (p. 19).

Our objective is to follow this route and develop a comprehensive model, that includes the cooperative and the noncooperative cases at the extremes, but allows for a continuum of intermediate "semi-cooperative" cases. This comprehensive model is based on a "mechanism design" reformulation of the Lindahl equilibrium for Nash-implementation (see Hurwicz, 1979, and Walker, 1981)

[3] Household money management systems have been extensively studied both empirically, essentially in the sociological literature (e.g. Pahl, 2008; see also the two surveys of the International Social Survey Programme of 1994 and 2002, analyzing representative samples of 38 countries) and experimentally (for recent studies see Ashraf, 2009, and Schaner, 2010).

and its extension to semi-cooperation. In order to implement the household collective objectives while preserving budgetary autonomy and feasibility, the proposed mechanism (formulated for simplicity in terms of a two-person household) is defined by non-manipulable personalized pricing rules for the public goods and by individual budget constraints preserving some autonomy to each spouse. Given this mechanism, the spouses choose strategically their individual contributions to the various public goods as well as their desired consumption of all goods. The solution of this game will be called a household (noncooperative) equilibrium.

In addition, testable (local) restrictions of household demand will be investigated, such as those studied under the assumption of efficiency by Browning and Chiappori (1998), relying on the decomposition of the pseudo-Slutsky matrix as the sum of a matrix with the Slutsky properties and a "deviation matrix" (of rank equal to 1 for a two-person household). Lechene and Preston (2011) derive a similar test for the fully non-cooperative model showing that the deviation matrix has generally a larger rank than in the collective model, increasing with the number of public goods. Our paper gives a derivation of the pseudo-Slutsky matrix allowing to separate different kinds of effects, each effect increasing the maximum possible rank of the deviation matrix. The implementation of such tests becomes more and more demanding in terms of the required number of goods. As we will see, however, the non-cooperative and the semi-cooperative models are distinguishable as soon as there is joint contribution to more than one public good.

In Section 2, we present a two-person household semi-cooperative decision model, defining the non-manipulable mechanism as well as two related approaches to the concept of "household equilibrium" characterized, either endogenously or exogenously, by the degrees of autonomy of each spouse for each public good. We illustrate via an example the implications of varying the degrees of autonomy. In Section 3, we examine the generic local properties of the household equilibrium and derive the pseudo-Slutsky matrix of the household demand, then discussing the testability of the various models. We conclude in

Section 4.

# 2 The household decision model

We study a two-adult household, consuming goods that are recognized by both spouses as being either private or public (within the household). Denote by $A$ (the wife) and $B$ (the husband) the two household members, and let $\left(q^A, q^B\right) \in \mathbb{R}_+^{2n}$ be the vector of consumption by the two members of the $n$ private goods and $Q \in \mathbb{R}_+^m$ the consumption vector of the $m$ public goods. The preferences of each spouse $J$ $(J = A, B)$ are represented by a utility function $U^J\left(q^J, Q\right)$, which is defined on $\mathbb{R}_+^{n+m}$, increasing and strongly quasi-concave. Each spouse $J$ is supposed to receive an initial income $Y^J \geq 0$, the total household income being $Y = Y^A + Y^B > 0$. Also assumed is an agreed upon mechanism which determines the game played by the household when deciding on its total consumption given the vector of private good prices $p \in \mathbb{R}_{++}^n$ and the vector of public good prices $P \in \mathbb{R}_{++}^m$. The first private good, assumed to be desired in any household environment, is taken as numéraire ($p_1 = 1$).

## 2.1 The Lindahl mechanism for cooperative household decisions

We start from the concept of Lindahl equilibrium, which is the best-known "decentralized" allocation mechanism[4] to allocate efficiently the cost of public goods within a group. The Lindahl scheme consists in supposing that there exists a pair of personalized (Lindahl) prices $\left(P^A, P^B\right) \in \mathbb{R}_+^{2m}$, satisfying $P^A + P^B = P$, which are posted within the household. However, the version we give of the concept is not the standard one. It is a "mechanism design" version *à la* Hurwicz (1979) and Walker (1981). We suppose that there is enough cooperation so that the two spouses can agree on some mechanism to share the expenses

[4] Introduced by Lindahl (1919) and popularized by Samuelson (1954). Cherchye, De Rock and Vermeulen (2007) also use Lindahl prices to analyze household decisions.

for financing the public goods on the basis of voluntary contributions $\left\{g_k^J\right\}$ and desired aggregate quantities $\left\{Q_k^J\right\}$ for each spouse $J$ and each public good $k$. This mechanism consists in posting *personalized pricing rules* $\left(\widehat{P}^A, \widehat{P}^B\right)$ for the public goods and in specifying individual budget constraints. As in Walker (1981), these personalized pricing rules have to be non-manipulable. In the present collective decision problem involving a husband-wife pair, we cannot use the personalized pricing rules proposed by Walker since they require at least three agents. But the following will do:

$$\widehat{P}_k^J\left(g_k^{-J}, Q_k^{-J}, P_k\right) \equiv \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k, \text{ for } J = A, B \text{ and any public good } k. \tag{1}$$

Given this mechanism a game is defined where the payoffs are the spouses' utility functions. The strategies of each spouse $J$ are the quantities $\left(q^J, g^J, Q^J\right) \in \mathbb{R}_+^{n+2m}$, denoting respectively the quantities of private goods, the voluntary contributions and the desired aggregate quantities for the various public goods. For each spouse $J$, these strategies have to respect two constraints. First there is a budget constraint on the spouse's expenses on private goods (at the market prices $p$) and on public goods (at the personalized prices $P^J$). Second there is a feasibility constraint, whereby the desired quantities $Q^J$ should be equal to the aggregate contributions $g_k^J + g_k^{-J}$. A Lindahl equilibrium for the household is then defined as the following Nash equilibrium of the game defined by the personalized pricing rules $\left(\widehat{P}^A, \widehat{P}^B\right)$:

**Definition 1** *A vector* $\left(q^A, g^A, Q^A, q^B, g^B, Q^B,\right) \in \mathbb{R}_+^{2n+4m}$ *is a* Lindahl (household) equilibrium *if the vector* $\left(q^J, g^J, Q^J\right)$ *solves, for each* $J = A, B$*, the program*

$$\max_{\left(\widetilde{q}^J, \widetilde{g}^J, \widetilde{Q}^J\right) \in \mathbb{R}_+^{n+2m}} U^J\left(\widetilde{q}^J, \widetilde{Q}^J\right) \tag{2}$$

$$\text{s.t.} \qquad p\widetilde{q}^J + \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J \leq Y^J$$

$$\text{and} \qquad \widetilde{Q}^J = \widetilde{g}^J + g^{-J}.$$

Observe that the Lindahl prices are taken as given by each spouse $J$ because of the Nash assumption of taking as given the equilibrium strategies of spouse $-J$. Also, we have $Q^A = g^A + g^B = Q^B$ at equilibrium, implying

$$P_k^J = \widehat{P}_k^J \left( g_k^{-J}, Q_k^{-J}, P_k \right) = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \text{ for } J = A, B \text{ and any public good } k. \tag{3}$$

This property (which implies that $P_k^A + P_k^B = P_k$ for every $k$) ensures the equivalence of this household equilibrium to the standard definition of a Lindahl equilibrium where individualized contributions $g_k^J$ are not introduced, but instead each individual chooses a desired total consumption $Q_k^J$ of public good $k$. At a Lindahl (household) equilibrium, $g_k^A$ and $g_k^B$ are either both positive or both nil for any public good $k$.[5]

For the sake of later comparisons, the first order conditions for a Lindahl household equilibrium (for $J = A, B$) can be easily derived (we can eliminate $Q^J$ since $Q^J = g^J + g^{-J}$ ):

$$\begin{aligned}
\frac{1}{\partial_{q_1} U^J \left(q^J, g^J + g^{-J}\right)} \partial_q U^J \left(q^J, g^J + g^{-J}\right) &\leq p, \qquad (4) \\
\frac{1}{\partial_{q_1} U^J \left(q^J, g^J + g^{-J}\right)} \partial_Q U^J \left(q^J, g^J + g^{-J}\right) &\equiv \tau^J \left(q^J, g^J + g^{-J}\right) \leq P^J, \\
pq^J + P^J \left(g^J + g^{-J}\right) &= Y^J,
\end{aligned}$$

$$\text{with } P_k^J = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \text{ for every public good } k,$$

and an inequality becoming an equality for any private good $i$ s.t. $q_i^J > 0$ or any public good $k$ s.t. $Q_k^J > 0$. These first order conditions entail the Bowen-Lindahl-Samuelson conditions for any interior Pareto efficient allocation.

[5] The case where spouse $J$ would prefer to diminish the consumption of public good $k$ at a Lindahl (household) equilibrium, but cannot because of a binding non-negative constraint on $g_k^J$, is eliminated. Indeed, $g_k^J = 0$ and $g_k^{-J} > 0$ imply $P_k^J = 0$, and hence a contradiction since, with $P_k^J = 0$, $g_k^J = 0$ could not be optimal since $U^J \left(q^J, Q^J\right)$ is increasing in $Q_k^J$.

## 2.2 Taking into account individual autonomy

From the point of view of Nash implementation, the mechanism we have defined to implement the Lindahl equilibrium has good properties, since it ensures both non-manipulability of the personalized pricing rules and Pareto efficiency. Also, at equilibrium, we have $P^J Q^J = P g^J$, hence $pq^J + P g^J = Y^J$ for each spouse $J$, implying that the chosen contributions are feasible for the household budget: they could actually be bought autonomously by each spouse instead of being paid by the Lindahl transfers to the common purse. However this autonomous feasibility might not hold out of equilibrium where aggregate contributions could be unfeasible even for the household total budget.

To always ensure such autonomous feasibility (and hence aggregate budgetary feasibility), we modify our Lindahl mechanism by introducing *individual autonomy* in each budget constraint, while keeping the same personalized pricing rules $\left(\widehat{P}^A, \widehat{P}^B\right)$. Each spouse $J$, when choosing a contribution $g_k^J$ to any desired public consumption $Q_k^J$ should keep the budgetary freedom either to pay the Lindahl tax $\widehat{P}_k^J Q_k^J$ to the common purse or to spend instead $P_k g_k^J$ directly in the market. In other words, any deviation by spouse $J$ should not only be autonomously feasible and comply with the obligation to take the agreed upon share of the household expenditure on each public good $k$ (*i.e.* the share resulting from the personalized pricing rule of the accepted mechanism), it should in addition preserve the individual autonomy for the partner (who can always go and buy $g_k^J$ in the market).[6] Of course, as for the Lindahl equilibrium, the equality of the "desired" quantities of public goods to aggregate contributions have to hold for both spouses only at equilibrium.

The non-cooperative equilibrium of the corresponding modified game, called a *household equilibrium*, still based on the personalized pricing rules $\left(\widehat{P}^A, \widehat{P}^B\right)$, can be defined as follows:

[6] Some money management systems could facilitate this type of individual autonomy. Each spouse having its own bank account and the household having one joint account is such a system (called 'partial pool' by Vogler, Brockmann and Wiggins, 2006). Of course this is neither necessary nor sufficient to ensure individual autonomy.

**Definition 2** *A vector* $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2n+4m}$ *is a* household equilibrium *if, for* $J = A, B$*, the strategy* $\left(q^J, g^J, Q^J\right)$ *solves the program:*

$$\max_{\left(\widetilde{q}^J, \widetilde{g}^J, \widetilde{Q}^J\right) \in \mathbb{R}_+^{n+2m}} U^J\left(\widetilde{q}^J, \widetilde{Q}^J\right) \tag{5}$$

$$\text{s.t.} \quad p\widetilde{q}^J + \sum_{k=1}^{m} \max\left\{\frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J, P_k \widetilde{g}_k^J\right\} \leq Y^J,$$

$$\text{and} \quad \widetilde{Q}^J = \widetilde{g}^J + g^{-J}. \tag{6}$$

This modification of the mechanism has considerable consequences. First, as will be shown in the following, there is now a continuum of household equilibria (that can be meaningfully parameterized), as opposed to the generic finiteness of the Lindahl equilibria. Second, it will follow that the set of equilibria includes the Lindahl equilibria at one extreme (the only efficient ones) and, at the other extreme, the noncooperative equilibria of the "*game with voluntary contributions to public goods*" (analyzed for household behavior by Lundberg and Pollak, 1993, Lechene and Preston, 2005, 2011, and Browning, Chiappori and Lechene, 2010). In this game, each spouse $J$ chooses a strategy $\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}$ ($q^J$ denoting $J$'s private consumptions and $g^J$ his/her contributions to public goods) in order to solve the programme:

$$\max_{\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}} U^J\left(q^J, g^J + g^{-J}\right) \tag{7}$$

$$\text{s.t.} \quad pq^J + Pg^J \leq Y^J.$$

A Nash equilibrium of this game can be characterized by the first order conditions (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^A + g^B\right) &\leq p \\ \tau^J\left(q^J, g^A + g^B\right) &\leq P \\ pq^J + Pg^J &= Y^J, \end{aligned} \tag{8}$$

with an equality for any private good $i$ s.t. $q_i^J > 0$ or any public good $k$ s.t. $g_k^J > 0$.

## 2.3 Characterization of a household equilibrium

The concept of household equilibrium can be alternatively formulated. In the previous definition of a household equilibrium, each spouse $J$ faces a nonsmooth optimization program (because of the max function in the budget constraint). This program can be replaced by an equivalent smooth program with $2^m$ budget constraints indexed by $\delta^J \in \Delta^J \equiv \{0,1\}^m$:

$$p\widetilde{q}^J + \sum_{k=1}^{m} \left( \delta_k^J P_k \widetilde{g}_k^J + \left(1 - \delta_k^J\right) \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J \right) \leq Y^J. \tag{9}$$

Each $\delta^J \in \Delta^J$ describes a configuration specifying the public goods for which spouse $J$ goes to the market ($\delta_k^J = 1$) and those for which $J$ accepts to contribute to the common purse ($\delta_k^J = 0$). All such configurations have to be budgetary feasible for spouse $J$. We then have the following result.

**Lemma 1** *A household equilibrium is characterized by the following system of inequalities and equalities for $J = A, B$:*

$$\frac{1}{\partial_{q_1} U^J \left(q^J, g^J + g^{-J}\right)} \partial_q U^J \left(q^J, g^J + g^{-J}\right) \leq p \tag{10}$$

*and, for all $k$ and some $\theta_k^J \in [0,1]$,*

$$\begin{aligned} & \frac{1}{\partial_{q_1} U^J \left(q^J, g^J + g^{-J}\right)} \partial_{Q_k} U^J \left(q^J, g^J + g^{-J}\right) \equiv \tau_k^J \left(q^J, g^J + g^{-J}\right) \\ \leq \quad & \theta_k^J P_k + \left(1 - \theta_k^J\right) P_k^J, \text{ with } P_k^J = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \\ & pq^J + \sum_{k=1}^{m} \left( \theta_k^J P_k g_k^J + \left(1 - \theta_k^J\right) P_k^J \left(g_k^J + g_k^{-J}\right)\right) = Y^J, \end{aligned} \tag{11}$$

*an inequality becoming an equality for any private good $i$ s.t. $q_i^J > 0$ or any public good $k$ s.t. $Q_k^J > 0$.*

**Proof.** Suppose that, in the definition of a household equilibrium, we replace the budget constraint in program (5) by the set of constraints (9). The program of each spouse consists then in maximizing a strongly quasi-concave utility function under linear constraints. Taking first order conditions we immediately get the wanted inequalities for the private goods and, for any public good $k$ and spouse $J$, and for some vector of Lagrange multipliers $\lambda^J_{\delta^J} \in \mathbb{R}^{2^m}_+$,

$$\begin{aligned} & \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_{Q_k} U^J\left(q^J, g^J + g^{-J}\right) \\ \leq \quad & \frac{1}{\sum\limits_{\delta^J \in \Delta^J} \lambda^J_{\delta^J}} \left( \sum_{\delta^J \in \left\{\Delta^J : \delta^J_k = 1\right\}} \lambda^J_{\delta^J} P_k + \sum_{\widehat{\delta}^J \in \left\{\Delta^J : \widehat{\delta}^J_k = 0\right\}} \lambda^J_{\widehat{\delta}^J} \widehat{P}^J_k\left(g^{-J}_k, Q^{-J}_k, P_k\right) \right) \\ \equiv \quad & \theta^J_k P_k + \left(1 - \theta^J_k\right) P^J_k \text{ (with } \theta^J_k \text{ the normalized Lagrange multiplier)}, \end{aligned}$$

so that we get the wanted inequality for public good $k$ using the fact that $\widehat{P}^J_k\left(g^{-J}_k, Q^{-J}_k, P_k\right) = P^J_k$ at equilbrium. Finally, because $P^J_k\left(g^J_k + g^{-J}_k\right) = P_k g^J_k$ for every $k$, we get the equality

$$pq^J + \sum_{k=1}^{m} \left( \theta^J_k P_k g^J_k + \left(1 - \theta^J_k\right) P^J_k \left(g^J_k + g^{-J}_k\right) \right) = Y^J,$$

and the result follows. ■

Comparing the characterization of a household equilibrium with the one of a Lindahl equilibrium, we immediately see that the two coincide for $\theta^J_k = 0$, for all $k$ and $J = A, B$. Similarly, the household equilibrium coincides with the Nash equilibrium of the game with voluntary contributions to public goods for $\theta^J_k = 1$, for all $k$ and $J = A, B$. But there are many such equilibria, at least one for each $\theta = \left(\theta^A, \theta^B\right) \in [0,1]^{2m}$, as it will be shown in the next Subsection. Considering the "smooth" definition of a household equilibrium with $2^m$ budget constraints (9) for $J = A, B$, we see that each $\theta^J_k$ is the (normalized) sum of the shadow prices associated with those $J$'s budget constraints corresponding to an autonomous purchase of public good $k$ in the market ($\delta^J_k = 1$). Each $\theta^J_k$ can thus be interpreted as the price $J$ would be ready to pay to relax anyone of the constraints. In that sense $\theta^J_k$ measures the *degree of autonomy* of spouse $J$ with respect to public good $k$. This interpretation is reinforced by the way the budget constraint is written in the characterization of a household equilibrium given in the lemma. The expenses of spouse $J$ to finance its contribution to public good $k$ are divided into two parts, one autonomous, valued at market price

$P_k$, the other collective, valued at the personalized price $P_k^J$. This suggests an alternative approach to the household game.

### 2.4 The game with given degrees of autonomy

In our analysis, up to now, the degrees of autonomy of both spouses are fixed endogenously, *ex post*, as characteristics of a specific equilibrium. We may however invert the approach, and take the parameters as preliminarily and conventionally fixed, *ex ante,* within the household. Otherwise, we keep the same mechanism, with the same strategies and the same payoffs. This leads to the following definition of equilibrium relative to fixed degrees of autonomy $\left(\theta^A, \theta^B\right) \in [0,1]^{2m}$ and based on the personalized pricing rules $\left(\widehat{P}^A, \widehat{P}^B\right)$:

**Definition 3** *A vector* $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2n+4m}$ *is a* household $\theta$-equilibrium *with degrees of autonomy* $\left(\theta^A, \theta^B\right) \in [0,1]^{2m}$ *if, for* $J = A, B$*, the strategy* $\left(q^J, g^J, Q^J\right)$ *solves the program:*

$$\max_{\left(\widetilde{q}^J, \widetilde{g}^J, \widetilde{Q}^J\right) \in \mathbb{R}_+^{n+2m}} U^J\left(\widetilde{q}^J, \widetilde{Q}^J\right) \tag{12}$$

$$\textit{s.t.} \qquad p\widetilde{q}^J + \sum_{k=1}^{m}\left(\theta_k^J P_k \widetilde{g}_k^J + \left(1-\theta_k^J\right)\frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J\right) \leq Y^J,$$

$$\textit{and} \qquad \widetilde{Q}^J = \widetilde{g}^J + g^{-J}.$$

In this concept, the division in two parts of the expenses of each spouse on each public good is made explicit through the budget constraint, according to the pre-established degrees of autonomy for that public good. The following proposition clarifies the relationship between the two concepts of household equilibrium and of household $\theta$-equilibrium.

**Proposition 1** *A vector* $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2n+4m}$ *with associated normalized Lagrange multipliers* $\theta \in [0,1]^{2m}$ *is a household equilibrium for the personalized pricing rules* $\left(\widehat{P}^A, \widehat{P}^B\right)$ *if and only if it is a household* $\theta$*-equilibrium. For each* $\theta \in [0,1]^{2m}$*, there is such an equilibrium. When* $\theta_k^J = 0$ *for any* $J$ *and any* $k$*, this is a Lindahl equilibrium. When* $\theta_k^J = 1$ *for any* $J$

*and any $k$, it is a Nash equilibrium of the game with voluntary contributions to public goods.*

**Proof.** In both cases, the program of each spouse consists in maximizing a strongly quasi-concave utility function under linear constraints. Taking the first order conditions for a household $\theta$-equilibrium with degrees of autonomy $\left(\theta^A, \theta^B\right)$, we see that they coincide with those for a household equilibrium with associated normalized Lagrange multipliers $\left(\theta^A, \theta^B\right)$, as given by Lemma 1. These conditions also coincide with the first order conditions for a Lindahl (household) equilibrium, as given by (4), when $\theta_k^J = 0$ for any $J$ and $k$, and with the first order conditions for a Nash equilibrium of the game with voluntary contributions to public goods, as given by (8), when $\theta_k^J = 1$ for any $J$ and $k$.

As to existence of a household $\theta$-equilibrium for every $\theta \in [0,1]^{2m}$, consider this equilibrium as one of a generalized game as in Debreu (1952). The strategies $\left(q^J, g^J, Q^J\right)$ of $J$ are constrained to be in a compact, convex, non-empty set $S^J$ defined by $0 \leq q_i^J \leq Y^J/p_i$ for $i = 1, ..., n$, and $0 \leq g_k^J \leq Q_k^J \leq Y/P_k$ for $k = 1, ..., m$. We further define for each spouse $J$ a choice correspondence associating with each strategy $\left(q^{-J}, g^{-J}, Q^{-J}\right) \in S^{-J}$, the subset of $S^J$ satisfying $Q^J = g^J + g^{-J}$ and

$$pq^J + \sum_{k=1}^{m} \left( \theta_k^J P_k g_k^J + \left(1 - \theta_k^J\right) P_k \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} \left(g_k^J + g_k^{-J}\right) \right) \leq Y^J.$$

This correspondence is continuous, and non-empty convex valued. Since the payoff functions $U^J$ are continuous and quasi-concave, Debreu's (1952) social equilibrium existence theorem applies. ■

Notice that, whenever it exists, a household $\theta$-equilibrium with *separate spheres*, namely with $g_k^A g_k^B = 0$ for all $k$, coincides with an equilibrium of the game with voluntary contributions to public goods. Indeed, if $g_k^{-J} = 0$ whereas $g_k^J > 0$, then $P_k^{-J} = 0$ and $P_k^J = P_k$. As a consequence, the planned unit expenditure on public good $k$ for a deviation by spouse $J$ is $\theta_k^J P_k + \left(1 - \theta_k^J\right) P_k = P_k$, the same as in the game with voluntary contributions to public goods. As to spouse $-J$, who decides not to contribute to public good $k$ when the planned unit expenditure on this good is $\theta_k^{-J} P_k$, he/she would not want to deviate in the game with voluntary contributions to public goods when the unit expenditure is $P_k \geq \theta_k^J P_k$. This proves the following proposition.

**Proposition 2** *Let $0 < \theta_k^J \leq 1$ for any $J$ and $k$. Suppose $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2n+4m}$ is a household $\theta$-equilibrium such that $g_k^A g_k^B = 0$ for all $k$ (*separate spheres*). Then $\left(q^A, g^A, q^B, g^B\right)$ is a Nash equilibrium of the game with voluntary contributions to public goods.*

A last remark in this subsection concerns Pareto efficiency. Take the first order conditions relative to the public good $k$ for both spouses' programs, as given by Lemma 1. For efficiency, the Bowen-Lindahl-Samuelson condition requires that the sum $\tau_k^A + \tau_k^B$ of the two marginal willingnesses to pay be equal to the corresponding market price $P_k = P_k^A + P_k^B$. This condition is generally violated as soon as cooperation is less than full. Indeed, the sum of the two marginal willingnesses to pay is equal, if both spouses contribute to public good $k$, to $P_k + \theta_k^A P_k^B + \theta_k^B P_k^A$, larger than $P_k$ outside the case $\theta_k^A = \theta_k^B = 0$, and the more so the higher the degrees of autonomy of the two spouses. Also, if a spouse, say the wife, contributes alone to public good $k$, $\tau_k^A = P_k$, so that $P_k < \tau_k^A + \tau_k^B$, leading to a similar conclusion.

## 2.5 An example

In order to discuss the conditions leading to the regimes of separate spheres and of joint contribution to one or more public goods, we shall refer to the example used by Browning, Chiappori & Lechene (2010) in the context of full non-cooperation. In this example, the spouses have Cobb-Douglas preferences over one private and two public goods. We denote by $x$ and $z$ the private consumptions of spouses $A$ and $B$, respectively, and by $X$ and $Z$ the quantities of the two public goods. The utility functions are given by:

$$U^A(x, X, Z) = xX^a Z^\alpha \text{ and } U^B(z, X, Z) = zX^b Z^\beta, \tag{13}$$

with positive parameter values $a$, $\alpha$, $b$ and $\beta$. The wife $A$ is supposed to care more about the first public good, and the husband $B$ about the second, so that

$$s \equiv \frac{\alpha/a}{\beta/b} < 1, \tag{14}$$

where the term on the LHS can be taken as the *degree of symmetry* of the spouses' preferences for the two public goods. We use the normalization

$$p_x = p_z = P_X = P_Z = Y = 1, \tag{15}$$

with an income distribution given by $Y^A = \rho$ and $Y^B = 1 - \rho$. The degrees of autonomy are assumed to be symmetric with respect to the two public goods, so that they will be simply denoted $\theta^A$ and $\theta^B$.

Browning, Chiappori & Lechene (2010) show the existence of three kinds of regimes. Two kinds correspond to separate spheres, prevailing both for extremely unequal income distributions, where the spouse with the higher income contributes alone to both public goods, and for relatively equal income distributions, where each spouse contributes to her/his preferred public good. The third kind appears in intermediate cases of income distribution and is characterized by separate spheres up to one public good (and local income pooling): the spouse with the higher income contributes to both public goods, while the other spouse contributes solely to her/his preferred public good. A fourth kind of regime, with both spouses contributing to both public goods, is generically excluded under full non-cooperation, but not in the semi-cooperative case, as we shall show.

- Consider the regime of *separate spheres* where both spouses contribute, $A$ to her preferred public good ($X$) and $B$ to his ($Z$). Hence, $P_Z^A = P_X^B = 0$ and $P_X^A = P_Z^B = 1$. By the first order conditions for public goods,

$$\begin{aligned} ax/X &= \theta^A + \left(1 - \theta^A\right) P_X^A = 1,\ \alpha x/Z < \theta^A + \left(1 - \theta^A\right) P_Z^A = \theta^A, \\ \beta z/Z &= \theta^B + \left(1 - \theta^B\right) P_Z^B = 1,\ bz/X < \theta^B + \left(1 - \theta^B\right) P_X^B = \theta^B. \end{aligned} \tag{16}$$

Using the equilibrium budget equations

$$x + X = \rho,\ z + Z = 1 - \rho, \tag{17}$$

we easily obtain the solution

$$x = \frac{\rho}{1+a},\ X = \frac{a\rho}{1+a},\ z = \frac{1-\rho}{1+\beta},\ Z = \frac{\beta(1-\rho)}{1+\beta}. \tag{18}$$

This solution is constrained by the two first order conditions on the non-contributed goods, expressed as inequalities:

$$\frac{\alpha}{\beta}\frac{1+\beta}{1+a}\frac{\rho}{1-\rho} < \theta^A,\ \frac{b}{a}\frac{1+a}{1+\beta}\frac{1-\rho}{\rho} < \theta^B. \tag{19}$$

Clearly, one of these two conditions will be violated for small enough or large enough values of $\rho$, so that separate spheres with both spouses contributing to public consumption can prevail only if the income distribution between the two spouses is not too unequal. Also, by multiplying both sides of the first inequality by the corresponding sides of the second, we obtain

$$0 < s < \theta^A \theta^B \leq 1. \tag{20}$$

Hence, existence of this regime requires a relatively high average degree of autonomy of the two spouses, the higher the larger the degree of symmetry of their preferences for the two public goods. The fully non-cooperative case, where $\theta^A\theta^B = 1$, always satisfies this condition, provided there is no full symmetry in the spouses' preferences.

- Now consider the regime of *joint contribution to both public goods*, which is generically excluded under full autonomy of the two spouses. By first order conditions, the marginal willingnesses-to-pay for the public goods are:

$$\begin{aligned} ax/X &= \theta^A + \left(1-\theta^A\right)P_X^A,\ \alpha x/Z = \theta^A + \left(1-\theta^A\right)P_Z^A, \\ \beta z/Z &= \theta^B + \left(1-\theta^B\right)P_Z^B,\ bz/X = \theta^B + \left(1-\theta^B\right)P_X^B. \end{aligned} \tag{21}$$

Division of both sides of the second and third equations by the corresponding sides of the first and fourth, respectively, leads to

$$\frac{a}{\alpha}\theta^A \leq \frac{a}{\alpha}\frac{\theta^A + \left(1-\theta^A\right)P_Z^A}{\theta^A + \left(1-\theta^A\right)P_X^A} = \frac{X}{Z} = \frac{b}{\beta}\frac{\theta^B + \left(1-\theta^B\right)P_Z^B}{\theta^B + \left(1-\theta^B\right)P_X^B} \leq \frac{b}{\beta}\frac{1}{\theta^B}, \tag{22}$$

the two inequalities being easily checked to be true (by taking the extreme values $P_X^A = 1$ and $P_Z^A = 0$). We thus obtain

$$0 \leq \theta^A\theta^B \leq s < 1, \tag{23}$$

an existence condition just opposite to the one we found for separate spheres. For both spouses to contribute to both public goods their average degree of autonomy must be small enough, the smaller the more asymmetric their preferences for the public goods.

Thus, for not too unequal income distributions, separate spheres appear as a characteristic of high individual autonomy in household decisions. As the spouses become less and less autonomous, the regime prevailing when their incomes are not too different is rather the one where both contribute to both public goods, which is the rule under full cooperation.

More generally, in order to represent the parameter configurations leading to the different regimes, we take the same values as in Browning, Chiappori and Lechene (2010), namely $a = 5/3$, $\alpha = 8/9$, $b = 15/32$ and $\beta = 1/2$, leading to a degree of symmetry equal to 0.5, and we stick now to equal degrees of autonomy $\theta^A = \theta^B = \theta$. This is given in Figure 1, with $\theta$ and $\rho$ varying from 0 to 1 along the horizontal and the vertical axes, respectively. Six different regimes are possible: (I) where $B$ is the only spouse to contribute to (both) public goods, (II) where $A$ contributes to her preferred public good and $B$ still contributes to both, (III) where each spouse specializes on his/her preferred public good (separate spheres with both spouses contributing), (IV) and (V) symmetric to (II) and (I) respectively (with inverted roles of $A$ and $B$), and (VI) where both spouses contribute to both public goods.

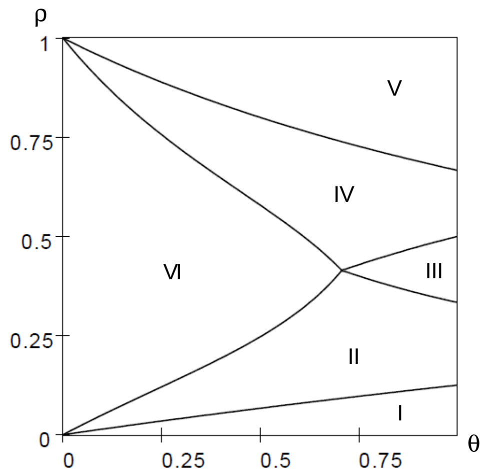


Figure 1: Regime switching values of $\rho$ as $\theta$ varies

As already mentioned, we see that the regime (III) of separate spheres exists only for a sufficiently high degree of autonomy (higher than the square root of the degree of symmetry $\sqrt{1/2} = 0.707$), and for relatively equal income shares. The corresponding regime (VI) for a lower degree of autonomy is the one where both spouses contribute to both public goods, allowing for more and more income disparities as we approach full cooperation. By contrast, the regimes (I) and (V) of exclusive contribution to public spending by the richer spouse are compatible with a lower and lower amplitude in income distribution as we approach full non-cooperation.

## 3 Local analysis of household $\theta$-equilibria

We shall now successively examine the local properties of the system of equations characterizing a household $\theta$-equilibrium (extending the analysis in Browning, Chiappori and Lechene, 2010), and the local properties of the associated household demand function (extending the analysis in Browning and Chiappori, 1998, and Lechene and Preston, 2010).

### 3.1 Local determinacy

For an environment $(p, P, Y)$ and an income distribution $\left(Y^A, Y^B\right)$, take a household $\theta$-equilibrium $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2n+4m}$, and refer to the partition $\left\{M^A, M^B, M^{AB}, M^0\right\}$ of the set $M$ of public goods, where $M^A$ and $M^B$ are the subsets of goods exclusively contributed by spouses $A$ and $B$, respectively, $M^{AB}$ is the subset of goods to which both spouses contribute and $M^0$ is the subset of goods that are not consumed by the household at this equilibrium. Denote by $m^A$, $m^B$, $m^{AB}$ and $m^0$ the cardinals of the respective subsets in this partition.

The variables $Q^A$ and $Q^B$ can be ignored in the following, since they are equal to the the sum $g^A + g^B$. Also, $m^A + m^B + 2m^0$ variables characterizing the equilibrium are trivially determined, namely $g_k^J = 0$ for $k \in M^{-J} \cup M^0$, $J = A, B$. Thus, we are left with $2n + 2m - \left(m^A + m^B + 2m^0\right)$ unknowns. There are 2 budget equations, $2(n-1)$ equations expressing the first order conditions for the private goods,[7] and $m^A + m^B + 2m^{AB}$ equations expressing the first order conditions for public goods. Hence, the number of unknowns is equal to the number of equations, so that an equilibrium is (generically) locally determinate.

Now, observe that the contributions to public consumption appear in the first order conditions only in the sums $g_k^A + g_k^B$ as soon as there is full autonomy for public good $k$: $\theta_k^A = \theta_k^B = 1$. Let $M^\theta$ be the subset of $M^{AB}$ (with cardinal $m^\theta$) for which this is the case. Then, the equation system reduced to the $2(n-1) + m^A + m^B + 2m^{AB}$ first order conditions has only $2n + m^A + m^B + 2m^{AB} - m^\theta$ unknowns, namely $q_i^J$ (for $J = A, B$ and $i = 1, ..., n$), $g_k^J$ (for $J = A, B$ and $k \in M^J \cup \left(M^{AB} \diagdown M^\theta\right)$) and $g_k^A + g_k^B$ (for $k \in M^\theta$). Hence, there is an excess of the number of equations over the number of unknowns equal to $m^\theta - 2$, so that there is generically *over*determinacy if $m^\theta > 2$, or even if $m^\theta = 2$, since the household consumption is entirely determined in this case by the sole first

[7] We are assuming for simplicity that all the $n$ private goods are consumed by both spouses at the equilibrium we refer to.

order conditions, independently of any budget constraint. Determinacy thus implies either $m^\theta = 0$ or $m^\theta = 1$: there is at most one public good for which the autonomy of the two spouses is full which is jointly contributed. If there is full autonomy with respect to public good $k$, which is jointly contributed, we may ignore the decomposition of $Q_k$ into $g_k^A$ and $g_k^B$, and completely determine the system by just adding the household budget equation. In other words, there is *local income pooling*.

This analysis can be straightforwardly applied to the game with voluntary contributions to public goods (where $m^\theta = m^{AB}$). Generically, either $m^{AB} = 0$ (*separate spheres*), with the two individual budget equations making the system determinate, or $m^{AB} = 1$ (*separate spheres up to one public good*), with *local income pooling*, as shown by Browning, Chiappori and Lechene (2010).

### 3.2 Foundations of the spouses' demand functions

In order to pursue our local analysis, let us establish, for exogenous degrees of autonomy, the foundations of the spouses' demand functions to be aggregated into the household demand function. The Marshallian demand function of spouse $J \in \{A, B\}$, conditional to a given choice $g^{-J} \in \mathbb{R}_+^m$ of the other spouse, can be straightforwardly derived from his/her utility maximization program:

$$x^J\left(p, \mathcal{P}^J, \mathcal{Y}^J, g^{-J}\right) \equiv \arg \max_{\substack{(q^J, g^J) \in \mathbb{R}_+^{n+m} \\ pq^J + \mathcal{P}^J g^J \leq \mathcal{Y}^J,}} U^J\left(q^J, g^J + g^{-J}\right) \tag{24}$$

with $\mathcal{P}_k^J \equiv \theta_k^J P_k + \left(1 - \theta_k^J\right) P_k^J = P_k - \left(1 - \theta_k^J\right) P_k^{-J}$ for $k = 1, ..., m$, and $\mathcal{Y}^J \equiv Y^J - \sum_k \left(1 - \theta_k^J\right) P_k^J g_k^{-J}$, equal at a household $\theta$-equilibrium to $Y^J - \sum_k \left(1 - \theta_k^J\right) P_k^{-J} g_k^J$.

We shall limit our local analysis to an open set $\Omega \subset \mathbb{R}_+^{n+m+1}$ of environment values $\omega \equiv (p, P, Y)$. We assume that the sharing rule $\rho \in (0, 1)$, defined by $\left(Y^A, Y^B\right) \equiv \left(\rho^A, \rho^B\right) Y \equiv (\rho, 1 - \rho) Y$, and the degrees of autonomy $\left(\theta^A, \theta^B\right) \in (0, 1)^{2m}$ of the two spouses are differentiable functions of the environment, which are defined on $\Omega$. We further assume equilibrium *uniqueness*,[8] as well as *no regime switching* over $\Omega$: the private goods purchased by each spouse and the public goods to which she/he *actually* contributes (corresponding to the non-zero elements of equilibrium vectors $\left(q^A, g^A\right)$ and $\left(q^B, g^B\right)$) are the same for each element of $\Omega$. This allows us to refer to differentiable functions $g^J : \Omega \to \mathbb{R}_+^m$ $(J = A, B)$ associating with each environment value $\omega$ the equilibrium values of the individual contributions $g^J(\omega)$. Because of no regime switching, we may admit without loss of generality (by simple redefinition of the utility function) that all private and public goods are consumed at equilibrium. We keep the notation of the preceding Subsection: $M^J$ and $M^{AB}$ for the sets of public goods contributed by spouse $J$ and by both spouses, respectively, $m^J$ and $m^{AB}$ for the corresponding cardinals (with $m^A + m^B + m^{AB} = m$).

By differentiability of the functions $\rho$, $\theta^J$ and $g^J$ (implying that of the functions $P^J$ with $P_k^J(\omega) = P_k g_k^J(\omega) / \left(g_k^J(\omega) + g_k^{-J}(\omega)\right)$), we obtain from the differentiable Marshallian conditional demand functions $x^J$, as defined by (24), differentiable functions $\xi^J : \Omega \to \mathbb{R}_+^{n+m}$, depending directly upon the environment. We take their sum $\xi = \xi^A + \xi^B$ as the household demand function.

## 3.3 The pseudo-Slutsky matrix of household demand at a household $\theta$-equilibrium

At any equilibrium, we can straightforwardly compute the pseudo-Slutsky matrix $\Psi = \left[\partial_{(p,P)}\xi\right] + \left[\partial_Y \xi\right]\left[{}^T\left(x^A + x^B\right)\right]$ of the household demand function as the sum of the two corresponding matrices $\Psi^J = \left[\partial_{(p,P)}\xi^J\right] + \left[\partial_Y \xi^J\right]\left[{}^T\left(x^A + x^B\right)\right]$ of the individual demand functions $\xi^J$, for $J = A, B$. By using the notations $\widetilde{P}_k^{-J} = \left(1 - \theta_k^J\right) P_k^{-J}$, so that $\mathcal{P}^J = P - \widetilde{P}^{-J}$ and $\mathcal{Y}^J = \rho^J Y - \widetilde{P}^{-J} g^J$, $\mathbf{\Pi}^J \equiv \left[\partial_{(p,P)}\widetilde{P}^J\right] + \left[\partial_Y \widetilde{P}^J\right]\left[{}^T\left(x^A + x^B\right)\right]$, $\mathbf{\Gamma}^J \equiv \left[\partial_{(p,P)} g^J\right] + \left[\partial_Y g^J\right]\left[{}^T\left(x^A + x^B\right)\right]$

[8] Lechene and Preston (2010), studying the fully non-cooperative case, also rely on the uniqueness assumption.

and $\mathbf{R} = \left[\partial_{(p,P)}\rho\right] + (\partial_Y \rho)\left[^T \left(x^A + x^B\right)\right]$, we obtain:

$$
\begin{aligned}
\Psi \quad = \quad & \overbrace{\left[\partial_{(p,\mathcal{P})}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[^T x^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_{(p,\mathcal{P})}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[^T x^B\right]}^{\Sigma^B} \qquad (25) \\
& - \overbrace{\left(\left[\partial_{\mathcal{Y}}x^A\right] - \left[\partial_{\mathcal{Y}}x^B\right]\right)\left[^T \left(\rho^B x^A - \rho^A x^B\right) - Y\mathbf{R}\right]}^{\Delta} \\
& + \overbrace{\begin{pmatrix} \left(\left[\partial_g x^A\right] - \left[\partial_{\mathcal{Y}}x^B\right]\left[\widetilde{P}^A\right]\right)\mathbf{\Gamma}^B + \\ \left(\left[\partial_g x^B\right] - \left[\partial_{\mathcal{Y}}x^A\right]\left[\widetilde{P}^B\right]\right)\mathbf{\Gamma}^A \end{pmatrix}}^{\Xi} \\
& - \overbrace{\begin{pmatrix} \left(\left[\partial_{\mathcal{P}}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[^T g^A\right]\right)\mathbf{\Pi}^B + \\ \left(\left[\partial_{\mathcal{P}}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[^T g^B\right]\right)\mathbf{\Pi}^A \end{pmatrix}}^{\Theta}.
\end{aligned}
$$

The two matrices $\Sigma^A$ and $\Sigma^B$ are the Slutsky matrices of the individual demand functions $\xi^A$ and $\xi^B$. Their sum $\Sigma$ has all the properties of a Slutsky matrix. However, the pseudo-Slutsky matrix $\Psi$ differs from $\Sigma$ by a deviation matrix $\Sigma - \Psi = \Delta - \Xi + \Theta$, which is the sum of three matrices. The first one, $\Delta$, the only element of $\Sigma - \Psi$ in the fully cooperative case, is an outer product, with rank no larger than 1, as shown by Browning and Chiappori (1998). It expresses an *aggregation effect*, working in the general case where there is no "representative consumer", independently of the existence of public goods. The second matrix, $-\Xi$, expresses an *externality effect*, working through the presence of the contributions of spouse $-J$ in the second argument of the utility function of spouse $J$. This effect appears in the fully non-cooperative case, in addition to the aggregation effect and has been analysed by Lechene and Preston (2010). This externality effect extends in our framework to the net income $\mathcal{Y}^J$ because of Lindahl taxation. The third matrix, $\Theta$, expresses the sum of the *substitution and income effects* of changes in the transformed personalized prices $\widetilde{P}^J$. The following proposition establishes the maximum rank of the deviation matrix $\Sigma - \Psi$ at different regimes.

**Proposition 3** *At an equilibrium associated with an environment* $\omega = (p, P, Y)$, *the household demand function has a pseudo-Slutsky matrix* $\Psi$ *which deviates* *from a Slutsky matrix* $\Sigma = \Sigma^A + \Sigma^B$*: (i) under separate spheres, by a matrix* $\Delta - \Xi$ *of rank at most equal to*

$$r_{\Delta-\Xi} = m + \min\left\{n - \max\left\{m^A - m^B, 1\right\}, 1\right\},$$

*where* $m^A - m^B$ *is assumed non-negative WLOG; (ii) under joint contribution to* $m^{AB}$ *public goods, by a matrix* $\Delta - \Xi + \Theta$ *of rank at most equal to*

$$r_{\Delta-\Xi+\Theta} = m + 3m^{AB} + \min\left\{n - \max\left\{m^A - m^B - 1, 1 + 3m^{AB}\right\}, 1\right\}.$$

**Proof.** See Appendix. ■

## 3.4 Empirical testing

The upper bound imposed upon the rank of the deviation matrix can be used to test the different models of household behavior. Browning and Chiappori (1998) have used this upper bound to discriminate between the unitary model (which predicts that the matrix $\Psi - \left({}^T\Psi\right)$ has rank 0 because of the symmetry of $\Psi = \Sigma$) and the collective model (which predicts that $\Psi - \left({}^T\Psi\right)$ has rank at most 2, since $\Psi = \Sigma - \Delta$, $\Delta$ having a rank at most equal to 1). They have also shown that this test requires at least 5 goods, a requirement that stems from the fact that the rank of $\Psi - \left({}^T\Psi\right)$ cannot be higher than 2 if the number $n+m$ of goods is not larger than 4 (given the linear dependence of the columns of $\Psi$ introduced by the homogeneity of degree zero of the demand functions).

Lechene and Preston (2011) have shown that, in order to reject the non-cooperative model, one must have $n \geq m + 5$. Their Lemma A.1 shows indeed that, if $\Psi - \left({}^T\Psi\right)$ has rank at most $n + m - 1$, then $\Psi$ can always be expressed as the sum of a symmetric matrix and a matrix of rank not higher than $r$ such that $2r + 1 \geq n + m - 1$. For $r = m + 1$, as in the non-cooperative case, the test works only if $2(m + 1) + 1 < n + m - 1$, that is, if $n > m + 5$.

As previously emphasized, there is no possibility of discriminating between full and partial autonomy under separate spheres, since the non-cooperative and the semi-cooperative models are then observationally equivalent. Discrimination between the two models becomes possible under joint contribution to at least one public good. In the semi-cooperative case, if we apply Lechene and Preston (2011) Lemma A.1 to the preceding Proposition, with $r = m + 3m^{AB} + 1$, we see that $n > m + 2\left(3m^{AB} + 2\right)$ is needed to discriminate between full and partial autonomy.[9] Moreover, as soon as there is joint contribution to more than one public good,[10] the non-cooperative model is generically excluded, and full cooperation is still rejected at a rank of $\Psi - \left({}^{T}\Psi\right)$ larger than 4, leaving the possibility of semi-cooperation.

# 4 Conclusion

Our purpose in this paper has been to develop models of the household that do not impose collective efficiency, an assumption often contradicted by empirical evidence,[11] and that do not limit non-cooperation to the pure voluntary contributions model, the consequences of which seem to be rather special (separate spheres or separate spheres up to one public good). However collective efficiency and pure non-cooperative behavior are not excluded but included as particular models in a more comprehensive approach trying to give account of the large variety of formal and informal contractual arrangements and decision procedures that are used by households all over the world. The concept of household equilibrium is meant to be flexible by allowing the household members to adopt various degrees of autonomy, either determined endogenously as characteristics of a specific equilibrium among many others, or taken as parameters preliminarily and conventionally fixed within the household and reflecting some selection process.

[9] If, for instance, we are in the special case of only one public good to which both spouses contribute, at least 12 private goods are required. The maximum possible rank of $\Psi - \left({}^{T}\Psi\right)$, given homogeneity of degree 0 of the demand functions, is then 12. As the observed rank increases from 0 to 10, the test successively rejects the unitary model (at 2), full cooperation (at 4), full autonomy (at 6) and the semi-cooperative model as a whole (at 12).

[10] In a recent empirical application of the non-cooperative model to data drawn from the Russia Longitudinal Monitoring Survey, Cherchye, Demuynck and De Rock (2011) could not reject joint contributions to two or three public goods by some households.

[11] See, for example, Udry (1996).

An important issue that is raised by these results is the identification of a good consumed by the household as being private or public (or semi-public). As we have seen, this is crucial for the testability of the various models.[12] The public or private nature of a good is of course linked to some objective characterictics, like the possibility of being non-exclusively consumed or the presence of external effects, but also linked to the recognition of such characteristics by the spouses themselves and their agreement to share the good. Another unsolved related issue is the fixing of the autonomy parameters. The techniques used by the New Empirical Industrial Organization when estimating conduct parameters could possibly be adapted.[13]Further work is obviously required.

# References


[1] N. Ashraf, Spousal Control and Intra-household Decision Making: An Experimental Study in the Philippines, Amer. Econ. Rev. 99 (2009), 1245–1277

[2] C. d'Aspremont, R. Dos Santos Ferreira, Price-quantity competition with varying toughness, Games Econ. Behav. 65 (2009), 62-82.

[3] M. Browning, P.-A. Chiappori, Efficient intra-household allocations: A general characterisation and empirical tests, Econometrica 66 (1998), 1241-1278.

[4] M. Browning, P.-A. Chiappori, V. Lechene, Distributional effects in household models: Separate spheres and income pooling. Econ. J. 120 (2010), 786–799.

[5] Z. Chen, F. Woolley, A Cournot-Nash model of family decision making. Econ. J. 111 (2001), 722-748.

[6] L. Cherchye, B. De Rock, F. Vermeulen, The collective model of household consumption: A nonparametric characterization, Econometrica 75 (2007), 553-


[12] This is not only true in the differentiable local approach we have explored here, but also in the revealed preference global approach such as the one used by Cherchye, De Rock and Vermeulen (2007).

[13] These so-called conduct parameters measure the relative weight of competitive toughness and play in the analysis of firm behavior a role similar to the degrees of autonomy in the analysis of household behavior. For more explanation and references see d'Aspremont and Dos Santos Ferreira (2009).

574.

[7] P.-A. Chiappori, Rational household labor supply. Econometrica 56 (1988), 63-90.

[8] P.-A. Chiappori, Collective labor supply and welfare. J. Polit. Economy 100 (1992), 437-46.

[9] G. Debreu, 1952. A social equilibrium existence theorem, Proc. Natl. Acad. Sci. 38 (1952), 886-893.

[10] L. Hurwicz, Outcome functions yielding Walrasian and Lindahl allocations at Nash equilibrium points, Rev. Econ. Stud. 46 (1979), 217-225.

[11] V. Lechene, I. Preston, Household Nash equilibrium with voluntarily contributed public goods, Econ. Series WP 226 (2005), Oxford.

[12] V. Lechene, I. Preston, Noncooperative household demand, J. Econ. Theory 146 (2011), 504-527.

[13] S. Lundberg, R. A. Pollak, Separate spheres bargaining and the marriage market. J. Polit. Economy 101 (1993), 988-1010.

[14] M. Manser, M. Brown, Marriage and household decision-making: A bargaining analysis, Intern. Econ. Rev. 21 (1980), 31-44.

[15] M. B. McElroy, M. J., Horney, Nash-bargained household decisions: Toward a generalization of the theory of demand, Intern. Econ. Rev. 22 (1981), 333-349.

[16] J. Pahl, Family finances, individualisation, spending patterns and access to credit. J. Socio-Econ. 37 (2008), 577-591.

[17] S. G. Schaner, 2010, Intrahousehold Preference Heterogeneity, Commitment, and Strategic Savings: Theory and Evidence from Kenya, Job Market Paper, MIT.

[18] C.R. Udry, 1996. Gender, Agricultural Production and the Theory of the Household. J. Polit. Economy 104 (1996), 1010–46.

[19] D. Ulph, D., A general non-cooperative Nash model of household consumption behaviour. University of Bristol, Economics Discussion Paper 88/205, 1988. French translation: Un modèle non-coopératif de consommation des ménages. L'Actualité économique 82 (2006), 53-85.

[20] C. Vogler, M. Brockmann, R.D. Wiggins, Intimate relationships and changing patterns of money management at the beginning of the twenty-first century, British J. Socio 57 (2006), 455-482.

[21] M. Walker, A simple incentive compatible scheme for attaining Lindahl allocations, Econometrica 49 (1981), 65-71.

# Appendix

## Proof of Proposition 3

(**Separate spheres**) This is the simpler case, in which semi-cooperation is observationally equivalent to non-cooperation. We first show that $\Theta = 0$. For any $h$ and $h'$, the generic element of $\Theta^A \equiv \left(\left[\partial_{\mathcal{P}} x^A\right] + \left[\partial_{\mathcal{Y}} x^A\right]\left[{}^T g^A\right]\right)\mathbf{\Pi}^B$, namely $\sum_k \left(\left[\partial_{\mathcal{P}_k} x_h^A\right] + \left[\partial_{\mathcal{Y}} x_h^A\right]\left[{}^T g_k^A\right]\right)\mathbf{\Pi}_{kh'}^B$, is equal to zero under separate spheres, since $\mathbf{\Pi}_{kh'}^B = 0$ if $k \in M^A$, and $\left(\left[\partial_{\mathcal{P}_k} x_h^A\right] + \left[\partial_{\mathcal{Y}} x_h^A\right]\left[{}^T g_k^A\right]\right) = 0$ if $k \in M^B$ (a variation in the price of a public good to which spouse $A$ does not contribute cannot induce changes in $A$'s demand for any good and $g_k^A = 0$ by definition). The same argument holds for the second term $\Theta^B$ of the matrix $\Theta$, so that $\Theta = 0$. Also, $\left[\widetilde{P}^J\right]\mathbf{\Gamma}^{-J} = \sum_k \overline{\theta}_k^{-J} P_k^J \mathbf{\Gamma}_k^{-J} = 0$, since $P_k^J = 0$ if $k \in M^{-J}$, and $\mathbf{\Gamma}_k^{-J} = 0$ if $k \in M^J$, so that $\Xi = \left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A$. Each matrix $\left[\partial_g x^J\right]$ has at most $n + m^J$ non-zero rows, which however cannot be linearly independent since $[(p,P)]\left[\partial_g x^J\right] = 0$ (consumption changes induced by the sole utility externality effect should not modify the expenditure $(p,P)\cdot x^J$, which has to be kept equal to $J$'s income). Hence, the rank of $\left[\partial_g x^J\right]$ is at most equal to $n + m^J - 1$. As to the matrix $\mathbf{\Gamma}^J$, it has at most $m^J$ non-zero rows, so that the rank of the matrix $\Xi$ cannot be higher than

$$r_\Xi = m + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}.$$

As already stated, the rank of matrix $\Delta$ is at most equal to $r_\Delta = 1$. Now, by applying Euler's identity to the functions $\xi$ and $x^J$, which are homogeneous of degree 0, we see that $[(p,P)]\left[{}^T\Psi\right] = [(p,P)]\left[{}^T\Sigma\right] = 0$, implying $[(p,P)]\left[{}^T(\Delta - \Xi)\right] = 0$, so that the columns of the matrix $\Delta - \Xi$ are not linearly independent. Hence, the rank of this matrix is at most equal to $n + m - 1$.

Taking this upper bound into account and simply adding $r_\Delta$ and $r_\Xi$, we obtain the same result as Lechene and Preston (2010, theorem 3.3) for the fully non-cooperative case:

$$\begin{aligned} r_{\Delta-\Xi} &= \min\left\{n+m-1, 1+m+\min\left\{n-1-\left(m^A-m^B\right), 0\right\}\right\} \\ &= m+\min\left\{n-\max\left\{m^A-m^B, 1\right\}, 1\right\}. \end{aligned}$$

**(Joint contribution to public consumption)** Under the regime of joint contribution to $m^{AB}$ public goods, the column $\left[\partial_{\mathcal{P}_k} x^J\right]+\left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T g_k^J\right]=0$ if $k \in M^{-J}$, as in the regime of separate spheres. Thus, in the product of matrix $\left[\partial_{\mathcal{P}} x^J\right]+\left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T g^J\right]$ with matrix $\mathbf{\Pi}^{-J}$ the corresponding $k$-th line of the latter might as well be zero. But $\mathbf{\Pi}^{-J}$ has $m^J$ further zero lines, namely any $j$-th line such that $j \in M^J$. Hence, the matrix product is upper bounded by $m^{AB}$ and the rank of $\Theta$ (the sum of two such products) by $r_\Theta=2m^{AB}$. As to matrix $\Xi$, we can apply the argument developed for the regime of separate spheres, only with an increase of $m^{AB}$ in the number of non-zero rows. We thus obtain for its maximum rank:

$$\begin{aligned} r_\Xi &= \min\left\{\min\left\{n+m^A+m^{AB}-1, m\right\}+1, m^B+m^{AB}\right\} \\ &\quad +\min\left\{\min\left\{n+m^B+m^{AB}-1, m\right\}+1, m^A+m^{AB}\right\} \\ &= m+m^{AB}+\min\left\{n-\left(m^A-m^B\right), 0\right\}. \end{aligned}$$

By taking again into account the homogeneity of degree zero of the demand functions $\xi$ and $x^J$, and adding the three maximum ranks $r_\Delta$, $r_\Xi$ and $r_\Theta$, we obtain the maximum rank of the deviation matrix:

$$\begin{aligned} r_{\Delta-\Xi+\Theta} &= \min\left\{n+m-1, 1+m+3m^{AB}+\min\left\{n-\left(m^A-m^B\right), 0\right\}\right\} \\ &= m+\min\left\{n-1, \min\left\{n-\left(m^A-m^B\right), 0\right\}+3m^{AB}+1\right\} \\ &= m+3m^{AB}+\min\left\{n-\max\left\{m^A-m^B-1, 3m^{AB}+1\right\}, 1\right\}. \end{aligned}$$

$\square$

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