# HOMEWORK 7

**Problem 1.** Vertices of a regular $n$-gon are colored black and white so that two black vertices are never adjacent. Denote $t_n$ the number of different colorings (vertices are numbered 1, 2, …, $n$). (a) Prove that $t_n \leq f_{n+1}$ where $f_k$ is the $k$-th Fibonacci number. (b) Express $t_n$ via Fibonacci numbers. (c) Prove that $t_{n+1} = t_n + t_{n-1}$ for $n \geq 5$.

A (finite) multiset is a string $[a_1, \dots, a_n]$ where elements can be repeated but their order is not important. Thus $[0, 1, 1] \neq [0, 0, 1]$ but $[0, 1, 1] = [1, 0, 1]$. To every sequence $a_1, \dots, a_n$ one can associate a multiset $[a_1, \dots, a_n]$ of its values.

**Problem 2.** Let $A_n$ be the set of multisets of values of all the parking functions $a_1, \dots, a_n$ (so, for example, $A_2 = \{[0, 0], [0, 1]\}$). (a) Find a direct one-to-one correspondence between the set $A_n$ and the set of Dyck paths of length $2n$. (b) Prove that the number of elements of $A_n$ is the $n$-th Catalan number.

**Hint.** For a parking function $a_1, \dots, a_n$ consider the sequence $p_0, \dots, p_{n-1}$ where $p_k$ is the number of $i$ such that $a_i = k$. Recall that $a_1, \dots, a_n$ is a parking function if and only if $p_k \leq n - k$ for all $k = 0, \dots, n - 1$. If you fail to solve 2(a) you can still try to find an independent solution of 2(b).