HOMEWORK 2

Problem 1 (LGF 3.1)**.** Is the generating function $A(t) = a_0 + a_1 t + a_2 t^2 + \dots$ rational if $a_0.a_1, \dots$ is the following sequence: (a) $1, 2, 3, 4, 5, \dots$; (b) $1, 4, 9, 16, \dots, k^2, \dots$; (c) $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{k^2}, \dots$; (d) f_n^2, where f_n are the Fibonacci numbers ? If yes, write $A(t)$ explicitly.

Problem 2 (LGF 3.2)**.** Let $A(t) = a_0 + a_1 t + a_2 t^2 + \dots$ be the generating function for a sequence $a_0, a_1, a_2, \dots$, and let b be a nonzero number. Express the generating functions for the sequences (a) $a_0 + a_1, a_1 + a_2, a_2 + a_3, \dots$; (b) $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$; (c) $a_0, a_1 b, a_2 b^2, a_3 b^3, \dots$; (d) $a_0, a_2, a_4, a_6, a_8, \dots$ in terms of A.

Problem 3 (LGF 3.3)**.** Prove the following properties of the Fibonacci sequence (a) $f_0 + f_1 + \dots + f_n = f_{n+2} - 1$; (b) $f_0 + f_2 + \dots + f_{2n} = f_{2n+1}$; (c) $f_1 + f_3 + \dots + f_{2n-1} = f_{2n} - 1$; (d) $f_0^2 + f_1^2 + \dots + f_n^2 = f_n f_{n+1}$. Try to use generating functions, not the direct induction.

Problem 4. Find the generating functions and explicit formulas for the elements of the sequences given by the following recurrence relations: (a) $a_{n+2} = 5a_{n+1} - 6a_n$, $a_0 = 2$, $a_1 = 5$; (b) $a_{n+2} = 2a_{n+1} - a_n$, $a_0 = p$, $a_1 = q$; (c) $a_{n+3} = -3a_{n+2} - 3a_{n+1} - a_n$, $a_0 = 1$, $a_1 = a_2 = 0$; (d) $a_{n+4} = -a_n$.