# HOMEWORK 11

Let a *$q$-factorial* mean $(q)_n \stackrel{\text{def}}{=} (1-q^n)\dots(1-q)$ where $n \in \mathbb{N}$ and $q \in \mathbb{C}$. Let also a *$q$-binomial coefficient* mean $\binom{n}{k}_q \stackrel{\text{def}}{=} \frac{(q)_n}{(q)_k(q)_{n-k}}$.

**Problem 1.** Prove the identities $\lim_{q\to 1} \frac{(q)_n}{(1-q)^n} = n!$ and $\lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k}$.

**Problem 2.** Prove the identitles of a "$q$-Pascal triangle": $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$ and $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q$

Let a *Young diagram* be a nondecreasing sequence of positive integers: $\mathcal{D} = (l_1, \dots, l_s)$, where $0 < l_1 \le \dots \le l_s$. It is convenient to draw a Young diagram as $s$ lines containing $l_1, l_2, \dots, l_s$ boxes, respectively.

**Problem 3.** Prove the identity $\binom{n}{k}_q = \sum_{m=0}^n q^m Y_m$ where $Y_m$ is the number of Young diagram containing $m$ boxes and fitting into a rectangle $k \times (n-k)$.

Denote $Z_n^k(q)$ the sum of all the expressions $q^{a_1+\dots+a_k}$ over the set of all the sets $\{a_1, \dots, a_k\}$ of $k$ integers, each integer satisfying the inequality $1 \le a_i \le n$.

**Problem 4.** Prove the identity $Z_n^k(q) = \binom{n}{k}_q q^{k(k+1)/2}$.

Denote $(1+z)^n|_q \stackrel{\text{def}}{=} (1+z)(1+zq)\dots(1+zq^{n-1})$ (a "$q$-power").

**Problem 5.** Prove the identity ("$q$-Newton's formula") $(1+z)^n|_q = \sum_{k=0}^n \binom{n}{k}_q q^{k(k-1)/2} z^k$.

**Problem 6.** Prove the following identity (Cauchy's formula): $1+\sum_{n=1}^\infty \frac{(1-a)^n|_q}{(q)_n} t^n = \prod_{m=0}^\infty \frac{1-atq^m}{1-tq^m}$.

**Hint.** Prove that both the left-hand side and the right-hand side of the formula satisfy the identity $F(t)(1-t) = F(tq)(1-at)$.

Take, by definition $(1+z)^\alpha|_q = \frac{(1+z)(1+zq)\dots}{(1+zq^\alpha)(1+zq^{\alpha+1})\dots}$, for any complex number $\alpha$ (a $q$-power with a complex exponent).

**Problem 7.** Derive from Cauchy's formula the following identity for the $q$-power with arbitrary complex exponent: $(1+z)^\alpha|_q = \sum_{k=0}^\infty q^{k(k-1)/2} \frac{(1-q^\alpha)(1-q^{\alpha-1})\dots(1-q^{\alpha-k+1})}{(q)_n} z^k$.