HOMEWORK 8

Problem 1. Find an explicit formula for the 2-variable generating function $F(x,y) = \sum_{n,m} u_{n,m} x^n y^m$ for the number of Dyck paths starting at the origin and ending at the point with coordinates (n,m).

Problem 2 (LGF 7.1)**.** Tchebyshev polynomials $T_n(x)$ are defined by the identity $\cos n\varphi = T_n(\cos\varphi)$ (so that $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, etc.). Prove the identity $\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}$.

Hint. Use the identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$ for $a = n\varphi$, $b = \varphi$. For example, it implies that $\cos 2\varphi = \cos(\varphi+\varphi) = \cos^2\varphi - \sin^2\varphi = \cos^2\varphi - (1 - \cos^2\varphi) = 2\cos^2\varphi - 1$, so $T_2(x) = 2x^2 - 1$.

Problem 3 (LGF 7.4)**.** Denote by $a_{n,k}$ the number of paths in the Dyck triangle consisting of n vectors such that the area under the path is k; $a_{2,1} = 1$, $a_{2,k} = 0$ for k even. Prove that $A(s,t) = \sum a_{n,k} s^n t^k = \frac{1}{1-\frac{s^2t}{1-\frac{s^2t^3}{1-\frac{s^2t^5}{1-\dots}}}}$.

Problem 4 (LGF 7.5ad)**.** Prove that (a) $\sum_{n=0}^{\infty} (2n-1)!!s^{2n} = \frac{1}{1-\frac{s^2}{1-\frac{2s^2}{1-\frac{3s^2}{1-\dots}}}}$ where $(2n-1)!! = 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)$, $(-1)!! = 1$; (b) $\sum_{n=0}^{\infty} n!s^n = \frac{1}{1-s-\frac{1^2s^2}{1-3s-\frac{2^2s^2}{1-5s-\frac{3^2s^2}{1-\dots}}}}$.