HOMEWORK 10

Problem 1. Find the Moebius function for the following partially ordered sets: (a) $A = \mathbb{N}$, $x \le y$ as usual; (b) $A = \mathbb{N}^k$, $(x_1, \dots, x_k) \le (y_1, \dots, y_k)$ means $x_1 \le y_1, \dots, x_k \le y_k$. (c) $A = \mathbb{N}$, $x \le y$ means that x divides y. (d) A is the set of all subsets of some finite set $U = \{u_1, \dots, u_n\}$, $x \le y$ means that $x \subseteq y$.

Problem 2. (a) Find the number d_n of permutations σ of the set $\{1, 2, \dots, n\}$ such that $\sigma(k) \ne k$ for all $k = 1, 2, \dots, n$. (b) Find the generating function $\sum_{n=1}^{\infty} d_n t^n/n!$.

Problem 3. (a) Let $\zeta(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} 1/n^s$ and $M(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \mu_k/k^s$ where $\mu_k = 0$ for any $k \in \mathbb{N}$ divisible by a square of a prime number, and $\mu_k = (-1)^m$ if $k = p_1 p_2 \dots p_m$ is a product of m different prime numbers. Prove that $\zeta(s)M(s) = 1$. (b) Prove that $M(s) = \prod_p (1 - 1/p^s)$ where the product is taken over the set of all prime numbers.

Hint. Use Moebius inversion formula and Problem 1(c).

Problem 4. Prove that $\max(a_1, \dots, a_n) = a_1 + \dots + a_n - \min(a_1, a_2) - \dots - \min(a_{n-1}, a_n) + \min(a_1, a_2, a_3) + \dots + (-1)^{n-1} \min(a_1, \dots, a_n)$ where $a_1, \dots, a_n$ are any real numbers.

Hint. Use Moebius induction formula and Problem 1(d)

Problem 5. Prove that $\prod_{n=0}^{\infty} (1 - x^n)^{-\mu_n/n} = e^x$ where μ_n is defined in Problem 3(a).

Hint. Take a logarithm of both sides of the equality,