HOMEWORK 3

Problem 1. Let $a = (a_1, \ldots, a_n)$ be a permutation of the numbers $1, \ldots, n$. Call a number i a descent, if $a_i > a_{i+1}$. The *major index* $\mathrm{maj}(a)$ of the permutation a is defined as the sum of all its descents. Let q_k be the number of permutations with $\mathrm{maj}(a) = k$. Prove that (a) $q_0 = 1$, (b) $q_1 = n-1$, (c) $q_2 = (n-1)(n+2)/2$, (d) $q_{n(n-1)/2} = 1$, (e) $q_k = q_{n(n-1)/2-k}$, for any k, (f) the generating function $\sum_{k=0}^{n(n-1)/2} q_k t^k$ is equal to the product $(1+t)(1+t+t^2)\ldots(1+t+\cdots+t^{n-1})$, and therefore q_k is equal to the number of permutations with k inversions.

Problem 2. Consider the set of paths on the plane consisting of vectors $(1,0)$, $(-1,0)$, $(0,1)$. Let a_n be the number of such paths of length n issuing from the origin and nonselfintersecting (this means that the path does not contain adjacent vectors $(1,0)$ and $(-1,0)$). Find the generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

Problem 3. Let $S(n,k)$ denote the number of ways to split the set $\{1,2,\ldots,n\}$ into k distinct nonempty pairwise nonintersecting subsets (e.g. $S(3,2) = 3$ because $\{1,2,3\} = \{1,2\} \cup \{3\} = \{1,3\} \cup \{2\} = \{2,3\} \cup \{1\}$). Prove that $\sum_{n \geq k} S(n,k) = x^k/(1-x)(1-2x)\ldots(1-kx)$.