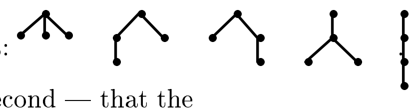
HOMEWORK 4

Problem 1. Let $a = (a_1, \dots, a_n)$ be a permutation of the numbers $1, \dots, n$. Call a pair (i, j) an *inversion* if $i < j$ but $a_i > a_j$. Let q_k be the number of permutations containing exactly k inversions. Prove that (a) $q_0 = 1$, (b) $q_1 = n - 1$, (c) $q_2 = (n-1)(n+2)/2$, (d) $q_{n(n-1)/2} = 1$, (e) $q_k = q_{n(n-1)/2-k}$, for any k, (f) the generating function $\sum_{k=0}^{n(n-1)/2} q_k t^k$ is equal to the product $(1+t)(1+t+t^2)\dots(1+t+\dots+t^{n-1})$, and therefore q_k is equal to the number of permutations with major index k.

Problem 2. Consider some operation $*$ with two operands: $(a, b) \mapsto a * b$. The operation is not associative, so the value of $a_1 * a_2 * a_3$ may depend on the order of operations: $(a_1 * a_2) * a_3 \neq a_1 * (a_2 * a_3)$ (for example, it is so if $a * b \stackrel{\text{def}}{=} a^b$). Prove that the number of possible interpretations of the expression $a_1 * a_2 * \dots * a_n$ is the $(n-1)$-th Catalan number.

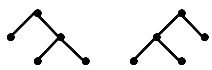
Example. The expression $a_1 * a_2 * a_3$ may have $c_2 = 2$ meanings: either $(a_1 * a_2) * a_3$ or $a_1 * (a_2 * a_3)$.

Problem 3. The *family tree* is a graph drawn as follows: any family member is represented by a point (vertex of the graph), and his/her children are vertices drawn immediately below it, left to right according to their ages; children are joined with their parents by lines (edges of the graph). Prove that the number of different family trees for a family with n members (in all generations) is equal to the $(n-1)$-th Catalan number.

Example. There are $c_3 = 5$ family trees for a family with 4 members: 

The leftmost means that the family founder had 3 children, the second — that the family founder had 2 children, of which the older had a child, etc.

Problem 4. The *decision tree* is a graph drawn as follows: any its vertex is either terminal ("an answer") or has 2 attached vertices one level below it ("a yes/no question"); the left vertex corresponds to a negative answer, and the right vertex, to a positive one. Prove that the number of different decision trees with n questions is equal to the n-th Catalan number.

Example. There are $c_2 = 2$ decision trees with 2 questions:  .

The left one means

`Ask question 1; if the answer is negative then the decision is made. If the answer was positive then ask question 2, and make a decision depending on the answer.`

The right one means that question 2 should be asked if the answer to question 1 is negative.