

HOMEWORK 9

Let $\{1, 2, \dots, n\} = A_1 \sqcup A_2 \sqcup \dots \sqcup A_k$ be a representation of the set $\{1, 2, \dots, n\}$ as a union of k disjoint subsets. Associate to such a representation a Motzkin path: starting in the origin, we make the i -th segment ascending if $i \in A_j$ and i is the smallest number in A_j ; descending if i is the biggest number in A_j ; horizontal if $A_j = \{i\}$ or i is neither smallest nor biggest.

Problem 1. Prove that the line obtained is indeed a Motzkin path.

Problem 2. Compute the number of representations $\{1, 2, \dots, n\} = A_1 \sqcup A_2 \sqcup \dots \sqcup A_k$ corresponding to a given Motzkin path.

Problem 3. Find the generating function $\sum_{n=1}^{\infty} P_n t^n$ where P_n is the total number of representations $\{1, 2, \dots, n\} = A_1 \sqcup A_2 \sqcup \dots \sqcup A_k$ (for all k).