FINAL EXAM

Problem 1. Which is bigger: the number of trees with n vertices numbered $1, 2, \dots, n$ such that the vertex number 1 is hanging or the number of such trees where the vertex number 1 is *not* hanging ?

Problem 2. Let I_n denote the number of ways to select several pairs (possibly zero) out of n people (so that $I_1 = 1$, $I_2 = 2$, $I_3 = 4$, etc.). Prove the identity $\sum_{n=0}^{\infty} I_n s^n = \cfrac{1}{1 - s - \cfrac{s^2}{1 - s - \cfrac{2s^2}{1 - s - \cfrac{3s^2}{1 - \dots}}}}$.

Problem 3. Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be a decomposition of the integer n into primes. Prove that the number of integers $m < n$ such that m and n are coprime is equal to $n(1 - 1/p_1)(1 - 1/p_2) \dots (1 - 1/p_k)$.

Hint. n and m are coprime if m is not divisible by any of the p_i.

Problem 4. How many are there ways to color three sides of an octagon black, three sides, white and two sides, blue ? Two colorings are considered the same if they can be made identical by a rotation of the octagon.